Branching processes with reinforcement
Exercises

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1 The Yule process

The Yule process $(Y_t : t \geq 0)$ with rate η is a process of immortal particles starting with one particle. At any time every particle independently gives birth to a new particle with rate η. Y_t is the number of particles alive at time t. The Yule process of parameter η is characterised as follows. Let τ be an exponential random variable of parameter η, then

$$Y_t = \begin{cases} 1, & \text{for } t < \tau, \\ Y^{(1)}_{t-\tau} + Y^{(2)}_{t-\tau}, & \text{for } t \geq \tau, \end{cases}$$

where $Y^{(1)}$ and $Y^{(2)}$ are two independent copies of Y.

Let $(Y_t : t \geq 0)$ be a Yule process with rate η.

(a) Let $a > 0$ and show that $(Y_at : t \geq 0)$ is a Yule process with rate $a\eta$.

(b) Show that that $(e^{-\eta t} Y_t : t \geq 0)$ is a martingale.

(c) Infer that there exists a random variable ξ such that, almost surely,

$$\lim_{t\uparrow\infty} e^{-\eta t} Y_t = \xi.$$

(d) Show that ξ is exponentially distributed with parameter one.

(e) Show that $\sup_{t\geq 0} \mathbb{E}[e^{-2\eta t} Y_t^2] < \infty$.

2 The empirical degree distribution in the Bianconi-Barabasi Tree

Assume we are in the situation of Example 2. Then

$$\Theta_t := \frac{1}{M(t)} \sum_{n=1}^{M(t)} \delta_{Z_n(t)}$$

is the empirical distribution of degrees in the network at time t.

(a) Show that under Assumption 1 we have

$$\lim_{t\uparrow\infty} \Theta_t = \nu \qquad \text{almost surely,}$$

where

$$\nu(k) = \int_0^1 \frac{\lambda^*}{kf + \lambda^*} \prod_{i=1}^{k-1} \frac{if}{if + \lambda^*} \mu(df).$$

(b) Show that $\lambda^* \in (1, 2)$ and that ν is a probability measure.

(c) Show that $\nu(k) = k^{-(1+\lambda^*)+o(1)}$ and hence the power law exponent ranges between the values of 2 and 3, which is sometimes referred to as the supercritical regime.

3 The size and fitness of the largest family

(a) Show that, as $t \to \infty$,

$$e^{-\gamma(t-T(t))} \max_{n\in\mathbb{N}} Z_n(t) \Rightarrow W^{-\frac{\gamma}{\lambda^*}},$$

where W is exponentially distributed with parameter $\Gamma(\alpha + 1)\Gamma(1 + \frac{\lambda^*}{\gamma})(\lambda^*)^{-\alpha}$.

(b) Let $V(t)$ the fitness of the family of maximal size at time t. Show that, under Assumption (cond), as $t \to \infty$, we have

$$t(1 - V(t)) \Rightarrow V,$$

where V is Gamma-distributed with rate parameter λ^* and shape parameter α.