Mean-field theory for neural networks

main goal: mathematical rigorous modelling of neural microcircuits Assumptions



 network should exhibit stable "balanced state", i.e. each cell operates close to threshold

Math challenges

- how to formalize the above assumptions?
- how to characterize "balanced states" in mathematical terms?
- how to derive mean-field limits and fluctuation theory (for finite-size corrections)?
- statistical analysis of sparse networks, e.g. autocorrelation
- ► information theory (?)

- Molecular

– Cellular

A brain-in-a-box simulation will have to capture every detail of neurons and nonneuronal glial cells, including the exact geometric shapes of the dendrites and exons that receive and send information.

- Circuits

A model of the neural connections between different brain areas and among neighboring cells may furnish clues to the origins of complex brain diseases such as autism and schizophrenia.

- Regions

Major neural substructures—the arrygdala (emotions), the hippocampus (memory), the frontal lobes (executive control) can be inspected alone or as they interact with one another.

- Whole Organ

An in slice brain might subsitute for the actual organ. By removing the computer code for a "gene," the virtual system can, for instance, minic the effects of a mutation, as scientists do today by "Knocking out" a gene in mice. The tool would evoid the lengthy breeding process and could simulates a multitude of experimental conditions.

Agenda

Part I: Building blocks of brain networks: neurons Part II: (Biological) neural networks Part III: (Stochastic) Mean-field theories

Main references

W. Stannat Stochastic Processes in Neuroscience, Lecture Notes, TU Berlin, Version: July 24, 2017

B. Ermentrout, H. Terman Mathematical Foundations of Neuroscience, Springer, Berlin, 2010.

W. Gerstner, W. Kistler Spiking Neuron Models, Cambridge University Press, 2002.

C. Van Vreeswijk, H. Sompolinsky Chaotic balanced state in a model of cortical circuits, Neural Comput 10, 1321--1371, 1998.

neural activity in living organisms occurs on different scales/levels:

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- (a) individual ion channels (microcospic)
- (b) single neurons (mezoscopic)
- (c) population of neurons (macroscopic)
 - neural activity is intrinsically noisy
 - single neurons exhibit a large variability

Membrane potential v

first neural activity that could be measured by physiologists **special dynamical feature** temporal spikes (action potential)



APs usually travel down the axon towards the axon terminals where they may be passed over to other neurons or muscles

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Regulating the membrane potential

v is driven by three types of electrical currents

$$C\frac{dv}{dt} = -F + I_{syn} + I_{ext} \tag{1}$$

- (i) F denotes the sum of currents as a result of ions flowing into or out of the cell membrane through ion channels, also called the membrane current
- (ii) *I*_{syn} denote the synaptic currents entering the cell
- (iii) $I_{e\times t}$ denotes externally injected currents (e.g. exterior signals).

 I_{syn} and I_{ext} can be seen as exterior controls

F denotes intrinsic regulation of v via electrically charged particles (ion) pouring through the membrane via ion channels

single ion channel currents have first been measured by Neher and Sakmann (Nobel Prize in the year 1991), measurements showed that the dynamics of single ion channels is intrinsically random

lon channel dynamics in the squid giant axon



(see Vandenberg et. al., Biophys J., 1991)

widely accepted in computational neuroscience today that adequate modeling of statistics of single ion channels requires (time-continuous) Markov chains on finite number of states and that the switching between these states, the transition rates, are voltage dependent

From microscopic to mezoscopic

simplest mathematical model for ion channels two-state Markov chains



well-known: p(t) = P(X(t) = 0) solves the Kolmogorov forward eq (resp. Fokker-Planck eq)

$$\frac{dp}{dt} = \alpha(v)(1-p) - \beta(v)p \tag{2}$$

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From microscopic to mezoscopic, ctd.

yields "first-order" approximation of the proportion of ion channels being in the open state in the sense of the law of large numbers:

given (independent) ion channels X_1, X_2, \ldots , then *P*-a.s.

$$p_N(t) := \frac{1}{N} \sum_{i=1}^N \mathbb{1}_O(X_i(t)) \to p(t) = E(\mathbb{1}_O(X_i(t)))$$
(3)

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important question in computational neuroscience: numerically efficient approximation of $p_N(t)$ and other statistics of large, but finite number of ion channels

classical method: diffusion approximation

$$dp_{N} = \alpha(v)(1-p_{N}) - \beta(v)p_{N} dt + \frac{1}{\sqrt{N}}\sqrt{\alpha(v)(1-p_{N}) + \beta(v)p_{N}} dB_{t}$$

$$\sim \alpha(v)(1-p_{N}) - \beta(v)p_{N} dt + \frac{1}{\sqrt{N}}\sqrt{\alpha(v)(1-p) + \beta(v)p} dB_{t}$$
(4)

where B denotes 1d-Brownian motion

Open problems

- ▶ (4) does not leave [0,1] forward invariant, as probabilities should do....
- (general) at stationary points of p(t) variance becomes small, hence diffusion approx. ill-posed

so there is a need for "faithful", yet computationally efficient, approximations

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Mathematical models for v

statistical models: point neurons

- binary neurons n(t) = 0 1, i.e. inactive-active
- (stochastic) integrate-and-fire (IF) neuron:

$$dv(t) = b(v(t)) dt + \sigma(v(t)) dB_t, v(0) = v_r$$

until first-passage time

$$T := \inf\{t \ge 0 \mid v(t) \ge v_{th}\}$$

for some given threshold value v_{th} , then $v(T)
ightarrow v_r$

conductance based neural models: e.g., Hodgkin-Huxley, Morris-Lecar, ...

Conductance-based neuronal models

coupling the membrane potential to the ion channel dynamics leads to conductance-based neural models first and most prominent example: Hodgkin-Huxley model introduced in 1952 consisting of three types of ion channels:

- *I_K* potassium channels (activating)
- I_{Na} sodium channels (activating)
- I_L sodium channel (inactivating)

$$C\frac{dv}{dt} = \bar{g}_k n^4 (v_K - v) + \bar{g}_{Na} m^3 h (v_{Na} - v) + \bar{g}_L (v_L - v) + I_{ext}$$

$$\frac{dn}{dt} = \alpha_n(v)(1 - n) - \beta_n(v)n$$

$$\frac{dm}{dt} = \alpha_m(v)(1 - m) - \beta_m(v)m$$

$$\frac{dh}{dt} = \alpha_h(v)(1 - h) - \beta_h(v)h$$
(5)

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Conductance-based neuronal models, ctd.

$$C\frac{dv}{dt} = \bar{g}_k n^4 (v_K - v) + \bar{g}_{Na} m^3 h (v_{Na} - v) + \bar{g}_L (v_L - v) + I_{ext}$$

$$\frac{dn}{dt} = \alpha_n(v)(1 - n) - \beta_n(v)n$$

$$\frac{dm}{dt} = \alpha_m(v)(1 - m) - \beta_m(v)m$$

$$\frac{dh}{dt} = \alpha_h(v)(1 - h) - \beta_h(v)h$$
(6)

▶ \bar{g}_{K} , \bar{g}_{Na} , \bar{g}_{L} maximal values of membrane conductances

$$\triangleright$$
 v_K , v_{Na} , v_L corr. reversal potentials

transition rates are given as

$$\begin{aligned} \alpha_n(v) &= \frac{10 - v}{100(e^{(10 - v)/10} - 1)} \qquad \beta_n(v) = \frac{1}{8}e^{-V/80} \\ \alpha_m(v) &= \frac{25 - v}{10(e^{(25 - v)/10} - 1)} \qquad \beta_m(v) = 4e^{-v/18} \\ \alpha_h(v) &= \frac{7}{100}e^{-v/20} \qquad \beta_h(v) = \frac{1}{e^{(30 - v)/10} + 1} \end{aligned}$$

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Conductance-based neuronal models, ctd.

the system of coupled differential equations exhibits a bifurcation w.r.t. the exterior input current I_{ext} . Depending on its value, one can observe a single or a finite number of spikes or even periodic spikes. More precisely: in the above parameter set:

- minimal current required for at least one spike: $I_{ext} = 2.5$
- threshold value for periodic spiking: $I_{ext} = 6.25$
- if $I_{ext} > 154$ the amplitude of the spikes decreases rapidly.



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Mathematical explanation

Simplified model: FitzHugh-Nagumo system

$$\dot{v} = f(v) - w + I$$
$$\dot{w} = \epsilon (v - \gamma w)$$

for bistable f, e.g., f(v) = v(1-v)(v-a), $a \in (0,1)$ basic feature - three different regimes: $\exists I_- < I_+$

- (suboscillatory, resp. excitable) $I < I_{-}$ at most one spike
- $I_- < I \le I_+$ periodic spiking
- ▶ (superoscillatory) *I* > *I*₊ at most one spike



Spatially extended models

taking into account spatial extensions of the neuron leads to PDEs

$$\begin{aligned} \tau \partial_t v &= \lambda^2 \partial_{xx}^2 v - g_{Na} m^3 h(v - v_{Na}) - g_K n^4 (v - v_K) - g_L (v - v_L) + I \\ \frac{dp}{dt} &= \alpha_P(v) (1 - p) - \beta_P(v) p \quad p \in \{m, n, h\} \end{aligned}$$

where

• v membrane potential, $v = v(t, x), t \ge 0, x \in [0, L]$

- \blacktriangleright m, n, h gating variables, $0 \le m, n, h \le 1$
- τ resp. λ specific time resp. space constants
- g_{Na}, g_K, g_L conductances
- v_{Na}, v_K, v_L resting potentials

•
$$\alpha_p(v) = a_p^1 \frac{v + A_p}{1 - e^{-a_p^2(v + A_p)}}, \ \beta_p(v) = b_p^1 e^{-b_p^2(v + B_p)}$$

typical shape of v



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In reality more like this ...



Figure 1: Membrane potential in one neuron (in a cortical slice preparation, observed in vitro, under different levels of potassium).

Höpfner, Math. Biosciences, 2007

due to fluctuations between open and closed states of ion channels regulating v

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Channel noise impact on APs

- spontaneous spiking (due to random opening of sufficient numbers of Na-channels)
- time jitter spike time distribution increases with time
- APs can split up or annihilate
- propagation failure

places limits on the axon diameter (around $0.1 \mu m$), hence also on the wiring density

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e.g., White, et al., Trends Neurosci. 2000, Faisal, et al., Current Biology 2005, Faisal, et al., PLOS 2007

$\mathsf{PDE} \to \mathsf{SPDE}$: additive noise

yields a stochastic pde: Current noise

$$\tau \partial_t v = \lambda^2 \partial_{xx} v - g_{Na} m^3 h(v - E_{Na}) - g_K n^4 (v - E_K) - g_L (v - E_L) + I + \sigma \partial_t \xi(t, x)$$

$$\frac{dp}{dt} = \alpha_P(v)(1-p) - \beta_P(v)p, \quad p \in \{m, n, h\}$$
(7)



features: subthreshold excitability (well-known already in the point neuron) due to spatial extension: spontaneous spiking, backpropagation, annihilation, propagation failure

Illustration - subthreshold excitability

(already known from the point neuron case)





 $I = 6.0, \sigma = 0.0$

 $I = 6.0, \sigma = 0.025$

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Illustration - spontaneous activation, backpropagation



 $I = 6.0, \sigma = 0.25$

 $I = 2.0, \sigma = 0.36$

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Part II: (Biological) neural networks

modeling of activity of neural circuits requires modeling the interneural communication



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[Izhikevich, Dynamical Systems in Neuroscience]

Self-organizing neural dynamics

already the generation of APs is an example of self-organization of ion-channels similar mechanisms of self-organization are also observed in neural fields

Illustration: Local field potentials in cats during wake and sleep



[Destexhe, et al., Scholarpedia, 2013]

Self-organizing neural dynamics

major open questions for neural systems, and in systems biology in general:

establish **theories for the collective behavior** of neural networks in terms of their local specifications, that is, the specification of the single neurons and their connections.

clearly, this requires **global** rules, similar to the case of kinetic gas theory, where the global statistical behavior of a gas can be deduced from its local interactions using simple thermodynamical rules

the **difficulty** in biological systems in general and in biological neural networks in particular is to determine simple but nevertheless relevant global rules that are responsible for the rich observed phenomenology of these complex highly nonlinear systems

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Self-organizing neural dynamics - Illustration

linearly coupled two dimensional FitzHugh-Nagumo systems



and on each grid point (i, j) the following two-dimensional FitzHugh-Nagumo system linearly coupled to neighboring neurons:

$$\frac{dv_{ij}}{dt} = v_{ij}(1 - v_{ij})(v_{ij} - a) - w_{ij} + \frac{1}{2h}(v_{i+1,j} - v_{ij} + v_{i-1,j} - v_{ij}) \\
+ \frac{1}{2h}(v_{i,j+1} - v_{ij} + v_{i,j-1} - v_{i,j})$$
(8)
$$\frac{dw_{ij}}{dt} = b(v_{ij} - a + w_{ij}) \\$$
where $a \in (0, 1), \ b \in \mathbb{R}$ and $h \sim \frac{1}{N}$

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Self-organizing neural dynamics - Illustration, ctd.

for certain parameters and certain initial conditions the system exhibits remarkable collective behavior



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Classical model - Binary neural networks

- reduced math. description $n_i(t) = 0 1$, i = 1, ..., N
- Markovian dynamics

$$\begin{cases} n_i: 0 \to 1 & \text{ with rate } f\left(\gamma \sum_j J_{ij}^{(N)} n_j + I_{ext}\right) \\ n_i: 1 \to 0 & \text{ with rate } 1 - f(\ldots) \end{cases}$$

for given $0 \leq f \leq 1$, $J_{ij}^{(N)} \in \{0,1\}$

- sparsity $\sum_{j} J_{ij}^{(N)} = \mathcal{O}(K), \ K \ll N$
- operation close to threshold $\gamma = \mathcal{O}\left(\frac{1}{\sqrt{K}}\right)$

features

- admits "asynchronous irregular" states
- ▶ MFT for the mean rate $\bar{n}(t) = \frac{1}{N} \sum_{i} n_i(t)$ combines Poissonian and central limit theorem, widely unexplored mathematically
- some own preliminary work on
 - series expansions of n

 in
 - algorithmic tractable approximations
 - sdes for fluctuations
 - extensions to heterogeneous $J_{ii}^{(N)}$

taking up current research on inhibitory neurons in cortical microcircuits

Part III - (Stochastic) MFT

class. simplifications for rigorous math analysis:

- symmetry $J_{ij} = J_{ji}$
- all-to-all couplings $K = \sum_{j} J_{ij} = \mathcal{O}(N)$

well understood with the help of (equilibrium) statistical mechanics (since in this case it becomes a gradient type dynamics) **motivation for asymmetry**

- symmetry lacks neurophysiological plausibility, because synapses operate unidirectional
- ▶ the majority of neurons either act excitatorily $(J_{ij} > 0)$ or inhibitorily $(J_{ij} < 0)$ which also contradicts symmetry
- symmetry creates additional attractors that do not correspond to memorized states (e.g., metastable mixture states, spin-glass attractor)

additional motivation for asymmetric couplings can be found in the survey article:

Kree, R. and Zippelius, A. (1991). Asymmetrically diluted neural networks, in Models of Neural Networks, ed. van Hemmen, et al., Springer

Classical simplifications

For rigorous math analysis:

- **•** symmetry $J_{ij} = J_{ji}$
- all-to-all couplings $K = \sum_{j} J_{ij} = \mathcal{O}(N)$

well understood with the help of (equilibrium) statistical mechanics (since in this case it becomes a gradient type dynamics)

motivation for dilution

neural connectivity is high, but far away from all-to-all

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allows for structural/hierarchical models

An exactly solvable asymmetric neural network model Derrida, et al., Europhys. Lett., 4, pp. 167-173 (1987)

$$J_{ij}=rac{1}{K}c_{ij}\sum_{\mu=1}^{
ho}\xi_{i}^{\mu}\xi_{j}^{\mu}$$

where

- ► $\xi_i^{\mu} = \pm 1$ value of neuron *i* in pattern μ , supposed to be independent random variables with $P(\xi_i^{\mu} = \pm 1) = \frac{1}{2}$
- ▶ $c_{ij} \in \{0,1\}$ random, independent, $P(c_{ij} = 1) = \frac{\kappa}{N}$

Dynamics

parallel - all neurons updated simultaneously

$$n_i(t+\delta t) = \begin{cases} +1 & \text{with prob.} (1+\exp(-2\beta u_i(t)))^{-1} \\ -1 & \text{with prob.} (1+\exp(+2\beta u_i(t)))^{-1} \end{cases}$$
(9)

where

▶ $\beta = \frac{1}{\overline{T}}$ has the interpretation of inverse temperature

•
$$u_i(t) = \sum_j J_{ij} n_j(t)$$

typical order of $\Delta t = \mathcal{O}(1)$

An exactly solvable asymmetric neural network model Derrida, et al., Europhys. Lett., 4, pp. 167-173 (1987)

$$J_{ij}=rac{1}{K}c_{ij}\sum_{\mu=1}^{
ho}\xi_{i}^{\mu}\xi_{j}^{\mu}$$

where

- ► $\xi_i^{\mu} = \pm 1$ value of neuron *i* in pattern μ , supposed to be independent random variables with $P(\xi_i^{\mu} = \pm 1) = \frac{1}{2}$
- $c_{ij} \in \{0,1\}$ random, independent, $P(c_{ij}=1) = rac{\kappa}{N}$

Dynamics

random sequential - choose a neuron *i* randomly (according to uniform distribution) and update its state according to (9) typical order of $\Delta t = O\left(\frac{1}{N}\right)$

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Problem How to compare both dynamics precisely?

- parallel update is deterministic
- sequential update is random

Main result: dynamical properties as $N \rightarrow \infty$

observable

$$m(t) = E\left(\frac{1}{N}\sum_{i=1}^{N}\xi_{i}^{\mu}n_{i}(t)\right)$$

overlap with stored pattern $(\xi_1^\mu,\ldots,\xi_N^\mu)$

parallel dynamics

$$m(t+\Delta t)=f(m(t))$$

sequential update

$$\frac{d}{dt}m(t) = -m(t) + f(m(t))$$

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where

$$f(m) = \sum_{k=0}^{\infty} \frac{K^k e^{-k}}{k!} \sum_{n=0}^k \sum_{s=0}^{k(p-1)} \frac{(1+m)^{k-n}(1-m)^n}{2^{kp}} \binom{k}{n} \binom{k(p-1)}{s} \cdot \tanh\left(\beta \left(kp - 2(n-s)\right)\right)$$

Main result: dynamical properties as $N \to \infty$ - fixed row sum

$$\sum_{j=1}^{N} c_{ij} \equiv K$$

observable

$$m(t) = E\left(\frac{1}{N}\sum_{i=1}^{N}\xi_{i}^{\mu}n_{i}(t)\right)$$

overlap with stored pattern $(\xi_1^\mu,\ldots,\xi_N^\mu)$

parallel dynamics

$$m(t+\Delta t)=f(m(t))$$

sequential update

$$\frac{d}{dt}m(t)=-m(t)+f(m(t))$$

where

$$f(m) = \sum_{n=0}^{K} \sum_{s=0}^{K(p-1)} \frac{(1+m)^{K-n}(1-m)^n}{2^{Kp}} \binom{K}{n} \binom{K(p-1)}{s}.$$

$$\cdot \tanh(\beta (Kp-2(n-s)))$$

General case - Heuristics

recall:

- reduced math. description $n_i(t) = 0 1$, i = 1, ..., N
- Markovian dynamics

$$\begin{cases} n_i: 0 \to 1 & \text{ with rate } f\left(\gamma \sum_j J_{ij}^{(N)} n_j + I_{ext}\right) \\ n_i: 1 \to 0 & \text{ with rate } 1 - f(\ldots) \end{cases}$$

for given $0 \leq f \leq 1$, $J_{ij}^{(N)} \in \{0,1\}$

as $N \to \infty$, but $K \ll N$, one may conjecture that $n_i(t)$ become independent, if initial conditions $n_i(0)$ are independent this conjecture has been verified in the case of the parallel update $K = \sqrt{2}(t - N)^{-1}$

 $K = O(\log N)$ in Derrida, et al., J. Physique 47, 1297-1303, 1986 suppose also that $n_i(t)$ are identically distributed, then

$$u_i^{(N)}(t) = \gamma \sum_{j=1}^N J_{ij}^{(N)} n_j(t) - m \sim \gamma U(t) - m$$

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with

$$U(t) \sim \operatorname{Bin}(K, m(t))$$

$$m(t) = E(n_i(t)) = E\left(\frac{1}{N}\sum_{j=1}^N m_j(t)\right)$$

General case - Heuristics, ctd.

the weak law of large numbers therefore implies

$$\frac{1}{N}\sum_{i=1}^{N}f(u_{i}^{(N)}(t))\sim E\left(f\left(\gamma U(t)-m\right)\right)$$

therefore

$$m^{(N)}(t) = rac{1}{N} \sum_{i=1}^{N} n_i(t) \sim m(t)$$

with

$$\frac{d}{dt}m(t) = -m(t) + E\left(f\left(\gamma U(t) - m\right)\right)$$

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CLT approximation - K large

for increasing K

CLT-approximation $Bin(K, m(t)) \sim N(Km(t), Km(t)(1 - m(t)))$ yields

$$\gamma U(t) - m \sim N(\mu_1(t), \mu_2(t))$$

with $\mu_1(t) = \gamma K m(t) - m$, $\mu_2(t) = \gamma^2 K m(t)(1 - m(t))$, and thus

$$\frac{d}{dt}m(t) \sim -m(t) + \frac{1}{\sqrt{2\pi\gamma^2 Km(t)(1-m(t))}} \int f(u) e^{-\frac{(u-\gamma Km(t)-m))^2}{2\gamma^2 m(t)(1-m(t))}} du$$

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obtained in Van Vreeswijk, et al., 1998.

2nd lecture: Stochastic mean field theories for brain networks

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Elements of MFT: martingale structure of Markov chains

General setting

 $(X(t))_{t\geq 0}$ - (time-homogeneous) time-continuous Markov chain on finite state space S, right-cont. trajectories $(P(t))_{t\geq 0}$ - family of transition probabilities Q - generator (rate) matrix, i.e., d

$$Q=rac{d}{dt}P(t)_{|t=0} \qquad P(t)=e^{tQ}\,,t\geq 0$$

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 $\mathcal{F}_t := \sigma\{X(s) \mid s \leq t\}, \ t \geq 0$, filtration generated by X

Martingale structure, ctd.

Theorem

Let $f : S \to \mathbb{R}$ be any bounded function. Then

$$f(X(t)) = f(X(0)) + M^{f}(t) + \int_{0}^{t} Qf(X(s)) \, ds \, t \ge 0 \,, \tag{10}$$

where

$$M^{f}(t) := f(X(t)) - f(X(0)) - \int_{0}^{t} Qf(X(s)) \, ds \, , t \geq 0 \, ,$$

is a right-continuous martingale w.r.t. $(\mathcal{F}_t)_{t\geq 0}$ with

$$E\left(M^{f}(t)^{2}\right) = E\left(\int_{0}^{t} \left(Q\left(f^{2}\right) - 2fQf\right)(X(s)) ds\right)$$
$$= \int_{0}^{t} E\left(\sum_{j \in S} q_{X(s)j}\left(f(X(s)) - f(j)\right)^{2}\right) ds$$
(11)

Moreover,

$$M^{f}(t)^{2} - \int_{0}^{t} \sum_{j \in S} q_{X(s)j} \left(f(X(s)) - f(j) \right)^{2} ds, t \geq 0.$$
 (12)

is again a right-continuous martingale w.r.t. $(\mathcal{F}_t)_{t\geq 0}$.

Remarks on Theorem

Remark

- (10) is called the semimartingale decomposition of the process f(X(t)), since it gives a decomposition into a martingale and a process of bounded variation ∫₀^t Qf(X(s)) ds.
- ▶ (10) is the analogue of the Ito-decomposition of f(X(t)) for f ∈ C² and X(t) being the solution of a stochastic differential equation

Corollary

Suppose that $P(X(0) = i_0) = 1$ for some initial state $i_0 \in S$. Then

$$E\left(\left(M^{f}\right)^{2}(t)\right) = \int_{0}^{t} \sum_{i,j\in S} p_{i_{0}j}(s)q_{ij}\left(f(i) - f(j)\right)^{2} ds$$

w.l.o.g. $P(X(0) = i_0) = 1$ for some initial state $i_0 \in S$ then Markov property implies for any bounded $g: S \to \mathbb{R}$

$$E\left(g(X(t+s)) \mid \mathcal{F}_s\right) = P(t)g(X(s)) \tag{13}$$

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in the sense that

$$P(t)g(X(s)) = \sum_{j \in S} p_{X(s)j}(t)g(j)$$

is a version of $E(g(X(t+s)) | \mathcal{F}_s)$

Indeed, the Markov property implies that

$$E(g(X(t+s)) | \mathcal{F}_s) = E(g(X(t+s)) | X(s)) \text{ and}$$

$$E(g(X(t+s)) | X(s) = i) = \sum_{j \in S} E(g(X(t+s))1_{\{X(t+s)=j\}} | X(s) = i)$$

$$= \sum_{j \in S} g(j)P(X(t+s) = j | X(s) = i)$$

$$= \sum_{j \in S} g(j)p_{ij}(t)$$

$$= P(t)g(i).$$

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Using $\frac{d}{dt}P(t) = QP(t) = P(t)Q$, the main theorem of calculus implies that

$$P(t)g(i) - g(i) = \int_0^t QP(s)g(i) \, ds = g(i) + \int_0^t P(s)Qg(i) \, ds$$

= $\int_0^t E(Qg(X(s))|X(0) = i) \, ds$
= $E\left(\int_0^t Qg(X(s)) \, ds \mid X(0) = i\right).$

From this identity it then follows that

$$E(f(X(t+s)) - f(X(s)) \mid \mathcal{F}_s) = P(t)f(X(s)) - f(X(s))$$

=
$$E\left(\int_s^{t+s} Qf(X(r)) \, dr \mid X(s)\right)$$
(14)

which implies the martingale property

$$E\left(f(X(t+s)) - \int_0^{t+s} Qf(X(r)) dr \mid \mathcal{F}_s\right)$$

= $f(X(s)) + E\left(\int_s^{t+s} Qf(X(r)) dr \mid X(s)\right) - E\left(\int_0^{t+s} Qf(X(r)) dr \mid \mathcal{F}_s\right)$
= $f(X(s)) - \int_0^s Qf(X(r)) dr$.

To derive the representation of the L^2 -norm we conclude similarly that

$$E\left(\left(M^{f}\right)^{2}(t)\right) = E\left(\left(f(X(t)) - f(X(0))\right)^{2} - 2\left(f(X(t)) - f(X(0))\right)\int_{0}^{t} Qf(X(s)) ds + \left(\int_{0}^{t} Qf(X(s)) ds\right)^{2}\right)$$

$$= E\left(\left(f(X(t)) - f(X(0))\right)^{2} + 2f(X(0))\int_{0}^{t} Qf(X(s)) ds - 2\int_{0}^{t} f(X(s))Qf(X(s)) ds\right)$$

$$= E\left(\int_{0}^{t} Q\left(f^{2}\right)(X(s)) - 2f(X(s))Qf(X(s)) ds\right)$$
(15)

using

$$\begin{split} E\left(\left(\int_{0}^{t} Qf\left(X(s)\right) \, ds\right)^{2}\right) &= 2\int_{0}^{t} \int_{s}^{t} E\left(Qp_{u-s}f(X(s))Qf(X(s))\right) \, du \, ds \\ &= 2\int_{0}^{t} E\left(\left(p_{t-s}f(X(s)) - f(X(s))\right) Qf(X(s))\right) \, ds \\ &= 2E\left(f(X(t))\int_{0}^{t} Qf(X(s)) \, ds - \int_{0}^{t} f(X(s))Qf(X(s)) \, ds\right) \, . \end{split}$$

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This proves (15).

For the proof of (11) note that for any state $i \in S$

$$Qf^{2}(i) - 2f(i)Qf(i) = \sum_{j \in S} q_{ij} \left(f^{2}(j) - 2f(i)f(j) \right)$$

= $\sum_{j \in S} q_{ij} \left(f^{2}(j) - 2f(i)f(j) + f^{2}(i) \right) = \sum_{j \in S} q_{ij} \left(f(i) - f(j) \right)^{2}$. (16)

The proof of (12) now follows from the Markov property and the previous two equalities (15) and (16), since

$$\begin{split} E\left(M^{f}(t)^{2} \mid \mathcal{F}_{s}\right) &- M^{f}(s)^{2} = E\left((M^{f}(t) - M^{f}(s))^{2} \mid \mathcal{F}_{s}\right) \\ &= E\left((M^{f}(t) - M^{f}(s))^{2} \mid X(s)\right) \\ &= E\left(\int_{s}^{t} \sum_{j \in S} q_{X(r)j} \left(f(X(r)) - f(j)\right)^{2} dr \mid X(s)\right) \\ &= E\left(\int_{0}^{t} \sum_{j \in S} q_{X(r)j} \left(f(X(r)) - f(j)\right)^{2} dr \mid \mathcal{F}_{s}\right) - \int_{0}^{s} \sum_{j \in S} q_{X(r)j} \left(f(X(r)) - f(j)\right)^{2} dr . \end{split}$$

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Binary neural networks: math. model

- ▶ network of N binary neurons $n(t) = (n_1(t), ..., n_N(t))$ with $n_i(t) \in \{0, 1\}$
- input $u_i(t)$ to the i^{th} neuron given as

$$u_i(t) = \gamma \sum_{j=1}^N J_{ij} n_j(t) - m_i, i = 1, \ldots, N,$$

with connectivity matrix $J_{ij} \in \{0, 1\}$ (e.g. J_{ij} i.i.d. Bernoulli $\left(\frac{K}{N}\right)$, or fixed row sum $\sum_{j=1} J_{ij} \equiv K \ J_{i} \subset \{1, \ldots, N\}$ uniformly distr. independent)

m_i denotes some mean input that will be specified later

the response of neuron *i* to given input u_i is determined in terms of a nonlinear increasing function $f : \mathbb{R} \to [0, 1]$ as follows: given $n_i(t) = 0$, $f(u_i(t))$ specifies the rate at which the neuron becomes active and conversely, given $n_i(t) = 1$, $1 - f(u_i(t))$ specifies the rate at which the neuron becomes inactive. The resulting dynamics of the network is then a time-continuous Markov chain on the state space $I_N = \{0, 1\}^N$ with generator matrix Q(n, m) = 0 if $|n - m| \ge 2$ and

$$Q(n,m) = \begin{cases} f(u_i) & \text{if } m-n=e_i \\ 1-f(u_i) & \text{if } m-n=-e_i \end{cases}$$

Here e_i denotes the i^{th} unit vector.

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Here e_i denotes the i^{th} unit vector.

Ex for f

Heaviside- function $f(u) = 1_{\{u \ge \theta\}}$ for some given threshold θ sigmoid-function $f(u) = \frac{1}{1+e^{-\gamma(u-\theta)}}$

Martingales

given $G: I_N: 0\{0,1\}^N \to \mathbb{R}$ the process

$$M_t = M_t^G = G(n(t)) - G(n(0)) - \int_0^t QG(n(s)) \, ds \, , t \ge 0$$

is a martingale w.r.t. the natural filtration generated by n(t) with

$$E\left(M_{t}^{2}\right) = \int_{0}^{t} E\left(\sum_{i:n_{i}=0}^{N} f(u_{i}) \left(G\left(n(s)+e_{i}\right)-G\left(n(s)\right)\right)^{2} + \sum_{i:n_{i}=1}^{N} (1-f(u_{i})) \left(G\left(n(s)-e_{i}\right)-G\left(n(s)\right)\right)^{2}\right) ds$$

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• $G(n) = \pi_i(n) = n_i$, we obtain that

$$M_t^i = n_i(t) - \int_0^t f(u_i(s)) - n_i(s) \, ds$$

• $G(n) = \pi_{ij}(n) = n_i n_j, i \neq j$, we obtain that $M_t^{ij} = n_i(t)n_j(t) - \int_0^t (-2n_i(s)n_j(s) + f(u_i(s))n_j(s) + f(u_j(s))n_i(s)) ds$

Elements of a mean-field theory

Laws of large numbers of the mean activity

$$\bar{n}(t) := \frac{1}{N} \sum_{i=1}^{N} n_i(t)$$

Scenario 1: Fixed row sum: $J_{ij} = J_{ij}^{(N)}$ such that $\sum_{j=1}^{N} J_{ij}^{(N)} \equiv K_N$ with $K_N \uparrow \infty$, $m_i^{(N)} \equiv m$, $f^{(N)}(u) = f(u)$, with $\gamma_N \ge 0$ and f Lipschitz In this case we will prove that the dynamics of any ensemble average

$$n^{J^{(N)}} := \frac{1}{|J^{(N)}|} \sum_{j \in J^{(N)}} n_j^{(N)}$$

with $|J^{(N)}|$ sufficiently large, is asymptotically equivalent to the solution $m^{(N)}(t)$ of the ordinary differential equation

$$\dot{m}^{(N)}(t) = -m^{(N)}(t) + f(\gamma_N K_N m(t) - m), m(0) = m_0$$
 (17)

for suitable initial conditions $n_i(0)$, e.g. $n_i(0)$ i.i.d. with $E(n_i(0)) = m_0$.

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Scenario 1: LLN

$$d_N(t) := \sup_{\substack{J \subset \{1,...,N\} \ |J| \ge K_N}} E\left(|n^J(t) - m^{(N)}(t)|^2
ight)^{\frac{1}{2}}$$

Theorem

$$d_N(t) \leq d_N(0) + \sqrt{rac{t}{\mathcal{K}_N}} + (\gamma_N \mathcal{K}_N \|f\|_{Li
ho} + 1) \int_0^t d_N(s) \, ds$$

Gronwall's inequality implies in particular,

$$d_N(t) \leq \left(d_N(0) + \sqrt{\frac{t}{K_N}} \right) e^{\left(\gamma_N K_N \|f\|_{Lip} + 1 \right)t}, t \geq 0.$$

Suppose now that $K_N \to \infty$, $\sup_{N \ge 1} \gamma_N K_N < \infty$ and initial conditions n(0) are chosen such that $\lim_{N\to\infty} d_N(0) \to 0$, e.g. $n_i(0)$ i.i.d. with $E(n_i(0)) = n_0$, then for every ensemble average $n^{J^{(N)}}$ with $|J^{(N)}| \ge K_N$ it follows that

$$\lim_{N\to\infty} E\left(|n^{J^{(N)}}(t)-m^{(N)}(t)|^2\right)=0.$$

Scenario 1: LLN, ctd.

Corollary If $\gamma_N K_N \rightarrow \gamma_*$, then

$$\lim_{N\to\infty} E\left(|n^{J^{(N)}}(t)-m(t)|^2\right)=0$$

where m is a solution to the ordinary differential equation

$$\dot{m}(t)=-m(t)+f(\gamma_*m(t)-m),m(0)=m_0$$
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We first consider N fixed. To simplify notations we can drop the dependence on N. Fix a subset $J \subset \{1, \ldots, N\}$ with $|J| \ge K_N$. $n^J(t)$ admits the following semimartingale decomposition

$$n^{J}(t) = n^{J}(0) + \int_{0}^{t} Qn^{J}(s) ds + M_{t}$$

with

$$\begin{split} E\left(M_{t}^{2}\right) &= \int_{0}^{t} \sum_{i \in J: n_{i} = 0} E\left(f(u_{i}(s))\left(\frac{1}{|J|}\right)^{2}\right) ds \\ &+ \int_{0}^{t} \sum_{i \in J: n_{i} = 1} E\left((1 - f(u_{i}(s)))\left(\frac{1}{|J|}\right)^{2}\right) ds \\ &= \frac{1}{|J|^{2}} \int_{0}^{t} E\left(\sum_{i \in J} (1 - n_{i}(s))f(u_{i}(s)) + n_{i}(s)(1 - f(u_{i}(s)))\right) ds \leq \frac{t}{|J|} \end{split}$$

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It follows that

$$\left(E \left(n^{J}(t) - m(t) \right)^{2} \right)^{\frac{1}{2}} \leq \left(E \left(n(0) - m(0) \right)^{2} \right)^{\frac{1}{2}}$$

$$+ \left(E \left(\int_{0}^{t} Q n^{J}(s) - \left(f(\gamma_{N} K_{N} m(s) - m) - m(s) \right) ds \right)^{2} \right)^{\frac{1}{2}}$$

$$+ \left(E \left(M_{t}^{2} \right) \right)^{\frac{1}{2}}$$

$$= I + II + III .$$

Let us estimate the three terms separately. From the definition $E(I) \leq d_N(0)$, from the above computations $E(III) \leq \sqrt{\frac{t}{|J|}} \leq \sqrt{\frac{t}{K_N}}$. It remains to estimate the second term:

$$II \leq \int_0^t \left(E\left(Qn^J(s) - f(\gamma_N K_N m(s) - m)\right)^2 \right)^{\frac{1}{2}} ds.$$

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Clearly,

$$\begin{aligned} Qn^{J}(s) &- \left(f(\gamma_{N}K_{N}m(s) - m) - m(s)\right) = \sum_{i \in J:n_{i}=0} f(u_{i}(s))\frac{1}{|J|} \\ &- \sum_{i \in J:n_{i}=1} (1 - f(u_{i}(s)))\frac{1}{|J|}) - \left(f(K_{N}(m(s) - m)) - m(s)\right) \\ &= \frac{1}{|J|} \sum_{i \in J} f(u_{i}(s)) - n^{J}(s) - \left(f(\gamma_{N}K_{N}m(s) - m)\right) - m(s)) \\ &= \frac{1}{|J|} \sum_{i \in J} (f(\gamma_{N}K_{N}n^{J_{i}}(s) - m) - f(\gamma_{N}K_{N}m(s) - m) - (n^{J}(s) - m(s)) \\ &= II_{a}(s) + II_{b}(s) \,, \end{aligned}$$

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so that

$$egin{aligned} & II \leq \int_{0}^{t} \left(E(II_{s}^{2}(s))^{rac{1}{2}} + \left(E(II_{b}^{2}(s))^{rac{1}{2}} \; ds
ight) \ & \leq (\gamma_{N} K_{N} \|f\|_{Lip} + 1) \int_{0}^{t} d_{N}(s) \; ds \, . \end{aligned}$$

Here $J_i := \{j \in \{1, \dots, N\} \mid J_{ij} = 1\}$ denotes the set of presynaptic neurons to neuron *i*.

Combining all three estimates we arrive at

$$d_N(t) \leq d_N(0) + \sqrt{rac{t}{\mathcal{K}_N}} + (\gamma_N \mathcal{K}_N \|f\|_{Lip} + 1) \int_0^t d_N(s) \, ds$$

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Remarks

- ► Th remains true for any sequence of connectivity matrices with row sum $\geq K_N$, converge is w.r.t. to the conditional probabilities $P\left(\cdot \mid \left(J_{ij}^{(N)}\right)\right)$ for any sequence $\left(J_{ij}^{(N)}\right)$
- ▶ remarkable implication: n_i become asymptotically uncorrelated: indeed, $f(u_i^{(N)}(t)) \rightarrow f(\gamma_* m(t) m)$ implies:

$$\begin{aligned} \frac{d}{dt} E(n_i(t)) E(n_j(t)) &\asymp \left(f(\gamma_* m(t) - m) - E(n_i(t)) \right) \left(f(\gamma_* m(t) - m) - E(n_j(t)) \right) \\ \frac{d}{dt} E(n_i(t)) E(n_j(t)) &\asymp -2E(n_i(t)n_j(t)) + f(\gamma_* m(t) - m)E(n_i(t)) \\ &+ f(\gamma_* m(t) - m)E(n_j(t))) \end{aligned}$$

implies

$$\frac{d}{dt}\left(E(n_i(t)n_j(t))-E(n_i(t))E(n_j(t))\right) \asymp -2\left(E(n_i(t)n_j(t))-E(n_i(t))E(n_j(t))\right)$$

so that $Cov(n_i(t), n_j(t)) \approx 0$ for t > 0 provided the same holds for the initial condition t = 0

Elements of a mean-field theory

The central limit theory for the mean activity

Scenario 1 (in addition $\gamma_N K_N \equiv \gamma_*$, and $f \in C_b^2$) hence

$$\bar{n}(t) = rac{1}{N} \sum_{i=1}^{N} \asymp m(t)$$

where

$$\dot{m}(t) = -m(t) + f(\gamma_* m(t) - m).$$
 (18)

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next define standardized ensemble averages

$$n^{j,*}(t):=\sqrt{|J|}\left(n^J(t)-m(t)
ight)=\sqrt{|J|}\left(rac{1}{|J|}\sum_{i\in J}n_i(t)-m(t)
ight)\,.$$

Scenario 1: CLT - small ensemble size

Theorem
Let
$$J^{(N)} \subset \{1, ..., N\}$$
, K_N and $d_N(0)$ be such that $|J^{(N)}|\infty$, but
 $\lim_{N\to\infty} \sqrt{|J^{(N)}|} \left(d_N(0) + \frac{1}{\sqrt{\kappa_N}} \right) \to 0$. Suppose that
 $P \circ \left(n^{|J^{(N)}|,*}(0) \right)^{-1} \to N(m_0, \sigma_0^2)$ weakly (e.g. $n_i(0)$ iid Bernoulli (m_0) , hence
 $\sigma_0^2 = m_0(1-m_0)$).

Then $n^{J^{(N)},*}(t) \to n_\infty(t)$ weakly on the Skorokhod space $\mathcal{D}([0,\infty))$, which is a sol. of the sde

$$dn_{\infty}(t) = -n_{\infty}(t) dt + \sigma(t) dW(t)$$

where W(t) is 1d-Brownian motion and

$$\sigma(t) := \sqrt{(1-m(t))f(\gamma_*m(t)-m)+m(t)(1-f(\gamma_*m(t)-m)))}$$

Scenario 1: CLT, ctd.

Rem

- f no longer enters the drift term, since the argument of f is "faster averaging" than n^{f(N)}
- the clt yields the following "finite size" correction

$$n^{J^{(N)}}(t) = m(t) + rac{1}{\sqrt{|J^{(N)}|}} n_{\infty}(t)$$

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in the LLN

start again with the semimartingale decomposition

$$n^{J^{(N)},*}(t) = n^{J^{(N)},*}(0) + \int_0^t \sqrt{|J^{(N)}|} \left(Q^{(N)}n^{J^{(N)}}(s) - \dot{m}(s)\right) ds + M_t^{(N)}$$

where

$$M_t^{(N)} := \sqrt{|J^{(N)}|} \left(n^{J^{(N)}}(t) - n^{J^{(N)}}(0) - \int_0^t Q^{(N)} n^{J^{(N)}}(s) \, ds \right)$$

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and apply the following martingale CLT

Martingale CLT

Theorem

For $n = 1, 2, ..., let (\mathcal{F}_t^n)_{t \ge 0}$ be a filtration and $(M_n(t))_{t \ge 0}$ be an $(\mathcal{F}_t^n)_{t \ge 0}$ -martingale with right-continuous sample paths, having left limits at t > 0 and starting at 0, i.e. $M_n(0) = 0$, such that

$$\lim_{n\to\infty} E\left(\sup_{0\leq s\leq t} |M_n(s)-M_n(s-)|\right)=0.$$

Assume that there exist nonnegative, nondecreasing, $(\mathcal{F}^n_t)_{t\geq 0}\text{-}adapted$ processes such that

$$M_n^2(t)-A_n(t), t\geq 0,$$

is an $(\mathcal{F}_t^n)_{t\geq 0}$ -martingale and that

$$\lim_{n o \infty} A_n(t) = \int_0^t \sigma^2(s) \, ds$$
 in probability

for some deterministic function $\sigma : [0,\infty) \to \mathbb{R}$. Then

$$\lim_{n\to\infty}M_n(t)=\int_0^t\sigma(s)\,dW(s)\,,t\ge 0\,,$$

weakly on the Skorokhod-space $D[0,\infty)$. Here, $(W(t))_{t\geq 0}$ is a 1d-Brownian motion.

Martingale CLT

For a proof see e.g. S. Ethier, T. Kurtz, Markov processes, Characterization and Convergence, 1984

note that $M_t^{(N)} - M_{t-}^{(N)} \neq 0$ if and only if there is a switch in one of the neurons at time t from active to inactive or conversely, so that

$$\sup_{t\leq T} \left| M_t^{(N)} - M_{t-}^{(N)} \right| \leq \frac{1}{\sqrt{J^{(N)}}} \to 0, N \to \infty.$$

limiting behaviour of the variance process -based on the previous LLN (!):

$$E\left(M_{t}^{(N),2}\right) = \frac{1}{|J^{(N)}|} \int_{0}^{t} E\left(\sum_{i \in J^{(N)}} \left((1 - n_{i}(s))f(u_{i}^{(N)}(s)) + n_{i}(s)(1 - f(u_{i}^{(N)}(s)))\right) ds\right)$$

 $|J^{(N)}| \ge K_N$ implies that

$$u_i^{(N)}(s) = \gamma_N \sum_{j:J_{ij}^{(N)}=1} n_j(s) - m o \gamma_* m(s) - m \,, N o \infty \,,$$

in $L^1(P)$, hence

$$E\left(M_t^{(N),2}
ight)
ightarrow \int_0^t \sigma(s)^2\,ds$$

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as $N \to \infty$ in $L^1(P)$

moreover, $M_t^{(N),2} - A_t^{(N)}$, $t \ge 0$, is a martingale, where

$$A_t^{(N)} = \int_0^t \sum_{i \in J^{(N)}} (1 - n_i(s)) f(u^{(N)}(s)) + n_i(s) (1 - f(u_i^{(N)}(s))) \, ds$$

so that the martingale CLT now implies

$$\lim_{N\to\infty} M_t^{(N)} = \int_0^t \sqrt{(1-m(s))f(\gamma_*m(s)-m) + m(s)(1-f(\gamma_*m(s)-m))} \, dW(s)$$

weakly on $\mathcal{D}([0,\infty))$

it remains to verify weak convergence of the drift term

$$\begin{split} \sqrt{|J^{(N)}|} \left(Q^{(N)} n^{J^{(N)}}(t) - \dot{m}(t) \right) \\ &= \sqrt{|J^{(N)}|} \left(\frac{1}{|J^{(N)}|} \sum_{i \in J^{(N)}} \left(f\left(\gamma_* n^{J^{(N)}_{i\cdot}}(t) - m \right) - f(\gamma_* m(t) - m) \right) \right) \\ &- n^{J^{(N)},*}(t) \\ &= I + II \end{split}$$

where

$$|I| \leq \sqrt{|J^{(N)}|} \gamma_* \|f\|_{Lip} |n^{J^{(N)}_{i}}(t) - m(t)|$$

thus

$$E\left(|I|
ight)\leq C(t,\|f\|_{Lip})\sqrt{|J^{(N)}|}\left(d_N(0)+\sqrt{rac{t}{K_N}}
ight)
ightarrow 0,N
ightarrow\infty.$$

so that

$$\sqrt{|J^{(N)}|} \left(Q^{(N)} n^{J^{(N)}}(t) - \dot{m}(t) \right) \asymp - n^{|J^{(N)},*}(t)$$

in $L^1(P)$

summarizing

$$n^{J^{(N)},*}(t) \asymp n^{J^{(N)},*}(0) - \int_0^t n^{J^{(N)},*}(s) \, ds + M_t^{(N)}$$

consider

$$\bar{n}^{J^{(N)},*}(t) := e^{-t} n^{J^{(N)},*}(0) + \int_0^t e^{-(t-s)} dM_s^{(N)} \\ := e^{-t} n^{J^{(N)},*}(0) + M_t^{(N)} - \int_0^t e^{-(t-s)} M_s^{(N)} ds$$

then $\bar{n}^{J^{(N)},*}(t) = \Phi(M^{(N)})(t)$, where $\Phi : \mathcal{D}([0,\infty)) \to \mathcal{D}([0,\infty))$ is the mapping

$$\Phi(\omega)(t) = e^{-t} n^{J^{(N)},*}(0) + \omega(t) - \int_0^t e^{-(t-s)} \omega(s) \, ds$$

since Φ is continuous w.r.t. the Skorokhod metric we conclude that

$$\lim_{N \to \infty} \bar{n}^{J^{(N)},*}(t) = e^{-t} n^{J^{(N)},*}(0) + \int_0^t e^{-(t-s)} \sigma(s) \, dW(s)$$

weakly, hence

$$\lim_{N \to \infty} n^{J^{(N)},*}(t) = e^{-t} n^{J^{(N)},*}(0) + \int_0^t e^{-(t-s)} \sigma(s) \, dW(s)$$

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weakly too on $\mathcal{D}([0,\infty))$

Scenario 1: CLT - population average activity

Additional assumption: $f \in C_b^2$

Theorem

Suppose that $P \circ \left(\bar{n}^{(N),*}(0)\right)^{-1} \rightarrow N(m_0, \sigma_0^2)$ weakly (e.g. $n_i(0)$ iid Bernoulli (m_0) , hence $\sigma_0^2 = m_0(1 - m_0)$).

Then

$$ar{n}^{(N),*}(t) := \sqrt{N}\left(rac{1}{N}\sum_{i=1}^N n_i(t) - m(t)
ight) o n_\infty(t)$$

weakly on the Skorokhod space $\mathcal{D}([0,\infty))$, which is a sol. of the sde

$$dn_{\infty}(t) = \left(\gamma_* f'(\gamma_* m(t) - m) n_{\infty}(t) - n_{\infty}(t)\right) dt + \sigma(t) dW(t)$$

where W(t) is 1d-Brownian motion and

$$\sigma(t) := \sqrt{(1 - m(t))f(\gamma_* m(t) - m) + m(t)(1 - f(\gamma_* m(t) - m))}$$

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start again with the semimartingale decomposition

$$\bar{n}^{(N),*}(t) = \bar{n}^{(N),*}(0) + \int_0^t \sqrt{N} \left(Q^{(N)} \bar{n}^{(N)}(s) - \dot{m}(s) \right) \, ds + M_t^{(N)}$$

where

$$M_t^{(N)} := \sqrt{N} \left(ar{n}^{(N)}(t) - ar{n}^{(N)}(0) - \int_0^t Q^{(N)} ar{n}^{(N)}(s) \, ds
ight)$$

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limiting behaviour of the variance process similar to the previous case

asymptotic of the drift term different:

$$\begin{split} \sqrt{N} \left(Q^{(N)} \bar{n}^{(N)}(t) - \dot{m}(t) \right) \\ &= \sqrt{N} \frac{1}{N} \sum_{i=1}^{N} \left(f \left(\gamma_* n^{J_i^{(N)}}(t) - m \right) - f(\gamma_* m(t) - m) \right) \\ &- n^{(N),*}(t) \\ &= I + II \end{split}$$

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Taylor expansion yields for the first term I

$$\begin{split} \sqrt{N} \frac{1}{N} \sum_{i=1}^{N} \left(f\left(\gamma_* n^{J_{i\cdot}^{(N)}}(t) - m\right) - f(\gamma_* m(t) - m) \right) \\ &= \sqrt{N} \frac{1}{N} \sum_{i=1}^{N} \left(\gamma_* f'(\gamma_* m(t) - m) \frac{1}{K_N} \sum_{j=1}^{N} J_{ij}^{(N)} n_j(t) - m(t) \right) \\ &+ \sqrt{N} \frac{1}{N} \sum_{i=1}^{N} \left(\frac{\gamma_*^2}{2} f''(\xi_i^{(N)}(t)) \frac{1}{K_N} \left(\sum_{j=1}^{N} J_{ij}^{(N)} n_j(t) - m(t) \right)^2 \right) \\ &= I_a + I_b \end{split}$$

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for certain values $\xi_i^{(N)}(t)$ between $u_i^{(N)}(t) - m$ and $\gamma_* m(t) - m$.

Asymptotic of *I*_a: rearranging terms

$$\begin{split} I_{a} &= \gamma_{*}f'(\gamma_{*}m(t) - m)\sqrt{N}\left(\frac{1}{N}\sum_{j=1}^{N}\left(\frac{1}{K_{N}}\sum_{i=1}^{N}J_{ij}^{(N)}\right)n_{j}(t) - m(t)\right) \\ &= \gamma_{*}f'(\gamma_{*}m(t) - m)\underbrace{\sqrt{N}\left(\frac{1}{N}\sum_{j=1}^{N}\frac{1}{K_{N}}\sum_{i=1}^{N}\left(J_{ij}^{(N)} - \frac{K_{N}}{N}\right)n_{j}(t)\right)}_{\to 0, N \to \infty, L^{2}(P)} \\ &+ \gamma_{*}f'(\gamma_{*}m(t) - m)\sqrt{N}\left(\frac{1}{N}\sum_{j=1}^{N}n_{j}(t) - m(t)\right) \end{split}$$

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because of the asymptotic uncorrelatedness of n_i

$$\begin{split} E\left(\left(\frac{1}{\sqrt{N}}\sum_{j=1}^{N}\frac{1}{K_{N}}\sum_{i=1}^{N}\left(J_{ij}^{(N)}-\frac{K_{N}}{N}\right)n_{j}(t)\right)^{2}\right)\\ &=\frac{1}{N}\sum_{j=1}^{N}E\left(\frac{1}{K_{N}^{2}}\left(\sum_{i=1}^{N}\left(J_{ij}^{(N)}-\frac{K_{N}}{N}\right)\right)^{2}E\left(n_{j}(t)^{2}\mid\left(J_{ij}\right)\right)\right)\\ &\leq\frac{1}{N}\sum_{j=1}^{N}E\left(\frac{1}{K_{N}^{2}}\sum_{i=1}^{N}\left(J_{ij}^{(N)}-\frac{K_{N}}{N}\right)^{2}\right)\\ &=\frac{N}{K_{N}^{2}}\frac{K_{N}}{N}\left(1-\frac{K_{N}}{N}\right)=\frac{1}{K_{N}}\left(1-\frac{K_{N}}{N}\right)\rightarrow0, N\rightarrow\infty\end{split}$$

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Proof of Theorem, ctd.

Asymptotic of

$$I_{b} = \frac{\gamma_{*}^{2}}{2} \frac{1}{N} \sum_{i=1}^{N} f''\left(\xi_{i}^{(N)}(t)\right) \left(\frac{1}{K_{N}} \sum_{j=1}^{N} J_{ij} n_{j}(t) - m(t)\right)^{2}$$

$$\begin{split} & E\left(\left(\frac{1}{K_N}\sum_{j=1}^N J_{ij}n_j(t) - m(t)\right)^2\right) \leq 2E\left(\left(\frac{1}{K_N}\sum_{j=1}^N \left(J_{ij}^{(N)} - \frac{K_N}{N}\right)n_j(t)\right)^2\right) \\ & + 2E\left(\left(\frac{1}{N}\sum_{j=1}^N n_j(t) - m(t)\right)^2\right) \\ & \leq 2\frac{1}{K_N^2}E\left(\sum_{j=1}^N \left(J_{ij}^{(N)} - \frac{K_N}{N}\right)^2\right) + 2d_N(t)^2 \to 0, N \to \infty \end{split}$$

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Proof of Theorem, ctd.

summarizing

$$\begin{split} \sqrt{N} \left(Q^{(N)} \bar{n}^{(N)}(t) - \dot{m}(t) \right) \\ &= \sqrt{N} \frac{1}{N} \sum_{i=1}^{N} \left(f \left(\gamma_* n^{J_i^{(N)}}(t) - m \right) - f(\gamma_* m(t) - m) \right) \\ &- \bar{n}^{(N),*}(t) \\ & \asymp \gamma_* f'(\gamma_* m(t) - m) \bar{n}^{(N),*}(t) - \bar{n}^{(N),*}(t) \end{split}$$

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remaining parts of the proof similar to the previous case