On preferential and uniform attachment random graphs

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1 Motivation
   - Preferential attachment random graphs

2 Preferential attachment and birth processes
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Motivation

**Figure:** A human brain networks

**Figure:** Internet Traffic Map
Preferential attachment random graphs. Motivation

- Not all things we measure are peaked around a typical value. A classic example of this type of behavior is the sizes of towns and cities.

**Figure**: Left: histogram of the populations of all US cities with population of 10,000 or more. Right: histogram of the same data, but plotted on logarithmic scales.
What does it mean?

Let $p(x) \, dx$ be the fraction of cities with population between $x$ and $x + dx$. If the histogram is a straight line on log-log scales, then

$$\ln p(x) = -\alpha \ln x + c,$$

where $\alpha$ and $c$ are constants.

Taking the exponential of both sides, this is equivalent to:

$$p(x) = C x^{-\alpha},$$

with $C = e^c$.

Distributions of this form are said to follow a power law.
Barabási–Albert model (BA)

- Start with an initial connected graph of $m_0$ vertices
- At every step we add a new vertex with $m \leq m_0$ edges
- The probability that a new vertex $v_{n+1}$ will be connected to a vertex $v_j, 1 \leq j \leq n + 1$, is proportional to the degree of $v_j$
- Usually $m$ edges are drawn independently or one by one.
- When $m = 1$ the random graph is a random tree.
Formally, let \( m \in \mathbb{Z}^+ \), \( n \in \mathbb{Z}^+ \cup \{0\} \), and let us define the process \((G^t_m)_{t \geq 1}\), with \( G^1_m \) the graph with a single vertex, without loops. Then,

1. at time \( t = n(m + 1) + 1 \) add a new vertex \( v_{n+1} \),

2. for \( i = 2, \ldots, m + 1 \) at each time \( t = n(m + 1) + i \) add a direct edge from \( v_{n+1} \) to \( v_j \), \( j = 1, \ldots, n + 1 \), with

\[
\mathbb{P}(v_{n+1} \rightarrow v_j) = \begin{cases} 
\frac{d(v_j,t-1)}{T_d(t-1)}, & v_j \neq v_{n+1} \\
\frac{d(v_j,t-1)+1}{T_d(t-1)}, & v_j = v_{n+1},
\end{cases}
\]  

(1)

where \( d(v_j,t) \) denotes the degree of \( v_j \) at time \( t \) and \( T_d(t) = \sum_{k=1}^{n+1} d(v_k,t) \).
Preferential attachment

Let $N_{k,t}^{BA}$ be the number of vertices with degree equal to $k$ in the BA model, and note that at time $t = n(m + 1)$, $G_m^t$ has exactly $n$ vertices.

**Theorem (Bollobás, Riordan, Spencer, Tusnády)**

Let $m \geq 1$ and $\epsilon > 0$ be fixed, and put $\alpha_{m,k} = \frac{2m(m+1)}{k(k+1)(k+2)}$. Then whp we have

$$(1 - \epsilon)\alpha_{m,k} \leq \frac{N_{m+k,n(m+1)}^{BA}}{n} \leq (1 + \epsilon)\alpha_{m,k},$$

for every $k$ in the range $m \leq k \leq n^{1/15}$. 
A continuous time birth process: The Yule process

Let \( \{N_\lambda(T) : T \geq 0\} \) be a pure birth process with \( N_\lambda(0) = b, \) \( b \geq 1, \) and

\[
\mathbb{P}(N_\lambda(T + h) = k + \ell \mid N_\lambda(T) = k) = \begin{cases} 
k\lambda h + o(h), & \ell = 1, 
o(h), & \ell > 1, 
1 - k\lambda h + o(h), & \ell = 0. 
\end{cases}
\] (2)
The Yule model $\{Y_{\lambda,\beta}^{a,b}(T)\}_{T \geq 0}$

- It makes use of two related Yule processes, $\{N_{\lambda}(T)\}_{T \geq 0}$ and $\{N_{\beta}(T)\}_{T \geq 0}$, of rates $\lambda > 0$ and $\beta > 0$, respectively, and with initial conditions $N_{\lambda}(0) = a$ and $N_{\beta}(0) = b$.

- The relation between them is such that when a new individual appears in the Yule process with parameter $\beta$, a new Yule process with parameter $\lambda$ starts.
The method of embedding when $m = 1$

- Let $\{Z_i(T) : T \geq 0\}_{i \geq 1}$ be independent and identically distributed copies of $\{N_1(T) : T \geq 0\}$, a Yule process $\{N_\lambda(T)\}_{T \geq 0}$, $\lambda = 1$.
- Let $\{\tau_n\}_{n \geq 0}$ be a convenient sequence of random times, with $\tau_0 = 0$, and so that $\{Z_1(T)\}_{T \geq \tau_0}$, $\{Z_2(T)\}_{T \geq \tau_0}$ and $\{Z_i(T - \tau_{i-2})\}_{T \geq \tau_{i-2}}$, $i \geq 2$.
- Let
  \[
  \tilde{d}(v_i, n) \equiv Z_i(\tau_n - \tau_{i-2}), 1 \leq i \leq n + 2
  \]
  and $\tilde{D}_n \equiv \{\tilde{d}(v_i, n), 1 \leq i \leq n + 2\}, n \geq 0$. 
The method of embedding when $m = 1$

**Theorem (Athreya (2007))**

Let $\{Z_i(T) : T \geq 0\}_{i \geq 1}$ and $\{\tau_n\}_{n \geq 0}$ be as above. Let

$$\tilde{d}(v_i, n) \equiv Z_i(\tau_n - \tau_{i-2}), 1 \leq i \leq n + 2$$

and $\tilde{D}_n \equiv \{\tilde{d}(v_i, n), 1 \leq i \leq n + 2\}, n \geq 0$. Consider the degree vector sequence for the random graph sequence $\{G_n\}_{n \geq 0}$, $D_n = \{d(v_i, n), 1 \leq i \leq n + 2\}$. Then the two sequences of random vectors $\{D_n : n \geq 0\}$ and $\{\tilde{D}_n : n \geq 0\}$ have the same distribution.
Through this technique of embedding a “discrete” sequence of graphs in a “continuous time” branching process

- K.B. Athreya (2006-2008) studied the empirical degree distribution for the linear, sub-linear and super-linear Preferential attachment model.
- S. Bhamidi (2007) used the results of Aldous (1991) of asymptotic “Fringe distribution” for general families of random trees to study more properties:
  - empirical degree distribution
  - the size of the subtree rooted at the $k$th vertex to the born.
  - degree of the root
  - maximum degree
  - the height of the tree
- S. Dereich, M. Ortgiese (2014): preferential attachment models with fitness
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The method of weak convergence

- Consider a Yule model \( \{Y_{1/2,1}^{m,1}(T)\}_{T \geq 0} \), that is, the initial conditions for the two Yule processes are \( N_\lambda(0) = m \), and \( N_\beta(0) = 1 \), \( \lambda = 1/2 \), \( \beta = 1 \).
- Let \( N_{T}^{m,1} \) be the size of a individual from \( N_\beta(T) \) chosen uniformly at random at time \( T \) in \( \{Y_{1/2,1}^{m,1}(T)\}_{T \geq 0} \).
- Let \( \text{deg}(v, n) \) denotes the degree of \( v \) at time \( t = n(m + 1) \) (i.e., when there are exactly \( n \) vertices) in the Barabási-Albert model.
Weak convergence theorem

Theorem (P., Polito, Sacerdote (2016))

For every \(i, j \in \mathbb{N}, j \geq i\), and \(w_1 < w_2 < \cdots < w_h \in \mathbb{R}^+\), there exists a convenient non-decreasing sequence \(z_1, \ldots, z_h \in \mathbb{N}\) of stopping times, so that

\[
\lim_{i \to \infty} \mathbb{P}[\text{deg}(v_{j, j+z_1}) = k_1, \ldots, \text{deg}(v_{j, j+z_b}) = k_h] = \mathbb{P}[N_{1/2}(\ln(1+w_1)) = k_1, \ldots, N_{1/2}(\ln(1+w_h)) = k_h],
\]

where \(N_{1/2}\) is a Yule process with \(N_{1/2}(0) = m\) and \(m \leq k_1 \leq \cdots \leq k_b\).
Weak convergence theorem

Let $d^U(V_t)$ indicates the degree of a vertex chosen uniformly at random at time $t$ in the BA model, and $N_{k,t}^{BA}$ be the number of vertices with degree equal to $k$ in the BA model.

**Theorem (P., Polito, Sacerdote (2016))**

Consider a Yule model $\{Y_{1/2,1}^m(T)\}_{T \geq 0}$. Then for $t = n(m + 1)$ we have

$$p_k := \lim_{n \to \infty} P(d^U(V_t) = k) = \lim_{T \to \infty} P(N_{T}^{m,1} = k).$$

(4)

where $N_{T}^{m,1}$ is the size of a individual from $N_1(T)$ chosen uniformly at random at time $T$ in $\{Y_{1/2,1}^m(T)\}_{T \geq 0}$.

Furthermore as $n \to \infty$,

$$\frac{N_{k,t}^{BA}}{n} \to p_k$$

in probability.
Proposition

Consider a Yule model \( \{ Y_{1/2,1}^m(T) \} \) for \( T \geq 0 \) and \( N_T^m,1 \) as above. Then,

\[
p_k = m(m+1) \frac{\Gamma(k)\Gamma(3)}{\Gamma(k+3)},
\]

(5)

where \( k \geq m \).
Some heuristics I

We interpret the BA model identifying two different processes in discrete time, one for the appearance of directed edges of a fixed vertex and the other for the creation of new vertices.

- For the first one, note that in the BA model, at time when there are \( n \) vertices, we have \( mn \) directed edges. If \( d(v, n) \) denotes the degree of an existing vertex \( v \) at time when there are \( n \) vertices in the BA model, then by the preferential attachment rule,

\[
P[d(v, n+1) = k + 1 \mid d(v, n) = k] \approx \frac{km}{2mn} = \frac{k}{2n}.
\]

From this we can see that the distribution of the time interval between the instants at which \( d(v, n) \) changes from \( k \) to \( k + 1 \) is approximately Geometric with parameter \( k/(2n) \).
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For the first one, note that in the BA model, at time when there are $n$ vertices, we have $mn$ directed edges. If $d(v, n)$ denotes the degree of an existing vertex $v$ at time when there are $n$ vertices in the BA model, then by the preferential attachment rule,

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$$

From this we can see that the distribution of the time interval between the instants at which $d(v, n)$ changes from $k$ to $k + 1$ is approximately Geometric with parameter $k/(2n)$. 
Some heuristics II

- **For the second one**, we see the deterministic process of appearance of vertices in the BA model in a different manner. To do that we wait up to see \( n \) vertices in the BA model, \( n > i, \ i > 1 \) and construct a set of \( i \) dependent but identically distributed birth processes (in discrete time). We call this the **planted model**. Then we consider the following experiment.

1. Choose one of the \( i \) birth processes with probability proportional to its number of vertices it has.
2. Choose a vertex uniformly at random, among the vertices belonging to the selected birth process.

- We prove that the planted model together with the previous experiment induce the uniform distribution for selecting a vertex on the set of \([n]\) vertices in the BA model. Furthermore, they induce the uniform distribution for choosing a birth process on the set of \( i \) birth processes of the planted model.
Some heuristics II

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The Uniform- Preferential attachment model (UPA)

We propose a generalization of the Barabási-Albert model which takes into account two different attachment rules for new nodes of the network. We investigate the degree distribution.

Motivation

- Consider a website where registered members can submit content, such as text posts.
- Furthermore, registered users can vote previous posts to determine their position on the site’s pages.
- Hence, the submissions with the most positive votes appear on the front page, together with a fixed number of the most recent posts (see www.reddit.com).

This is a network in which nodes are the posts and edges are votes.
The attachment rules:

- Imagine that when a user submits a new post, it also votes on some other previous submissions.
- It is reasonable to assume that the user tends to select and vote either on the most recent posts or the most popular posts.
- Hence, the user votes the posts according to two different rules:
  - with uniform probability if the user decides to select a post recently published, and
  - with probability proportional to the number of votes, otherwise.
The attachment rules:

- Imagine that when a user submits a new post, it also votes on some other previous submissions.
- It is reasonable to assume that the user tends to select and vote either on **the most recent posts** or **the most popular posts**.
- Hence, the user votes the posts according to two different rules:
  - with **uniform probability** if the user decides to select a post recently published, and
  - with **probability proportional** to the number of votes, otherwise.
The UPA model

Formally, suppose that every new node \( v_{t+1} \) selects a neighbor either within a limited window of nodes (the \( l \in \mathbb{N} \) youngest nodes of the network), or among all nodes \((v_0, \ldots, v_t)\) as follows: Let \( 0 \leq p \leq 1 \) and \( l \geq 1 \). Then,

(a) At the starting period \( t = l \), \( l \in \mathbb{N} \), the initial graph \( G^l \) has \( l + 1 \) nodes \((v_0, v_1, \ldots, v_l)\), where every node \( v_j \), \( 1 \leq j \leq l \), is connected to \( v_0 \).

(b) Given \( G^t \), at time \( t + 1 \) add a new node \( v_{t+1} \) together with an outgoing edge. Such edge links \( v_{t+1} \) with an existing node chosen either within a window, or among all nodes present in the network at time \( t \), as follows:

- with probability \( p \), \( v_{t+1} \) chooses its neighbour in the set \( \{v_{t-l+1}, \ldots, v_t\} \), and each node within this window has probability \( \frac{1}{l} \) of being chosen.
- with probability \( 1 - p \), the neighbour of \( v_{t+1} \) is chosen from the set \( \{v_0, \ldots, v_t\} \), and each node \( v_j \), \( j = 0, \ldots, t \), has probability \( \frac{\text{deg}(v_j)}{2t} \) of being chosen.
The UPA model

Formally, suppose that every new node $v_{t+1}$ selects a neighbor either within a **limited window of nodes** (the $l \in \mathbb{N}$ youngest nodes of the network), or among all nodes ($v_0, \ldots, v_t$) as follows: Let $0 \leq p \leq 1$ and $l \geq 1$. Then,

(a) At the starting period $t = l$, $l \in \mathbb{N}$, the initial graph $G^l$ has $l + 1$ nodes ($v_0, v_1, \ldots, v_l$), where every node $v_j$, $1 \leq j \leq l$, is connected to $v_0$.

(b) Given $G^t$, at time $t + 1$ add a new node $v_{t+1}$ together with an outgoing edge. Such edge links $v_{t+1}$ with an existing node chosen either within a window, or among all nodes present in the network at time $t$, as follows:

- with probability $p$, $v_{t+1}$ chooses its neighbour in the set $\{v_{t-l+1}, \ldots, v_t\}$, and each node within this window has probability $\frac{1}{l}$ of being chosen.
- with probability $1 - p$, the neighbour of $v_{t+1}$ is chosen from the set $\{v_0, \ldots, v_t\}$, and each node $v_j$, $j = 0, \ldots, t$, has probability $\frac{\deg(v_j)}{2t}$ of being chosen.
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Asymptotic degree distribution in the UPA model

**Theorem**
Let $N(k, t)$ denotes the number of nodes in the network with degree $k$ at time $t$. Then,

$$\frac{N(k, t)}{t} \rightarrow P(k)$$

(6)

in probability as $t \rightarrow \infty$, where for $l = 1$ it holds:

$$P(k) = \begin{cases} 
\frac{2(1-p)}{3-p} & \text{if } k = 1 \\
\frac{(1-p)^2}{(2-p)(3-p)} + \frac{p}{2-p} & \text{if } k = 2 \\
\left(\frac{2}{1-p} + 2\right)\left(\frac{2}{1-p} + 1\right)B\left(k, 1 + \frac{2}{1-p}\right)P(2) & \text{if } k > 2,
\end{cases}$$

(7)

where $B(x, y)$ is the Beta function,
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\left(\frac{2}{1-p} + 2\right)\left(\frac{2}{1-p} + 1\right)B\left(k, 1 + \frac{2}{1-p}\right)P(2) & \text{if } k > 2,
\end{cases}$$

where $B(x, y)$ is the Beta function,
while for $l > 1$ we have:

$$P(k) = \begin{cases} 
\frac{2}{(3-p)} \left(1 - \frac{p}{l}\right)^l & \text{if } k = 1 \\
\frac{2}{2+k(1-p)} \left(\frac{p}{l} (H_{k-1} - H_k) + \frac{(1-p)(k-1)}{2} P(k-1)\right) & \text{if } k = 2, \ldots, l+1 \\
\frac{B(k, l+2 + \frac{2}{1-p})}{B(l+1, k+1 + \frac{2}{1-p})} P(l+1) & \text{if } k > l+1,
\end{cases}$$

where

$$H_k = \begin{cases} 
(p/l)^{k-1} \sum_{m=1}^{l-(k-1)} (l-m-(k-1)) (1 - \frac{p}{l})^{l-m-(k-1)} & \text{if } k = 1, \ldots, l. \\
0 & \text{if } k > l.
\end{cases}$$

(9)
while for $l > 1$ we have:

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\frac{B(k,l+2+\frac{2}{1-p})}{B(l+1,k+1+\frac{2}{1-p})} P(l+1) & \text{if } k > l+1,
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0 & \text{if } k > l.
\end{cases}$$

(9)
Corollary

As \( k \to \infty \), for \( l = 1 \)

\[
\frac{N(k, t)}{t} \sim C_p \left[ k^{-\left(1+\frac{2}{1-p}\right)} - \frac{3 - p}{(1 - p)^2} k^{-\left(2+\frac{2}{1-p}\right)} \right],
\]

where \( C_p = \Gamma\left(1 + \frac{2}{(1 - p)}\right)\left(\frac{2}{(1 - p)} + 2\right)\left(\frac{2}{(1 - p)} + 1\right)P(2) \), and for \( l > 1 \),

\[
\frac{N(k, t)}{t} \sim C_{p,l} \left[ k^{-\left(1+\frac{2}{1-p}\right)} - \frac{3 - p}{(1 - p)^2} k^{-\left(2+\frac{2}{1-p}\right)} \right],
\]

where \( C_{p,l} = \Gamma\left(l + 2 + \frac{2}{(1 - p)}\right)\left(\Gamma(l + 1)\right)^{-1}P(l + 1) \).
Ideas of the proof

To prove the previous theorem we pursue the following steps:

(1) we determine recursively $\mathbb{E}[N(k, t)], k = 1, 2, \ldots$;

(2) we prove the existence of $P(k) := \lim_{t \to \infty} \frac{\mathbb{E}[N(t, k)]}{t}$;

(3) we determine an explicit expression for $P(k)$,

(4) we use the Azuma-Hoeffding Inequality to prove convergence in probability of $N(k, t)/t$ to $P(k)$. 
Future projects

- On the relation between generalizations of Yule’s model and random graphs.

- To explore variants of random graphs with birth and death mechanism in the creation of new links or vertices; migration effects; attachment rules proportional to given functions, Non-Markov hypothesis and long-range dependence.