Near critical preferential attachment networks

Marcel Ortgiese

joint work with **Maren Eckhoff** and **Peter Mörters** (Bath/Köln),

Probabilistic and statistical methods for networks
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Percolation on graphs

Let G_n be any graph with n vertices and fix $p \in [0,1]$. The *percolated graph* $G_n(p)$ is obtained from G_n by deciding independently for each edge e of G_n:

- keep e with probability p,
- otherwise delete e with probability $1-p$.

Two typical scenarios:

(a) The network is robust: removing the edges does not change the global features (e.g. the existence of a giant component), or

(b) a phase transition:

$$p < p_c \implies G_n(p) \text{ has no giant component,}$$

$$p > p_c \implies G_n(p) \text{ has a giant component.}$$

Example: (supercritical) configuration model with degree distribution $\mu_k \sim k^{-\tau}$.

$$\tau \in (3,\infty) \implies \text{phase transition.}$$

vs.

$$\tau \in (2,3) \implies \text{robust.}$$

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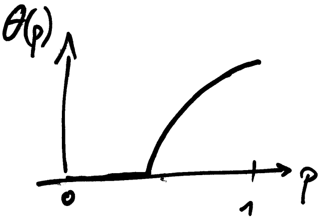
Approaching criticality

Plot $p \mapsto \theta(p)$, the asymptotic size of the giant component $\mathcal{C}_n^{(1)}(p)$ after percolation, i.e.

$$\frac{|\mathcal{C}_n(p)|}{n} \to \theta(p).$$



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Important question: How does $\theta(p)$ decay for $p \downarrow p_c$?

[Cohen, ben-Avraham, Havlin '02] show for the configuration model with tail exponent τ:

$$\theta(p) \sim (p - p_c)^{\beta'} \quad \text{for } p \downarrow p_c,$$

for

$$\beta' = \begin{cases} \frac{1}{\tau - 3} & \text{für } \tau \in (3, 4), \\ 1 & \text{für } \tau > 4. \end{cases}$$

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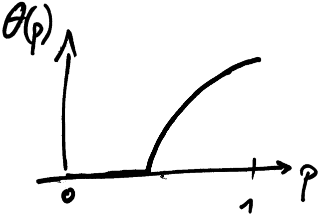

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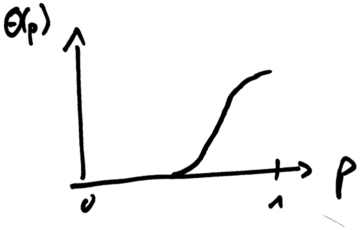

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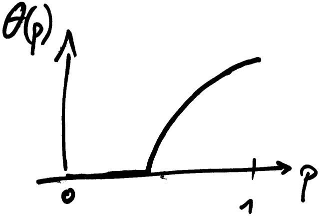

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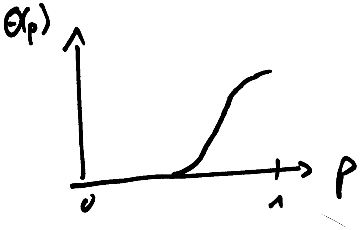

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The exponent β' is believed to be universal (and only depend on τ).

Preferential attachment models

- Preferential attachment models were proposed by [Barabási, Albert 1999] to model the growth of a network, such as the World Wide Web.
- Two essential ingredients:
  - *evolving network:* vertices are added to the system and connected to old vertices.
  - *Preferential attachment:* new vertices connect preferably to vertices that already have a high degree.
- [Barabási, Albert 1999] propose preferential attachment as a mechanism explaining that many real-world networks are *scale-free*: the degree distribution of a typical vertex converges to a power-law distribution:

$$X_k^{(n)} = \frac{1}{n} \sum_{i=1} \mathbb{1}_{\{\deg_n(i)=k\}} \to \mu_k \approx k^{-\tau},$$

  where τ is a power law exponent.
- First mathematical analysis by [Bollobás, Riordan, Spencer and Tusnády, 2001]. For a robust approach, see [Dereich, O. '14].
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The model

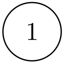
in a variation due to [Dereich, Mörters '09].

The strength of preferential attachment is governed by an *attachment rule* $f : \mathbb{N}_0 \to (0, \infty)$ (e.g. $f(k) = \gamma k + \beta$) such that $f(k) \le k + 1$.

- At time 1 the network consists of a single vertex without edges.
- At time $n+1$, the new vertex $n+1$ connects to each old vertex i independently with probability

$$\frac{f(\deg_n^-(i))}{n},$$

where $\deg_n^-(i)$ denotes the *in-degree* of vertex i at time n.



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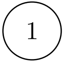

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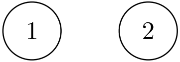

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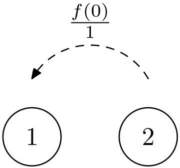

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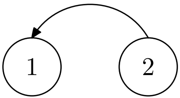

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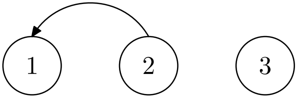

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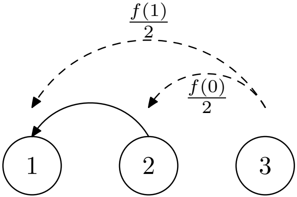

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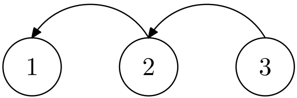

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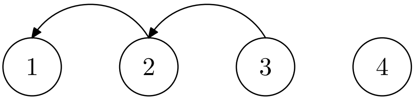

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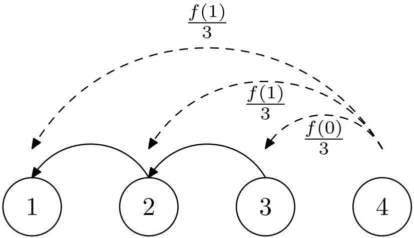

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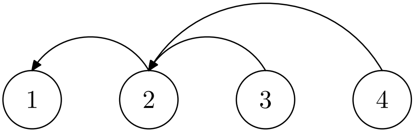

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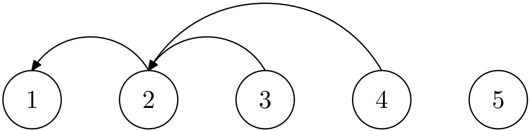

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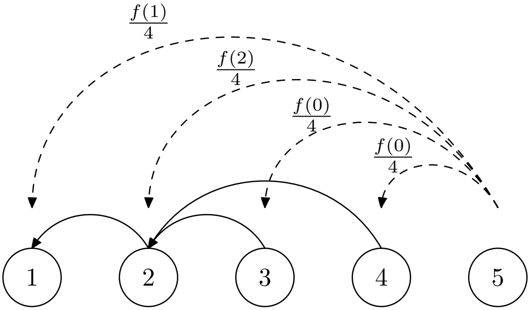

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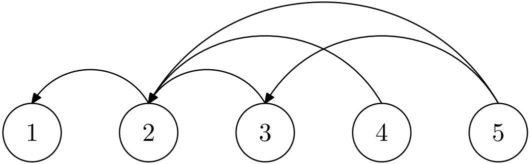

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Let f be increasing and $f(k) \le k + 1$. There exists a probability distribution $\mu = (\mu_k)_{k\in\mathbb{N}_0}$ such that, almost surely,

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If $f(k)/k \to \gamma \in (0, 1)$, then

$$\frac{\log \mu_k}{\log k} \to -(1 + \frac{1}{\gamma}) := -\tau.$$

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A sequence of random graphs $(G_n)_{n\geq 1}$ with largest connected components $(\mathcal{C}_n)_{ngeq1}$ has a *giant component* if

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Theorem 2 ([Dereich, Mörters '13])

For linear attachment rule $f(k) = \gamma k + \beta$ with $\gamma \in [0,1)$ and $\beta \in (0,1]$, there exists a giant component if and only if

$$\gamma \geq \frac{1}{2} \quad \text{or} \quad \beta > \frac{(\frac{1}{2} - \gamma)^2}{1-\gamma}. \tag{1}$$

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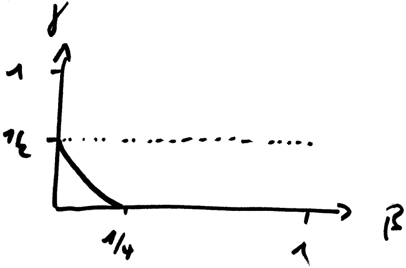

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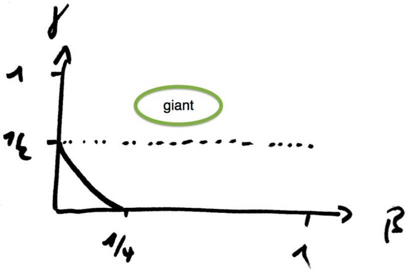

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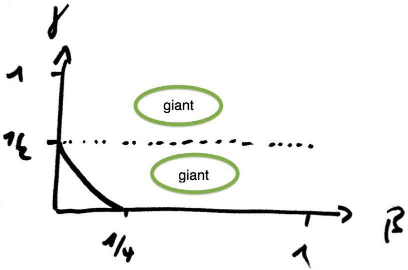

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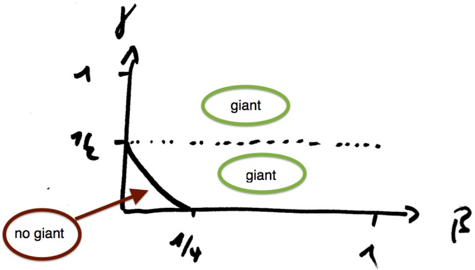

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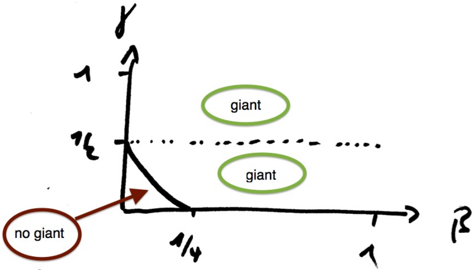


- Results also work for more general sublinear attachment rules.

The percolation threshold for preferential attachment

If G_n denotes a graph on n vertices, denote by $G_n(p)$ the *percolated graph*, where each edge is kept independently with probability p.

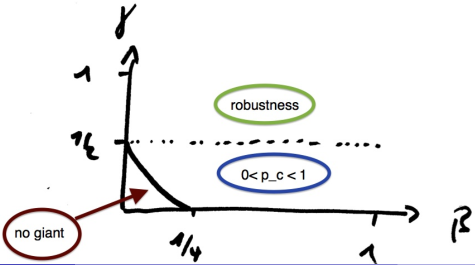
Theorem 3 ([Dereich, Mörters '13])

Let G_n be the preferential attachment graph with attachment rule $f(k) = \gamma k + \beta$.

(i) *The network is robust in the sense that $G_n(p)$ has a giant component for all $p \in (0, 1]$ if and only if $\gamma \geq \frac{1}{2}$.*

(ii) *If $\gamma \in [0, \frac{1}{2})$ and $\beta > \frac{(\frac{1}{2}-\gamma)^2}{1-\gamma}$, then there exists $p_c = p_c(\gamma, \beta) \in (0, 1)$ such that*

$$G_n(p) \text{ has a giant component } \iff p > p_c.$$



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If $p_c \in (0,1)$, what about $\theta(p)$ as $p \downarrow p_c$?

Theorem 4 (Eckhoff, Mörters and O.; in progress)

Let f be the linear attachment rule with $f(k) = \gamma k + \beta$, with $\gamma < \frac{1}{2}$ and $\beta > \frac{(\frac{1}{2}-\gamma)^2}{1-\gamma}$ and let $\theta(p,f)$ be the asymptotic size of the giant component of the p-percolated network. Then,

$$\lim_{p \downarrow p_c} \sqrt{p-p_c} \log \theta(p,f) = -\frac{1}{2\sqrt{2}} \pi p_c \sigma_{\beta,\gamma},$$

where $\sigma_{\beta,\gamma}$ is an explicit function of β and γ.

- **Slightly supercritical percolated preferential attachment networks are really small.**
- Does *not* show same behaviour as configuration model with the *same* tail exponent, where the decay is polynomially.

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- Instead of looking at size of the largest component, could also look at the size of giant component and let (γ, β) converge to the 'critical curve', so that $\theta(1, f) \to 0$.

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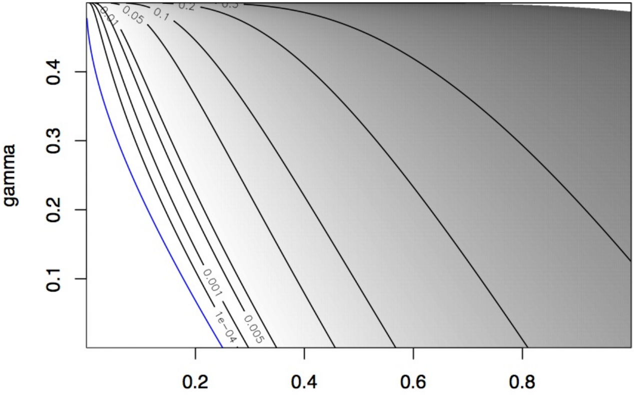

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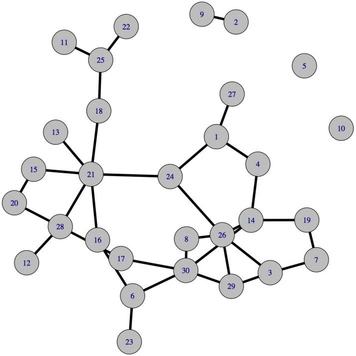
- Similar results also hold for any other $f_k \downarrow f$ with $\theta(1, f_k) > 0$, but $\theta(1, f) = 0$.
- Related work: [RIORDAN '05] shows for models that morally correspond to $\gamma = 1/2, \beta = 0$ and $\gamma = 0, \beta = 1/4$ that size of slightly super-critical component is exponentially small.

Local neighbourhoods in sparse random graphs

Run an exploration process on the graph

- Start in a uniformly chosen vertex
- Discover all its neighbours
- Discover all the neighbours of the neighbours, etc.



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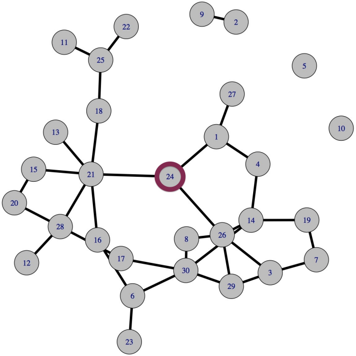

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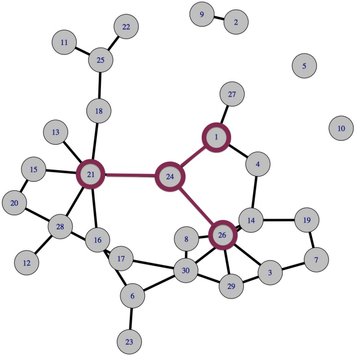

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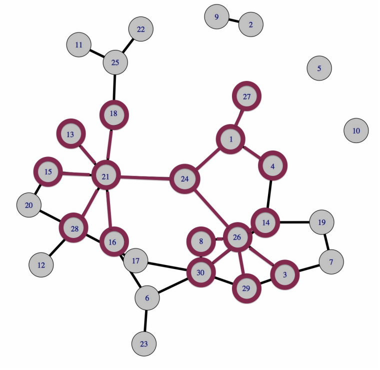

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General philosopy:

Existence of a giant component in the network

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the (idealized) random tree is infinite with positive probability.

This also extends to the *percolated network*.

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Local neighbourhoods for the preferential attachment model

Description by [Dereich and Mörters '13].

Consider a linear attachment rule f. The local neighbourhoods described by a **branching random walk**, where particles have:

- a position: time of birth (on a logarithmic scale) relative to newest vertex.
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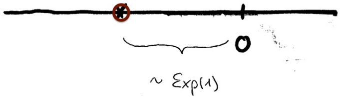

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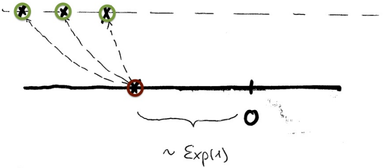

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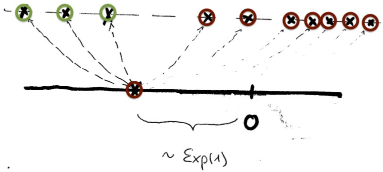

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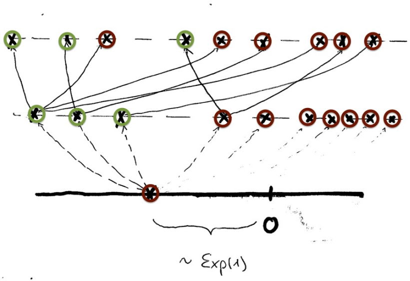

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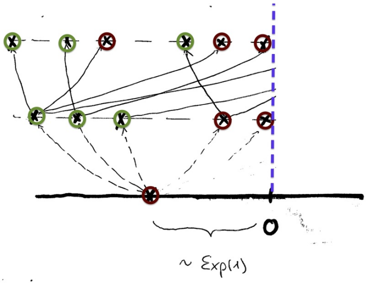
- Finally remove all particles with positive positions! $\rightsquigarrow$ Branching Random Walk (BRW) with absorption at 0



Proof of results

[Dereich, Mörters '13] show:

Existence of giant component

$\Updownarrow$

Branching random walk absorbed at 0 survives with positive probability.

Also, if $\mathcal{C}_n^{(1)}(p)$ denotes the p-percolated random graph, then

$$\frac{|\mathcal{C}_n^{(1)}(p)|}{n} \to \theta(p, f),$$

where $\theta(p, f)$ is the survival probability of the percolated branching random walk (absorbed at 0). Thus, our task is reduced to understanding

$$\theta(p, f) \sim ? \quad \text{as } p \downarrow p_c.$$

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We need to understand the asymptotics of the survival probability of a killed BRW as it becomes more and more critical.

- Our proof uses modern branching processes techniques. We adapt the proof of a result of [Gantert, Hu, Shi '09], who show asymptotics for a killed branching random walk.
  - Difference: we have infinitely many particles to the right.
  - They kill just above maximal speed; in our case the model approaches criticality!
- Idea: identify optimal strategy!
- Technical tools:
  - Many-to-one lemma: reduce question about branching random walk to question about random walk.
  - A robust version of Mogulskii's theorem: large deviations estimates for random walk to stay in a small corridor.

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  - A robust version of Mogulskii's theorem: large deviations estimates for random walk to stay in a small corridor.

Some proof ideas

Introduce a new measure, by setting for any test function f

$$\mathbb{E}^\alpha\Big[f(S_i, i = 1, \ldots, N)\Big] = \mathbb{E}\Big[\sum_{|x|=N} e^{-\alpha S_N(x)} f(S_i(x), i = 1, \ldots, N)\Big].$$

For the right α, $S_i, i = 1, \ldots, n$ is a centred random walk.

Then, take a sequence $p_n \downarrow p_c$, choose paramaters

$$N = (b(p_n - p_c))^{3/2}, \quad \text{for some } b > 0.$$

We can show

$$\begin{aligned}
\mathbb{P}(\text{survival}) &\approx \mathbb{P}(\exists |x| = N \,:\, S_i(x) \approx i\tfrac{b}{N^{2/3}} \forall i \in [N]) \\
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We looked at the decay of the relative size $\theta(p)$ of the giant component in the p-percolated network for $p \downarrow p_c$.

Configuration model
(degree exponent τ)
[Cohen et. al, 2002]

$$\theta(p) \sim (p - p_c)^{\beta}$$

where

$$\beta = \begin{cases} \frac{1}{\tau-3} & \tau \in (3,4) \\ 1 & \tau > 4 \end{cases}$$

Local description:
Galton-Watson tree

Preferential attachment model
(degree exponent $\tau \in (3, \infty)$)

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Outlook

I. Extension of the preferential attachment models

- Classic preferential attachment models do not describe real-world networks perfectly. Two shortcomings of PA:
  - Local neighbourhoods are trees.
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- Two possible modifications of the model:
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  - Introduce random fitness of nodes $\rightsquigarrow$ more interestingcompetition.

II. More complicated stochastic processes

- Percolation is the easiest random process defined on top of random network.
- Random walks, e.g. mixing and cover times. [Cooper, Frieze '07].
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