Lecture 3: Dynamic network models
Probabilistic and statistical methods for networks
Berlin Bath summer school for young researchers

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August, 2017

Outline

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8. Case study: Configuration model.
9. Conclusion: extensions and open problems.

All the work in this lecture joint with Nicolas Broutin, Sanchayan Sen and Xuan Wang.

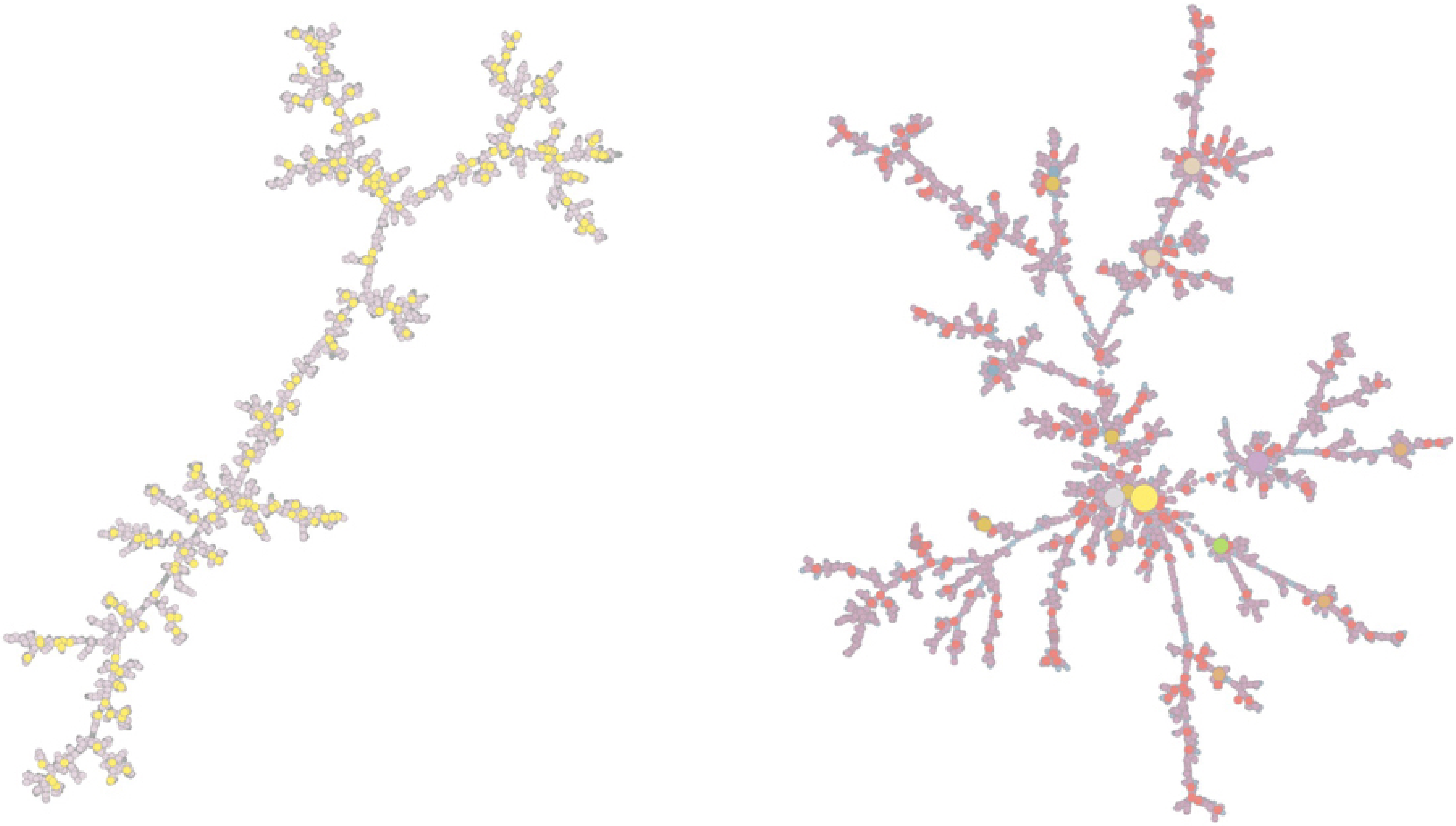
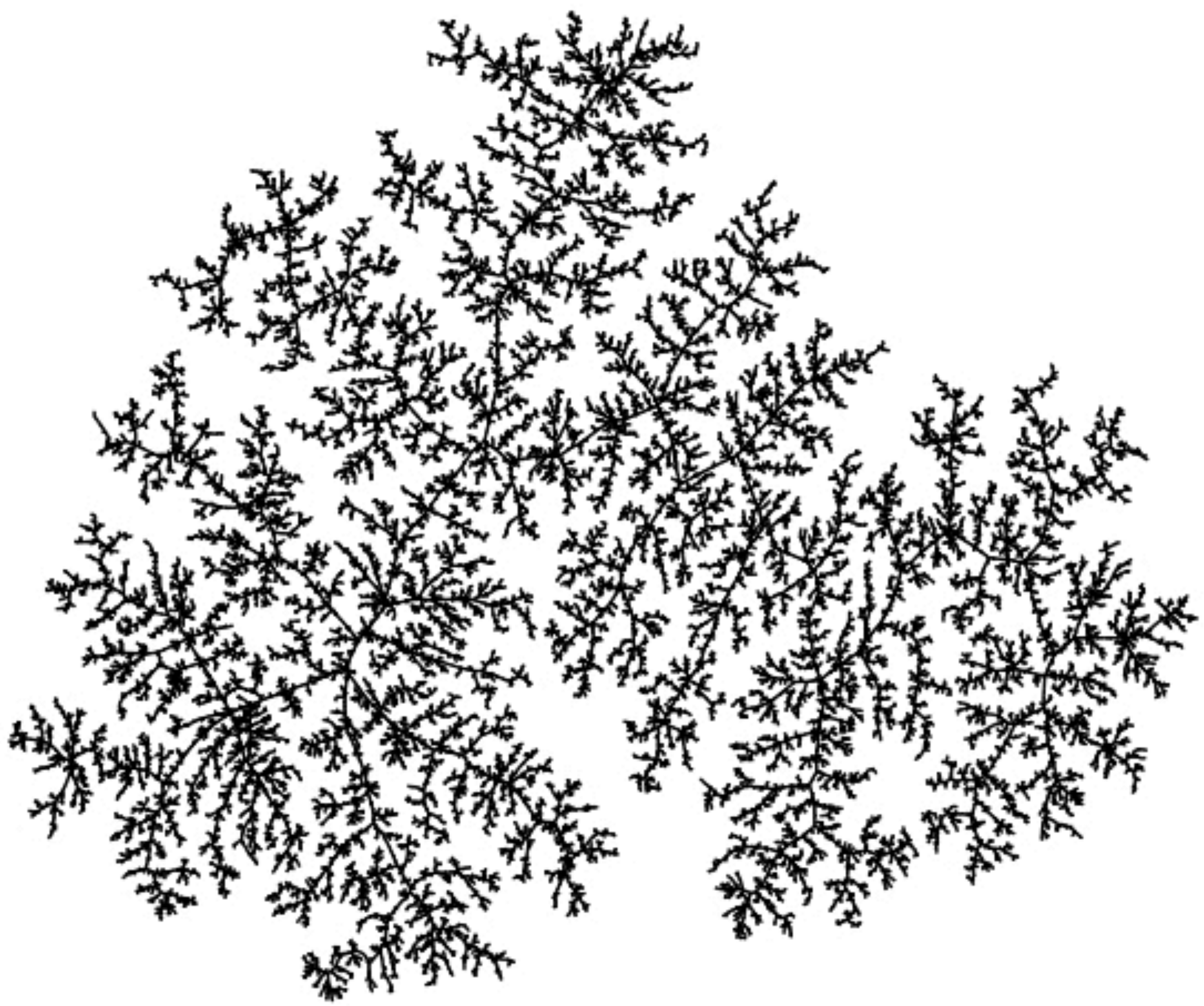


Figure: CRTs and *Inhomogeneous* CRTs

Minimal spanning tree (MST) and scaling limits

- Consider a network model
- Suppose each edge has a (random) edge length.
- Consider the minimal spanning tree (MST). (*Strong disorder*) How does this object scale? Precisely: suppose we view this tree as a metric space using graph distance. Does this tree appropriately rescaled converge to a limiting object?
- *How do these depend on the degree distribution? Is there universality?*

MST on the complete graph



MST on the complete graph on 100,000 vertices. Generated by Nicolas Broutin.

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Predictions from Statistical Physics (Braunstein et al, 2006)

- Phase transition at $\tau = 4$: When $\tau > 4$ distances scale like $n^{1/3}$. When $\tau \in (3, 4)$ distances scale like $n^{(\tau-3)/(\tau-1)}$.
- Also predict *universality*: Results should hold for a wide array of random graph models.

MST on the complete graph and critical Erdős-Rényi random graphs

Kruskal's algorithm

- **Setting:** Complete graph with uniform $[0,1]$ iid edge weights. Let $\mathcal{M}_n$ denote MST.
- Construction: Start with n isolated vertices. At each step, add unique edge of smallest weight joining two distinct components. Stop when all vertices connected.

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Erdős-Rényi random graph process

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- At each stage choose an edge at random and place it in the system.
- Think for yourself: easy to couple Kruskal's algorithm and Erdős-Rényirandom graph process.

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- At each stage choose an edge at random and place it in the system.
- Think for yourself: easy to couple Kruskal's algorithm and Erdős-Rényirandom graph process.
- A giant component of MST present when $cn/2$ edges in the system (for any $c > 1$). Most of the global structure of MST present at this stage.

Fundamental finding of Addario-Berry, Broutin, Goldschmidt, Miermont (ABGM)

MST and critical random graphs

- Recall from Lecture 1 that the "critical scaling window" corresponds to edges of the sort $n/2 + \lambda n^{2/3}$.
- ABGM in 2013 showed that the MST on the complete graph *looks* like the maximal component $\mathcal{C}_n^{(1)}(\lambda)$ "for large λ".
- Deep result and novel ideas to make the above notion precise since obviously $|\mathcal{C}_n^{(1)}(\lambda)|/n = (n^{-1/3})$ so has a very small fraction of the eventual MST.

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Conclusion

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Now "random objects" live in the space of compact metric spaces. So need proper notion of metric so as to talk about weak convergence.

Metric d_{GH}

Fix two metric spaces $X_1 = (X_1, d_1)$ and $X_2 = (X_2, d_2)$. For subset $C \subseteq X_1 \times X_2$, distortion of C is defined as

$$\mathrm{dis}(C) := \sup\left\{|d_1(x_1, y_1) - d_2(x_2, y_2)| : (x_1, x_2), (y_1, y_2) \in C\right\}. \tag{0.1}$$

A correspondence C between X_1 and X_2 is a measurable subset of $X_1 \times X_2$ such that for every $x_1 \in X_1$ there exists at least one $x_2 \in X_2$ such that $(x_1, x_2) \in C$ and vice-versa. The Gromov-Hausdorff distance between the two metric spaces (X_1, d_1) and (X_2, d_2) is defined as

$$d_{\mathrm{GH}}(X_1, X_2) = \frac{1}{2} \inf\left\{\mathrm{dis}(C) : C \text{ is a correspondence between } X_1 \text{ and } X_2\right\}. \tag{0.2}$$

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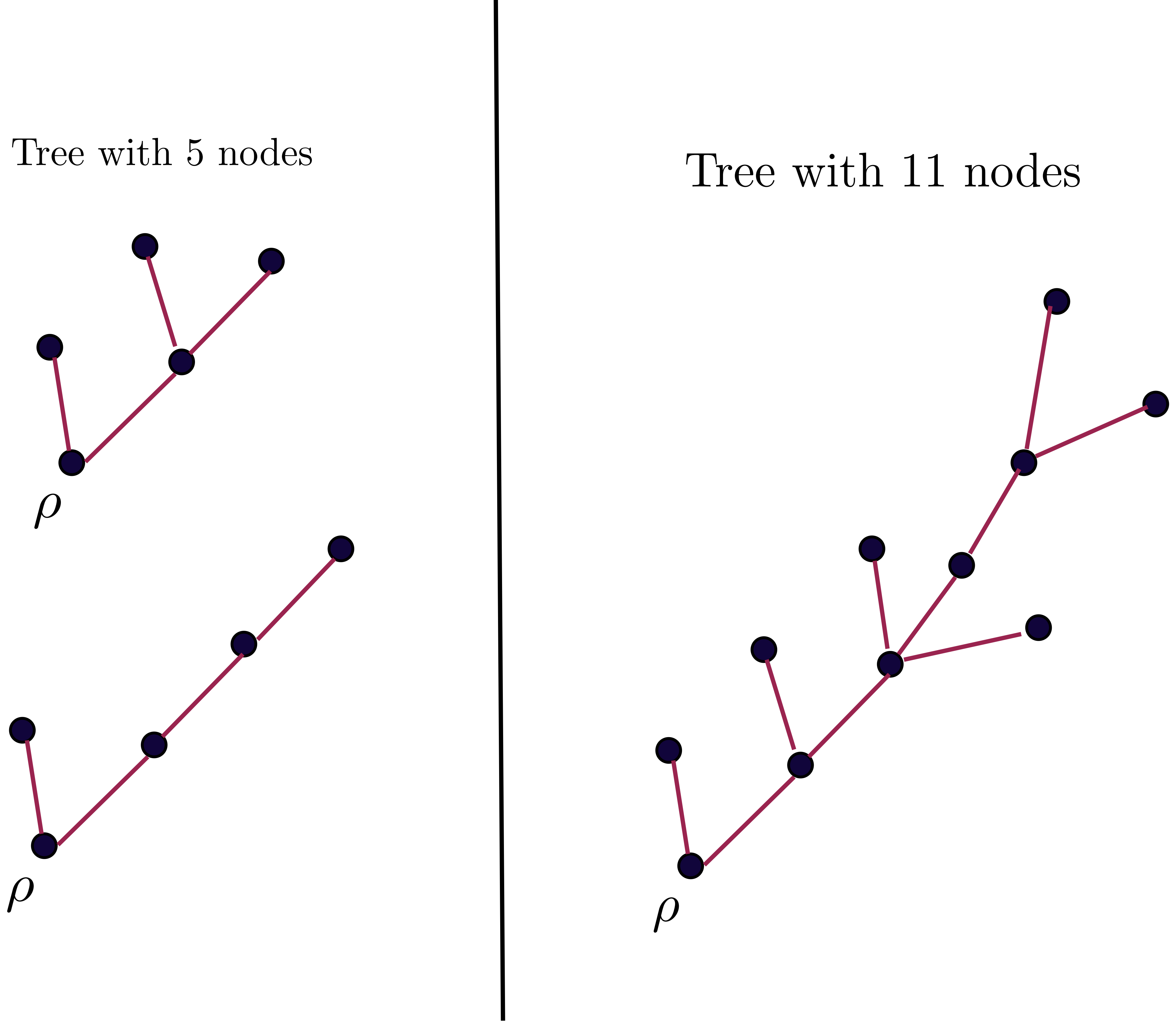
Bottom line

$\mathscr{S}$ space of compact metric spaces can be metrized via above metric (and results in a Polish space). Can talk about weak convergence of $\mathscr{S}$-valued random variables.

Starting point: Aldous's continuum random tree (CRT)

First: some motivation

- Random trees a very vast field


Tree with 5 nodes
ρ
ρ
Tree with 11 nodes
ρ

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Example arising in RNA studies

On the space of trees of size n consider probability measure

$$p_{n,\beta}(\mathbf{t}) \propto \exp(\beta \# \text{ leaves in } \mathbf{t})$$

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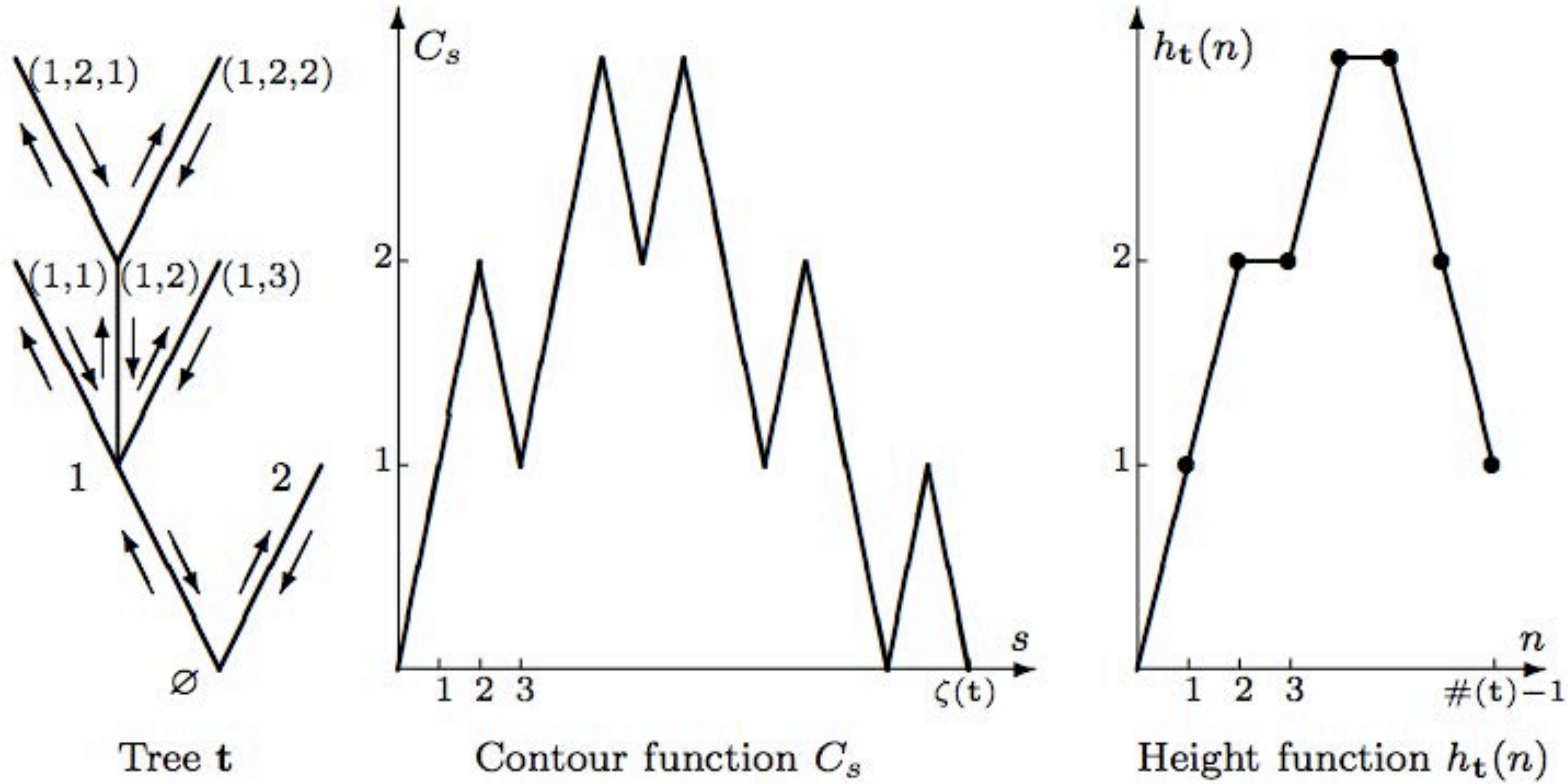

Courtesy the amazingly beautiful survey by J.F.Le Gall: Random trees and applications, Prob. Surveys, 2005

Contour functions and "real trees"

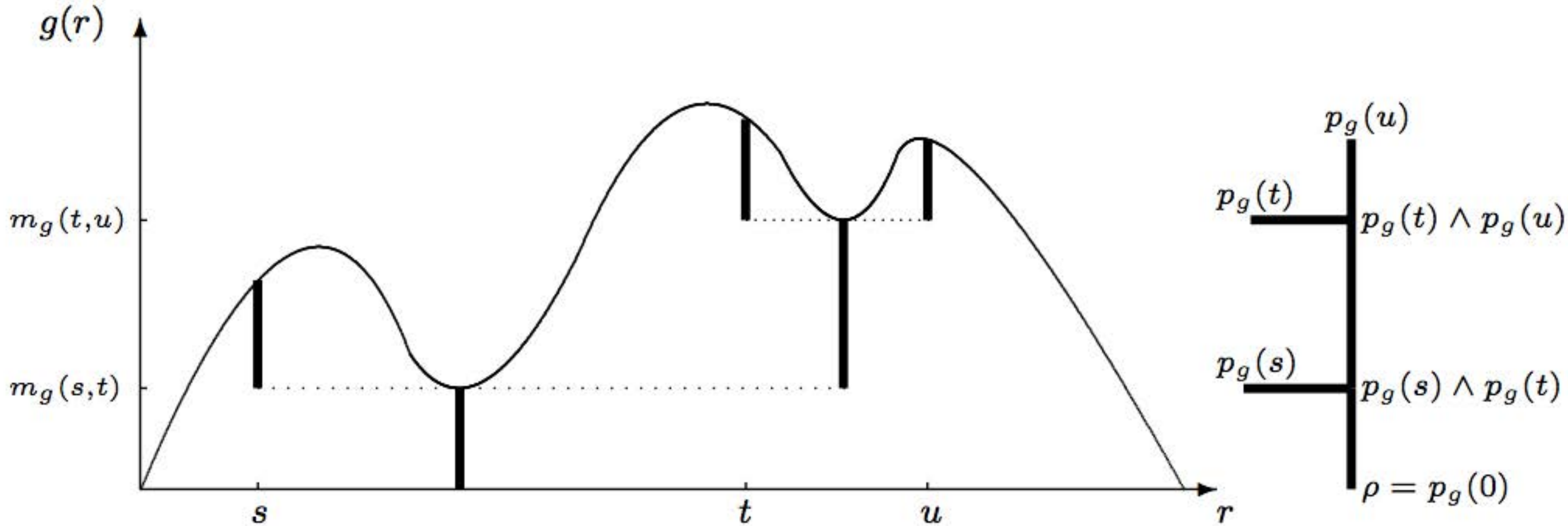

Figure 2

Can metrize support of function using the "distance"

$$d_g(s,t) = g(s) + g(t) - 2m_g(s,t)$$

Resulting metric space called the real tree corresponding to g.

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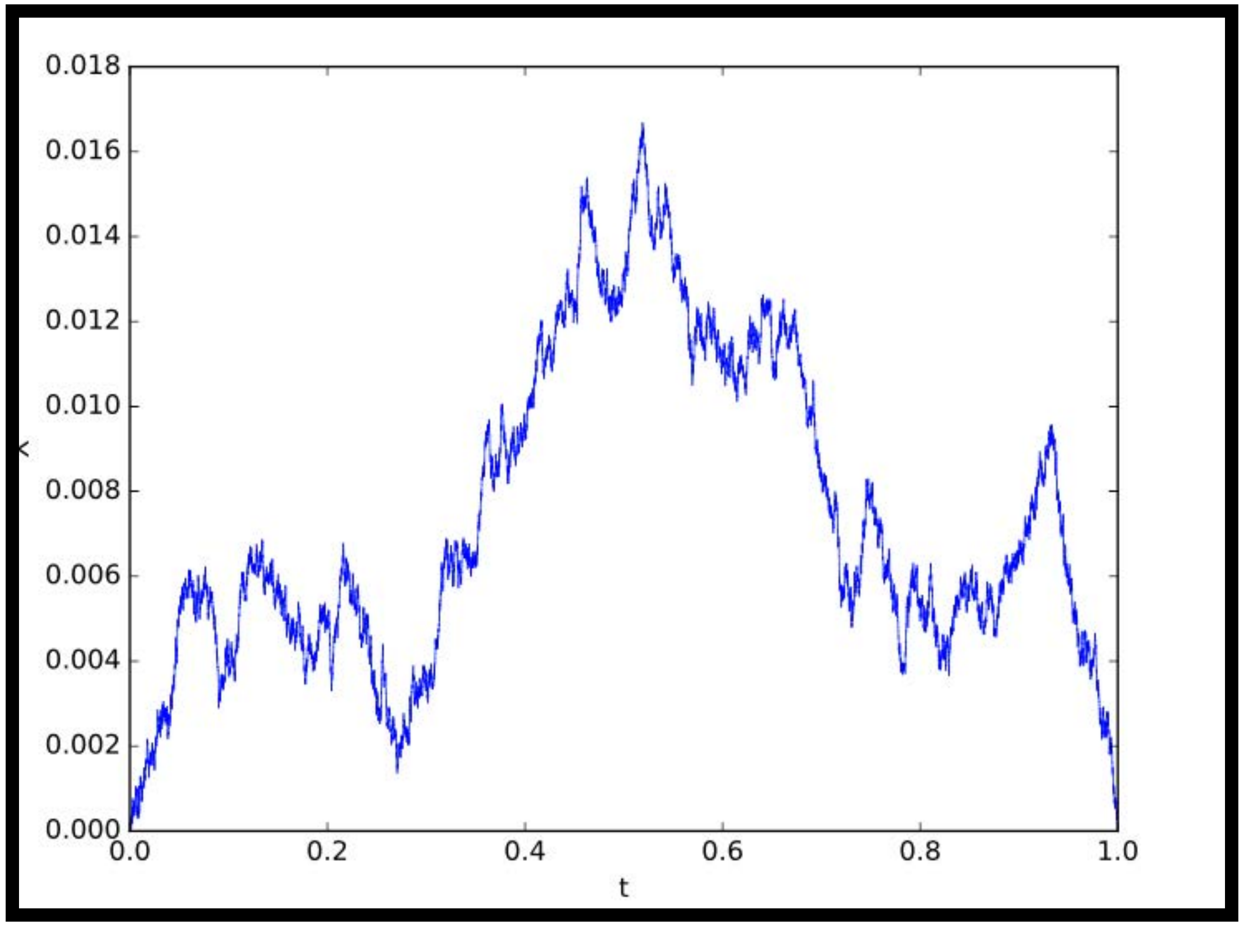
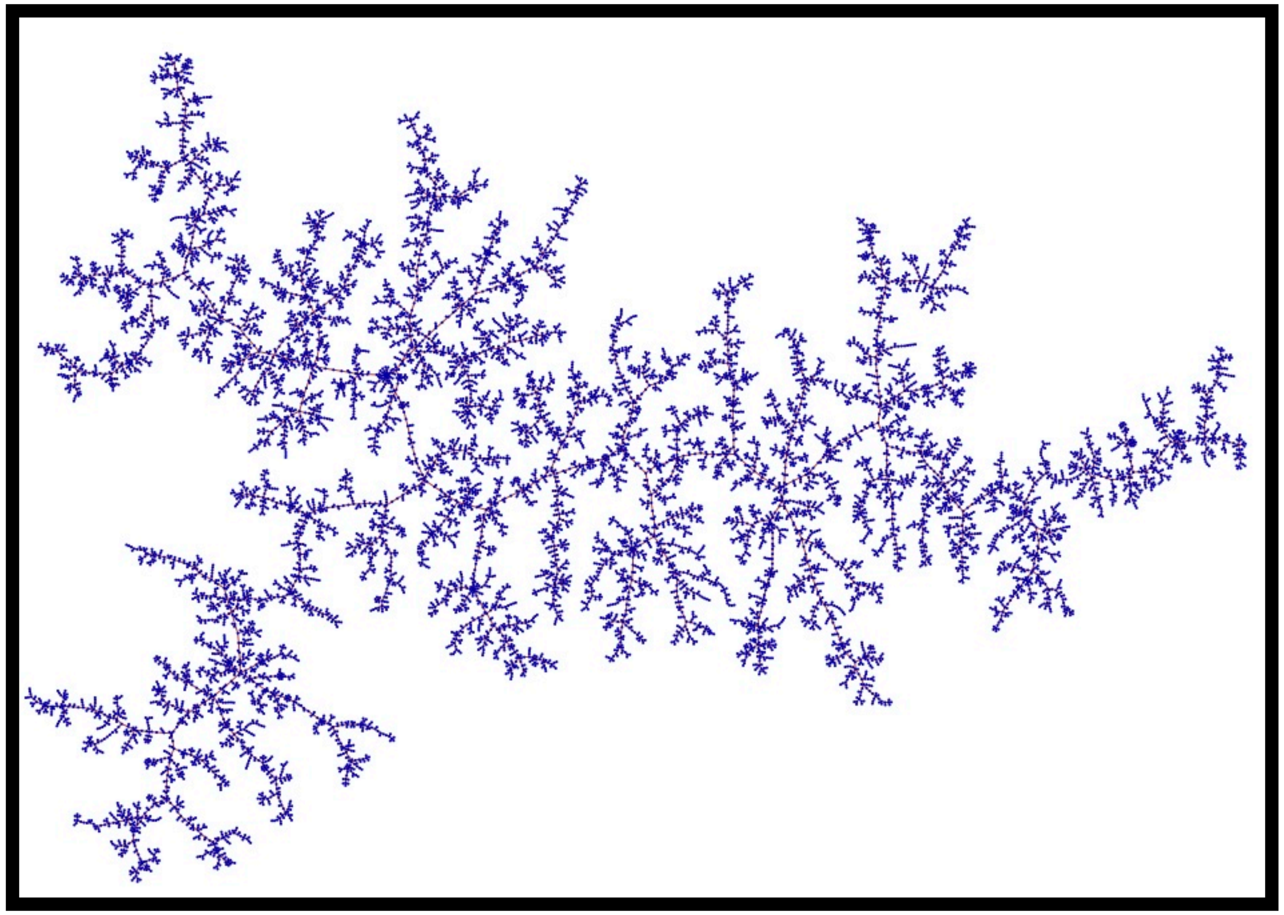
Brownian excursion and Aldous's continuum random tree



Brownian excursion simulation
By Shiyu Ji (Own work) [CC BY-SA 4.0 (http://creativecommons.org/licenses/by-sa/4.0)], via Wikimedia Commons



Approximation of Aldous's CRT.
By Igor Kortchemski
https://www.normalesup.org/~kortchem/english.html

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- Dyck path has same distribution as **conditioned** simple random walk
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- Proved that many of these trees, if you rescale each edge by $n^{-1/2}$ then the tree seen as metric spaces converges (with space of compact metric spaces metrized by the Gromov-Hausdorff metric d_{GH} as above) to a random fractal called **continuum random tree**.

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h_{ex} height of standard Brownian excursion.

Background

Criticality and emergence of the giant

- Fundamental problem in random graphs: connectivity and emergence of the giant.
- Many random graph models come with a parameter t (often related to **edge density**) and model dependent "critical time" t_c.
- If $t < t_c$ no giant component ($\mathcal{C}_1(t) = o_P(n)$).
- If $t > t_c$ then $\mathcal{C}_1(t) \sim f(t)n$. **Giant component.**

Current obsession

What happens in the critical regime?

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Current obsession

What happens in the critical regime? **What happens to the metric structure of the maximal components?**

Classical example: Erdos-Renyi random graph at criticality

History

- after initial work by [ER1960], further fundamental work in Luczak and [JKLP1994]. Form we will use finally proved by [Aldous1997].
- Formal existence of multiplicative coalescent.

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Problem statement

- Connection probability $p_n := \frac{1}{n}\left[1 + \frac{\lambda}{n^{1/3}}\right]$.
- $\mathcal{C}_n^{(i)}(\lambda)$ size of the i-th largest component.
- Surplus (Complexity) of a component

$$N_i^{(n)}(\lambda) = E(\mathcal{C}_n^{(i)}(\lambda)) - (\mathcal{C}_n^{(i)}(\lambda) - 1)$$

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- $l^2_{\downarrow} = \{(x_i)_{i\geqslant 1} : x_1 \geqslant x_2 \geqslant \cdots \geqslant 0, \sum_i x_i^2 < \infty\}$

- $$\mathbf{C}_n^*(\lambda) := n^{-2/3}(|\mathcal{C}_1(\lambda)|, |\mathcal{C}_2(\lambda)|, \ldots)$$
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Complexity

Surplus in maximal component $N_i^{(n)}(\lambda) = O_P(1)$. Nice point process description of the limit.
Punchline: Components almost tree-like.

In pictures

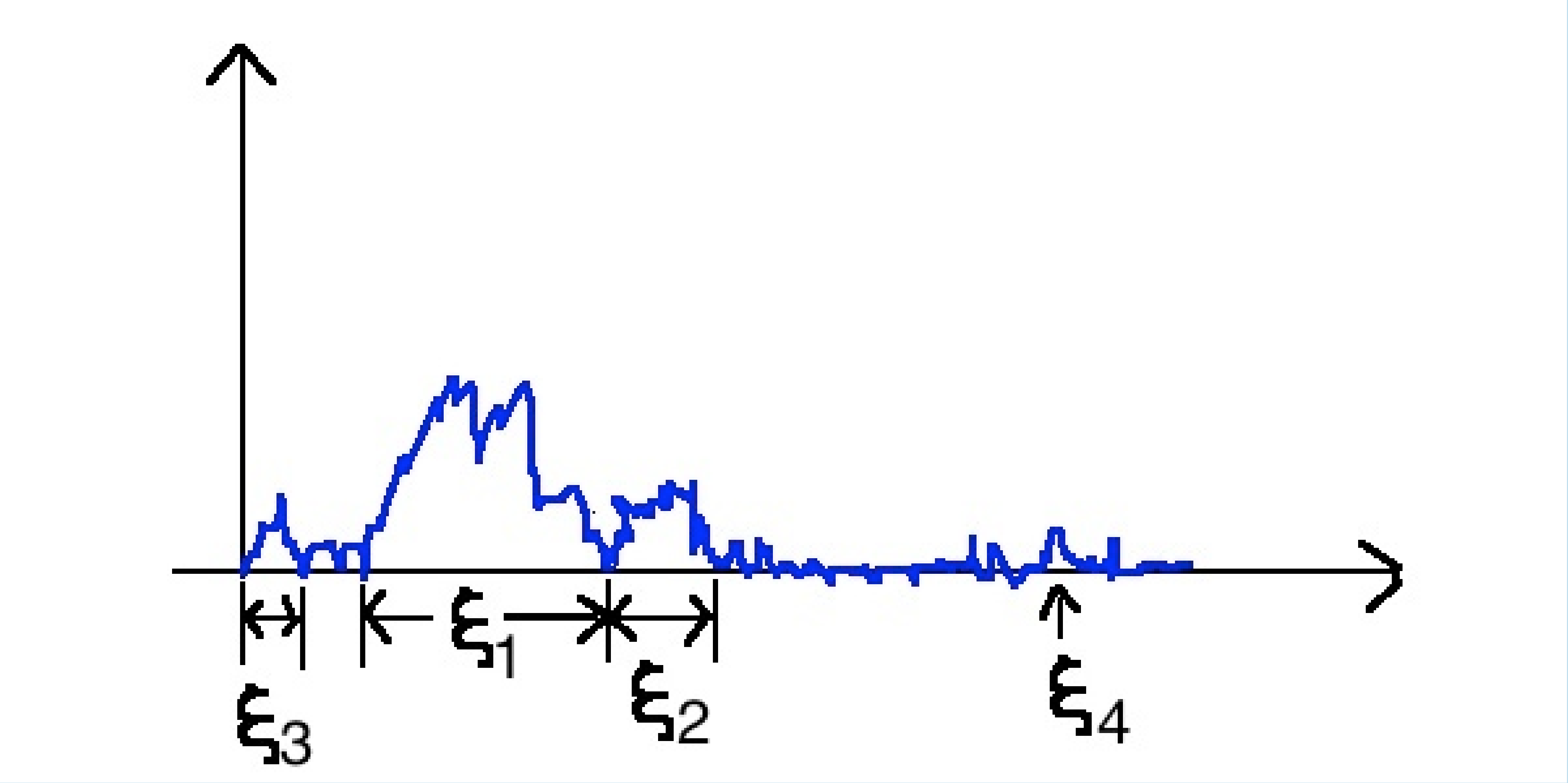


Figure: Reflected process

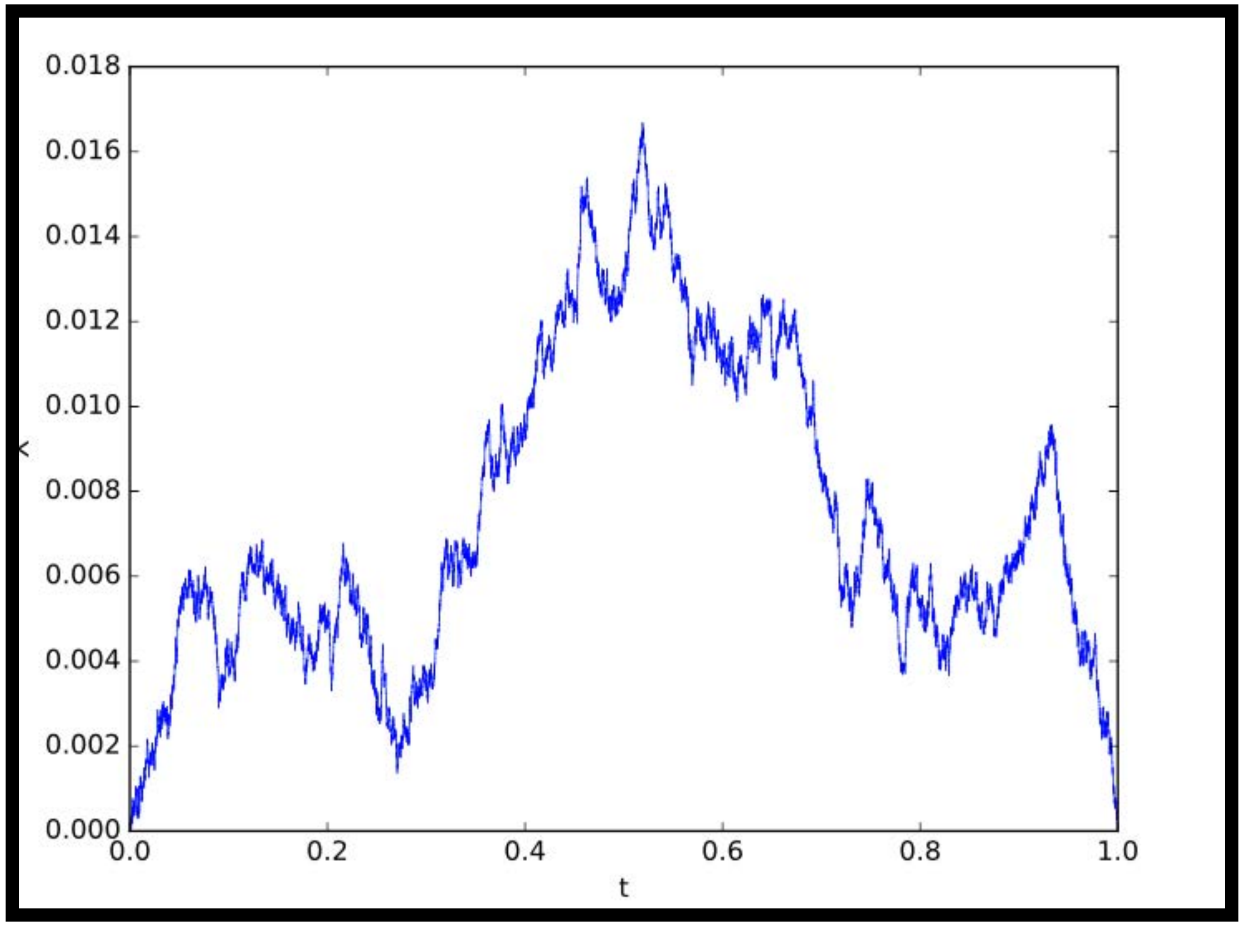
Intuition: $\mathcal{C}_i(\lambda)$

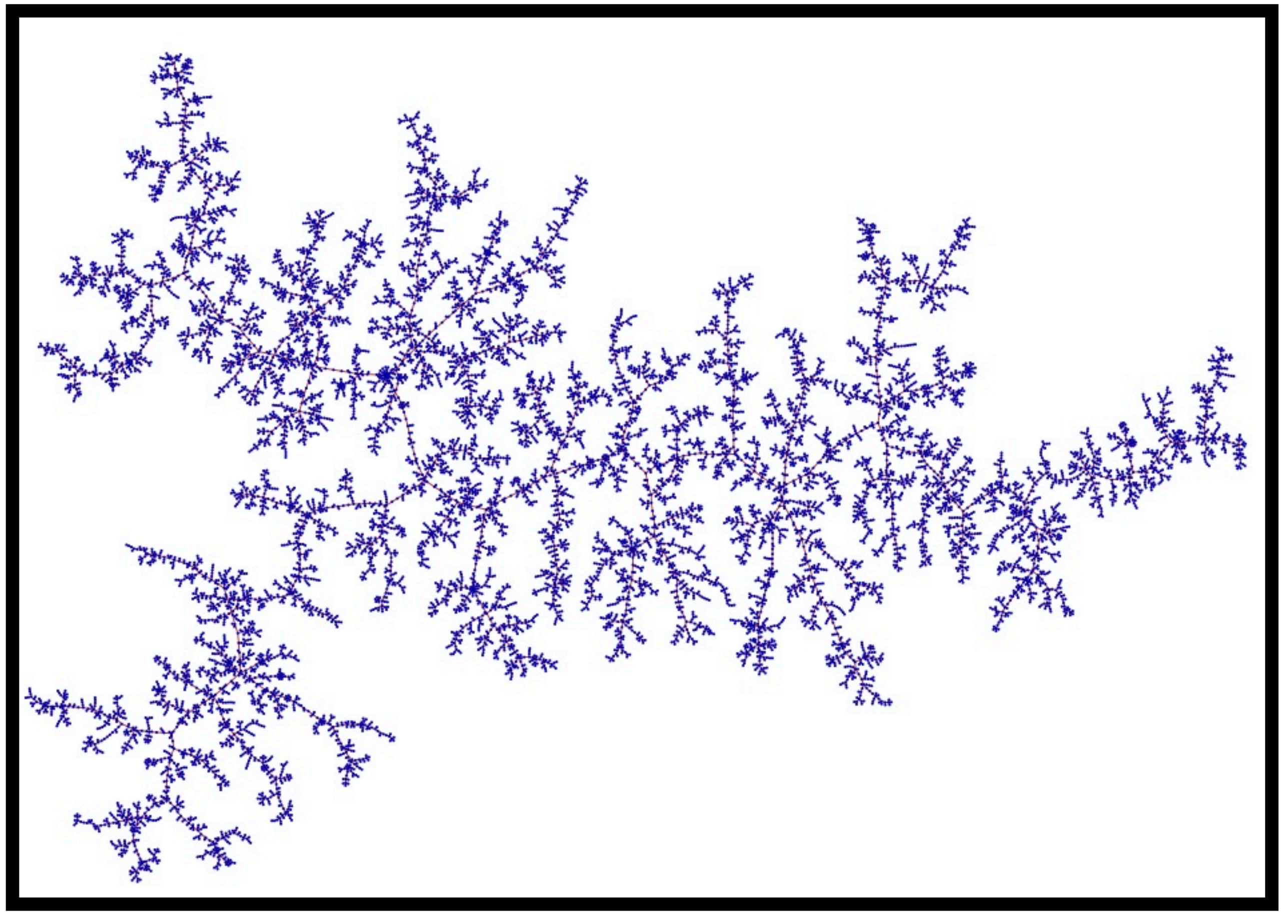
- Uniform random tree $\mathcal{T}_{cn^{2/3}}$ on $cn^{2/3}$ viewed as **metric space** then

$$n^{-1/3}\mathcal{T}_{cn^{2/3}} \xrightarrow{d_{GH},w} \mathrm{CRT}_c$$

- Recall that CRT random real tree encoded by Brownian excursion $2\mathbf{e}_c(\cdot)$.

Brownian excursion and Aldous's continuum random tree





Brownian excursion simulation
By Shiyu Ji (Own work) [CC BY-SA 4.0 (http://creativecommons.org/licenses/by-sa/4.0)], via Wikimedia Commons

Approximation of Aldous's CRT.
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Metric structure: Addario-Berry, Broutin, Goldschmidt

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- $\tilde{\mathcal{T}}_i$: Random random real tree encoded by this excursion. Pick a Poisson $\#$ of leaves $\mathcal{L}$ with density proportional to height.
- For each $x \in \mathcal{L}$ pick a uniform point on unique path from root ρ to x, U_x. Identify x and U_x.
- This gives limit object $\mathrm{Crit}_i(\lambda)$.

Motivation

- Last few years motivated by data, wide array of interesting random graph models proposed.
  1. **Configuration model**
  2. **Inhomogeneous random graph**
  3. **Bounded size rules**
- Tremendous amount of work on understanding phase transition especially above and below critical regime.
- Lot of work on **maximal component sizes** in the critical regime. Often match Erdos-Renyi in terms of size scaling and components being described via excursions of inhomogeneous BM.

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  3. **Bounded size rules**
- Tremendous amount of work on understanding phase transition especially above and below critical regime.
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Aims/questions of the research program

- Develop general techniques that enable one to prove scaling limits of maximal components in the critical regime at the **metric level** that can work in different settings.

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- **Probability theory:** Lots of invariance principles (Martingale FCLT, Donsker, Lindeberg-Levy-Feller-Lyapunov CLT, Continuum random tree etc).
- View the scaling limit for Erdos-Renyi limits as analog of the normal distribution/BM: what "Asymptotic negligibility conditions" do we need to ensure that for a random graph model in the critical regime, maximal components scale like $n^{1/3}$ and converge to $(\mathrm{Crit}_i(\cdot))$?

Organization/Aims of the talk

Why should you care?

1. Technique hopefully general enough to be useful in other regimes. Will show results in 3 major classes.
2. Scaling limit of critical components first step in understanding more complicated objects such as the MST.

Logical flow of talk

1. Give you basic idea of our attempts at this universality.
2. Hard to understand if I just state the abstract result so first will give you what this result (+ a lot of work!) gives for 3 major classes of random graphs
3. Then give intuition of why we started thinking along these lines
4. State abstract result and ramifications

Model I: Percolation on supercritical Configuration model

Model definition

- Fix pmf $\mathbf{p}_{\mathrm{deg}} = \{p_k : k \geqslant 0\}$. Assume $p_2 < 1$. Also assume
$$\nu = \frac{\sum_k k(k-1)p_k}{\sum_k kp_k} > 1, \qquad \beta = \sum_k k(k-1)(k-2)p_k$$
- Let $d \sim \mathbf{p}_{\mathrm{deg}}$. Assume exponential tails: for some $\gamma > 0$, $\mathbb{E}(e^{\gamma d}) < \infty$.

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- $[n] = \{1, 2, \ldots, n\}$. Let $d_i \sim_{iid} \mathbf{p}_{\deg}$. Start with n vertices with degree/# free/alive **half** edges d_i. Perform uniform matching of half-edges to get full edges.

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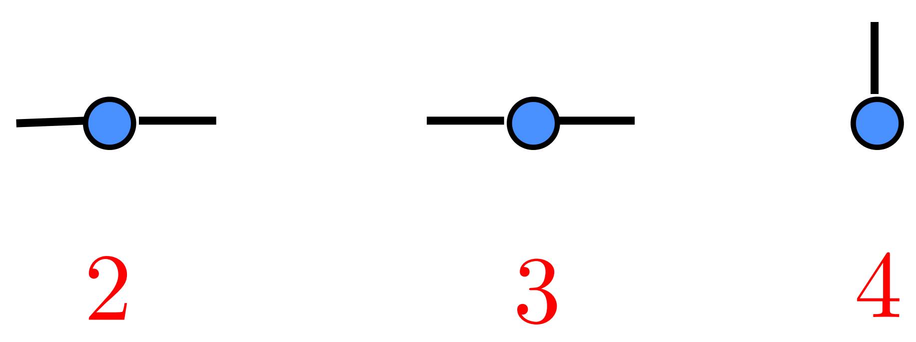
- Random graph CM_n

$$d_1 = 1, \quad d_2 = 2, \quad d_3 = 2, \quad d_4 = 1$$



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$$p(\lambda) = \frac{1}{\nu}$$

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- Random graph $\mathrm{CM}_n(\infty)$. Now consider critical percolation with edge retention probability

$$p(\lambda) = \frac{1}{\nu} + \frac{\lambda}{n^{1/3}}.$$

- Denote the corresponding graph $\mathrm{Perc}_n(\lambda)$.

Model I: Percolation on the configuration model

Known results

- Enormous amount of work (Bollobas, Janson, Molloy and Reed, Riordan....). Used also extensively in applications.
- $p > 1/\nu$: Giant component
- $p < 1/\nu$: $\mathcal{C}_1 = o_P(n)$
- $p = p(\lambda)$: All maximal component sizes $|\mathcal{C}_i| \sim \xi_i n^{2/3}$ [Nachmias-Peres (random regular graph); Joseph; Riordan (bounded degree).]

Model I: Percolation on the configuration model [Our results]

Theorem: Continuum scaling limits of metric structure for $\mathrm{Perc}_n(\lambda)$

For critical percolation on the CM_n we can show

$$\left(\frac{\beta^{2/3}}{\mu\nu}\frac{1}{n^{1/3}}\mathcal{C}_i^{(n)}(\lambda) : i \geqslant 1\right) \xrightarrow{\mathrm{w}} \mathbf{Crit}_\infty\left(\frac{\nu^2}{\beta^{2/3}}\lambda\right), \qquad \text{as } n \to \infty.$$

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- Distances in maximal components scale like $n^{1/3}$.
- Convergence not just in d_{GH} but in d_{GHP}.

Corollary: Random r-regular graph

$$p(\lambda) = \frac{1}{r-1} + \frac{\lambda}{n^{1/3}}.$$

Then the maximal components viewed as metric spaces satisfy

$$\left(\frac{(r(r-1)(r-2))^{2/3}}{r(r-1)}\frac{1}{n^{1/3}}\mathcal{C}_i^{(n)}(\lambda) : i \geqslant 1\right) \xrightarrow{\mathrm{w}} \mathbf{Crit}_\infty\left(\frac{(r-1)^2}{(r(r-1)(r-2))^{2/3}}\lambda\right),$$

Model II: Inhomogenous random graphs

Model definition (Bollobas, Janson, Riordan)

- **Vertex type space:** $\mathscr{X} = [K] = \{1, 2, \ldots, K\}$

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- n**-dependent kernel:** $\kappa_n : [K] \times [K] \to \mathbb{R}_+$.
- **Empirical distribution of types:** $\mu_n(x) = \# \{i \in [n] : x_i = x\} / n$.
- Connect vertex i, j with probability

$$p_{ij} := 1 - \exp\left(-\frac{\kappa_n(x_i, x_j)}{n}\right).$$

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Associated operator

$$(T_{\kappa_n} f)(x) := \sum_{y \in [K]} \kappa_n(x, y) f(y) \mu_n(y), \quad x \in [K], f \in \mathbb{R}^{[K]}.$$

By **BJR[05]:** Assume $\kappa_n \approx \kappa$, $\mu_n \approx \mu$. Let $||T_\kappa||$ operator norm of T_κ in $L^2([K], \mu)$.

- Supercritical regime: If $||T_\kappa|| > 1$ $\mathcal{C}_1 \sim \rho(\kappa, \mu) n$.
- Subcritical regime: If $||T_\kappa|| < 1$ $\mathcal{C}_1 = o_P(n)$.
- **Critical regime:** If $||T_\kappa|| = 1$: content of this talk.

Model II: Inhomogeneous random graphs at Criticality

Known results

- Amazing array of results in BJR[05], especially above and below criticality.
- Number of results on susceptibility functions by Janson and Riordan when $||T_\kappa|| = 1 - \varepsilon$ (barely subcritical regime).
- At this level of generality no results even for component sizes in the critical regime. Critical scaling window?
- One particular example: **rank one/Norros-Reittu/Chung-Lu/Britton-Deijfen**. Here type space is $\mathbb{R}_+$.

$$p_{ij} := 1 - \exp(-x_i x_j / n)$$

- Under moment conditions [SB, Hofstad, van Leeuwarden] and [Turova] showed that again maximal components scale like $|\mathcal{C}_1| \sim \xi_i n^{2/3}$.
- Will show up later. Original talk was supposed to be all about this model. Forms a key component in proving the results.

Model II: Inhomogeneous random graphs at Criticality

Assumptions

1. **Convergence of the kernels:** There exists a kernel $\kappa(\cdot,\cdot) : [K] \times [K] \to \mathbb{R}^+$ and a matrix $A = ((a_{xy}))_{x,y\in[K]}$ such that

$$\min_{x,y\in[K]} \kappa(x,y) > 0 \text{ and } \lim_n n^{1/3} \left(\kappa_n(x,y) - \kappa(x,y)\right) = a_{xy} \text{ for } x,y \in [K].$$

2. **Convergence of the empirical measures:** There exists a probability measure μ on $[K]$ and a vector $\mathbf{b} = (b_1, \ldots, b_K)^t$ such that

$$\min_{x\in[K]} \mu(x) > 0 \text{ and } \lim_n n^{1/3} \left(\mu_n(x) - \mu(x)\right) = b_x \text{ for } x \in [K].$$

3. **Criticality of the model**: The operator norm of T_κ in $L^2([K],\mu)$ equals one. **Equivalent to:** Matrix M having max-eigen value $\rho(M) = 1$ where $M = \mu(j)\kappa(i,j)$.

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Parameters required for main result

1. $\mathbf{u},\mathbf{v}$: right and left eigen-vectors of M; $D=\mathrm{Diag}(\boldsymbol{\mu})$; $B=\mathrm{Diag}(\mathbf{b})$.
2. $\alpha=\frac{1}{(\mathbf{v}^t\mathbf{1})\cdot(\boldsymbol{\mu}^t\mathbf{u})}$, $\beta=\frac{\sum_{i\in[K]}v_iu_i^2}{(\mathbf{v}^t\mathbf{1})\cdot(\boldsymbol{\mu}^t\mathbf{u})^2}$ and $\zeta=\alpha\cdot\left[\mathbf{v}^t(AD+\kappa B)\mathbf{u}\right]$.

Model II: Inhomogeneous random graphs at Criticality

Theorem: Continuum scaling limits of metric structure of critical IRG

Consider the critical IRG with assumptions as in previous slide. View it as a measured metric space with mass 1 to each vertex and usual graph metric. Then

$$\left(\operatorname{scl}\left(\frac{\beta^{2/3}}{\alpha n^{1/3}}, \frac{\beta^{1/3}}{n^{2/3}}\right) \mathcal{C}_i(\mathcal{G}_{\mathrm{IRG}}^{(n)}) : i \geq 1\right) \xrightarrow{\mathrm{w}} \mathbf{Crit}_\infty\left(\frac{\zeta}{\beta^{2/3}}\right)$$

Corollary: Sizes of components

We get scaling limits for component sizes as a by-product namely component sizes satisfy

$$\left(\frac{\beta^{1/3}}{n^{2/3}} |\mathcal{C}_i(\mathcal{G}_{\mathrm{IRG}}^{(n)})| : i \geq 1\right) \xrightarrow{\mathrm{w}} \boldsymbol{\xi}\left(\frac{\zeta}{\beta^{2/3}}\right)$$

Model III: Bounded size rules. Effect of limited choice

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- Each step, two candidate edges (e_1, e_2) chosen uniformly among all $\binom{n}{2} \times \binom{n}{2}$ possible pairs of ordered edges. If e_1 connect two singletons (component of size 1), then add e_1 to the graph; otherwise, add e_2.

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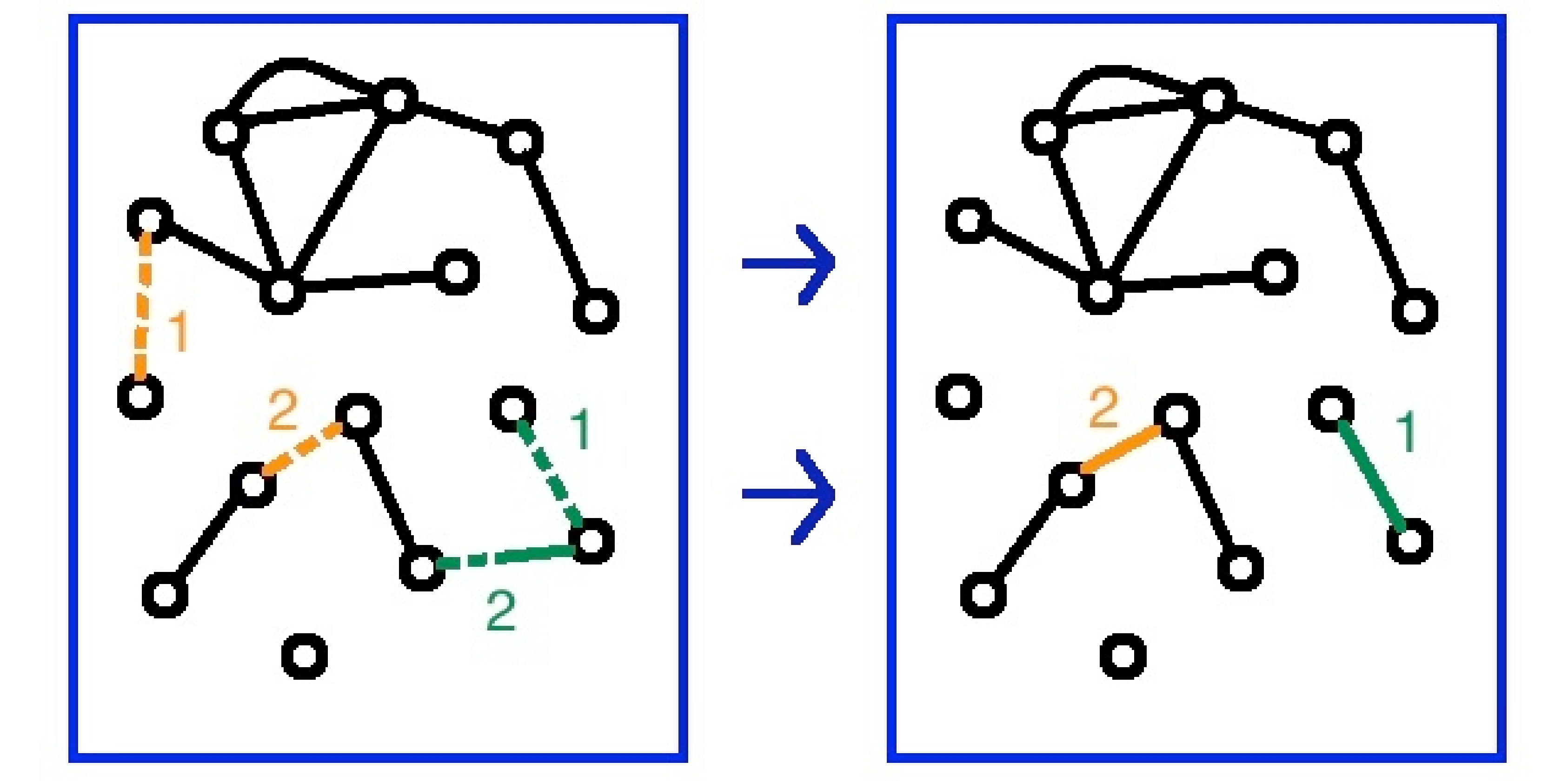

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[Bohman, Frieze 2001] The delay of phase transition

Consider the continuous time version $\mathcal{G}_n^{BF}(t)$, then there exists $\epsilon > 0$ such that at time $t_c^{ER} + \epsilon$,

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[Spencer, Wormald 2004] The critical time

- $t_c^{BF} \approx 1.1763 > t_c^{ER} = 1$.
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Near Criticality

- Janson and Spencer (2011) analyzed how $s_2(\cdot), s_3(\cdot) \to \infty$ as $t \uparrow t_c$.
- Kang, Perkins and Spencer (2011) analyze the near subcritical $(t_c - \epsilon)$ regime.

General bounded size rules

- Fix $K \geqslant 1$
- Let $\Omega_K = \{1, 2, \ldots, K, \omega\}$
- General bounded size rule: subset $F \subset \Omega_K^4$.
- Pick 4 vertices uniformly at random. If $(c(v_1), c(v_2), c(v_3), c(v_4)) \in F$ then choose edge e_1 else e_2

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BF model

$K = 1$, $F = \{(1, 1, \alpha, \beta)\}$.

Model III: Bounded size rules in the critical regime

Theorem (Bhamidi, Budhiraja, Wang, 2012)

Let $(\mathcal{C}_n^{(1)}(t), \mathcal{C}_n^{(2)}(t), ...)$ *be the component sizes of* $\mathcal{G}_n^{BSR}(t)$ *in decreasing order. Define the rescaled size vector* $\mathbf{C}_n(\lambda)$, $-\infty < \lambda < +\infty$ *as the vector*

$$\mathbf{C}_n(\lambda) := (\bar{\mathcal{C}}_i(\lambda) : i \geq 1) = \left(\frac{\beta^{1/3}}{n^{2/3}} \mathcal{C}_n^{(i)}(t_c + \frac{\beta^{2/3}\alpha\lambda}{n^{1/3}}) : i \geq 1 \right)$$

where α, β *are constants determined by the BSR process. Then*

$$\{\mathbf{C}_n(\lambda) : -\infty < \lambda < \infty\} \xrightarrow{d} \{\boldsymbol{\xi}(\lambda) : -\infty < \lambda < \infty\}$$

.

Model III: Bounded size rules [Critical regime]

BF constants

$$x'(t) = -x^2(t) - (1-x^2(t))x(t) \quad \text{for } t \in [0,\infty) \quad x(0)=1$$
$$s_2'(t) = x^2(t) + (1-x^2(t))s_2^2(t) \quad \text{for } t \in [0,t_c), \quad s_2(0)=1$$
$$s_3'(t) = 3x^2(t) + 3(1-x^2(t))s_2(t)s_3(t) \quad \text{for } t \in [0,t_c), \quad s_3(0)=1.$$

$$s_2(t) \sim \frac{\alpha}{t_c - t}, \qquad s_3(t) \sim \beta (s_2(t))^3 \sim \beta \frac{\alpha^3}{(t_c-t)^3} \qquad \text{as } t \uparrow t_c.$$

Final equation:

$$v'(t) := -2x^2(t)^2 y(t)v(t) + \frac{x^2(t)y^2(t)}{2} + 1 - x^2(t), \qquad v(0)=0.$$

Easy to check

$$\lim_{t\uparrow t_c} v(t) := \varrho \approx .811.$$

Model III: Bounded size rules [Critical regime]

Theorem: Metric space asymptotics

For the Bohman Frieze process we have

$$\left(\mathrm{scl}\left(\frac{\beta^{2/3}}{\varrho n^{1/3}}, \frac{\beta^{1/3}}{n^{2/3}}\right)\mathcal{C}_i^{(n)}\left(t_c + \frac{\beta^{2/3}\alpha}{n^{1/3}}\lambda\right) : i \geqslant 1\right) \xrightarrow{\mathrm{w}} \mathbf{Crit}_\infty(\lambda),$$

Theorem

Same is true for **any** bounded size rule with appropriate rule dependent constants α_F, β_F and ρ_F

Key principle 1: Dynamics and behavior after barely subcritical regime

- Other than as an artifact of the proof technique (Martingale FCLT) why do maximal components in the critical regime look like Erdos-Renyi?
- One reason: Dynamics after the barely subcritical regime.
- What do I mean?

Key principle 1: Dynamics and behavior after barely subcritical regime

Erdos-Renyi: dynamics

- Assign independent Poisson processes rate $1/n$ on each of the $\binom{n}{2}$ possible edges $\{i, j\}$. When process corresponding to an edge fires, place that edge.

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- For fixed λ

$$\mathbf{C}_n(\lambda) := n^{-2/3}(|\mathcal{C}_i(1+\lambda/n^{1/3})| : i \geqslant 1,) \xrightarrow{d} \boldsymbol{\xi}(\lambda) := \text{ Excursion lengths }.$$

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Important question

What happens to $\{\mathbf{C}_n^*(\lambda) : -\infty < \lambda < \infty\}$ as a a process in λ?

Rate of mergers

- Recall we are looking at the new time scale $t = 1 + \lambda/n^{1/3}$

Rate of mergers

- Recall we are looking at the new time scale $t = 1 + \lambda/n^{1/3}$
- In this time scale, in time interval $[\lambda, \lambda + d\lambda)$, components a and b merge at rate

$$\frac{1}{n^{1/3}} \times \frac{\mathcal{C}_a(1 + \lambda/n^{1/3})\mathcal{C}_b(1 + \lambda/n^{1/3})}{n} = \bar{\mathcal{C}_a}(\lambda)\bar{\mathcal{C}_a}(\lambda)$$

- Aldous showed there exists an $l^2_{\downarrow}$ valued Markov process $\{X(\lambda) : -\infty < \lambda < \infty\}$ called the **Standard multiplicative coalescent** such that

$$\{\mathbf{C}_n(\lambda) : -\infty < \lambda < \infty\} \stackrel{d}{\Longrightarrow} \{\boldsymbol{\xi}(\lambda) : -\infty < \lambda < \infty\}$$

Standard Multiplicative coalescent

Dynamics

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Standard Multiplicative coalescent

Dynamics

- For each fixed λ, $\boldsymbol{\xi}(\lambda)$ has distribution given by excursion lengths
- suppose $\mathbf{X}(\lambda) = (x_1, x_2, x_3, ...)$, each x_l is viewed as the size of a cluster.
- each pair of clusters of sizes (x_i, x_j) merges at rate $x_i x_j$ into a cluster of size $x_i + x_j$.
- if x_i, x_j is merging, then $(x_1, x_2, x_3, ...) \rightsquigarrow (x'_1, x'_2, x'_3, ...)$ where the latter is the re-ordering of $\{x_i + x_j, x_l : l \neq i, j\}$.

Standard Multiplicative coalescent

Dynamics

- For each fixed λ, $\boldsymbol{\xi}(\lambda)$ has distribution given by excursion lengths
- suppose $\mathbf{X}(\lambda) = (x_1, x_2, x_3, ...)$, each x_l is viewed as the size of a cluster.
- each pair of clusters of sizes (x_i, x_j) merges at rate $x_i x_j$ into a cluster of size $x_i + x_j$.
- if x_i, x_j is merging, then $(x_1, x_2, x_3, ...) \rightsquigarrow (x'_1, x'_2, x'_3, ...)$ where the latter is the re-ordering of $\{x_i + x_j, x_l : l \neq i, j\}$.
- *If your initial starting configuration at time "$\lambda = -\infty$" has good properties and follows the merging dynamics of the multiplicative coalescent then*

$$\{\mathbf{C}_n(\lambda) : -\infty < \lambda < \infty\} \stackrel{d}{\Longrightarrow} \{\boldsymbol{\xi}(\lambda) : -\infty < \lambda < \infty\}$$

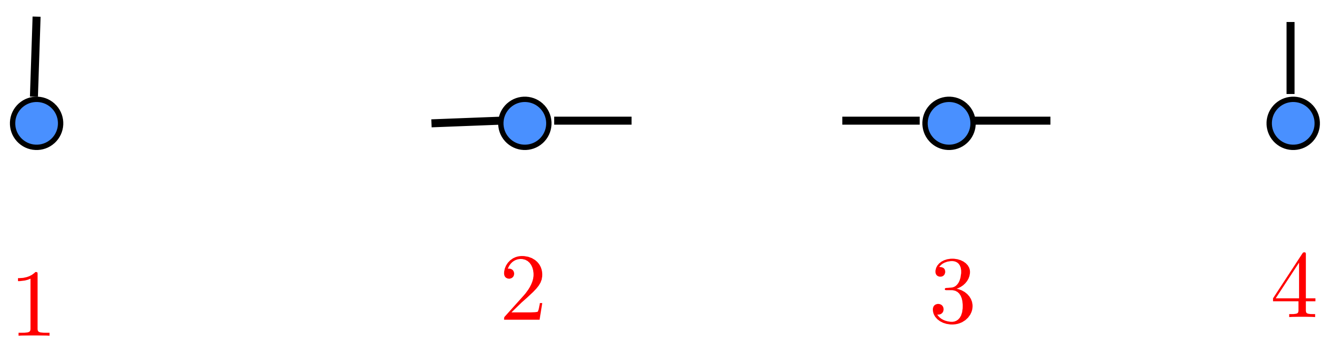
Key principle 1: Dynamics and behavior after barely subcritical regime

Recall CM_n: Related to Janson-Luczak dynamic construction

- Start with n vertices with d_i half-edges for $i \in [n]$. At time $t = 0$ start with n-isolated vertices.
- Each half-edge has exponential rate one clock. When clock rings, chooses one of the **alive** (active) half-edges, forms a full edge and both half-edges die (leave system).
- If you ran this process for $t = \infty$ then get full $\mathrm{CM}_n(\infty)$.
- $\{\mathrm{CM}_n(t) : t \geqslant 0\}$ *dynamic graph valued process.*
- Standard results imply critical time

$$t_c = \frac{1}{2} \log \frac{\nu}{\nu - 1}.$$

$$d_1 = 1, \quad d_2 = 2, \quad d_3 = 2, \quad d_4 = 1$$



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Phase transition

- $t < t_c$: $\mathcal{C}_1(t) = O(\log n)$.
- $t > t_c$: $\mathcal{C}_1(t) = f(t)n$. $\qquad f(t) \uparrow \rho(\nu)$.

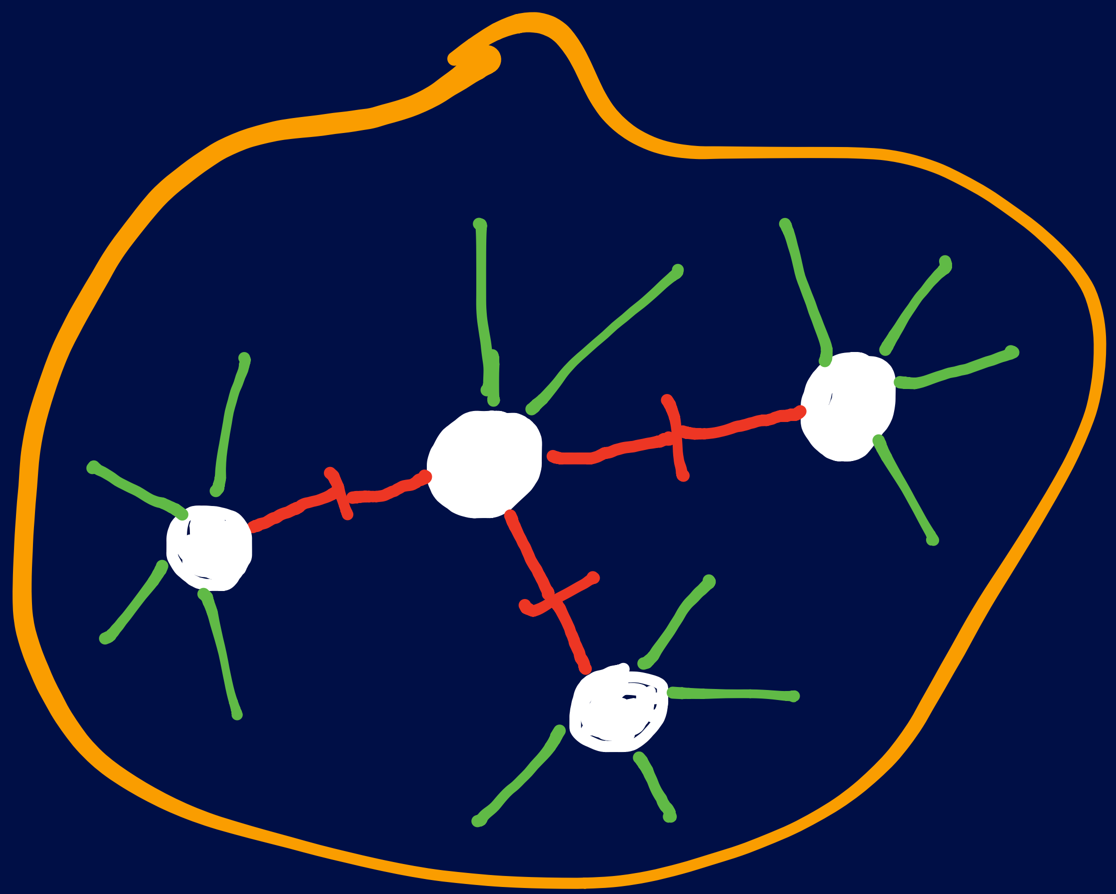
Key principle 1: Dynamics and behavior after barely subcritical regime

By results of Fountanakis and Janson

$$\mathrm{Perc}_n(p(\lambda)) \approx \mathrm{CM}_n\left(t_c + \frac{\nu}{2(\nu-1)}\frac{\lambda}{n^{1/3}}\right)$$

So what?

- Have transferred a nice static problem (percolation) into something about a dynamic graph valued process.



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- Have transferred a nice static problem (percolation) into something about a dynamic graph valued process.
- Components do not merge at rate proportional to **size** of components
- Abusing notation, let $f_i(t)$ be the number of **alive edges** in $\mathcal{C}_i(t)$ at time t. The $\mathcal{C}_i(t)$ and $\mathcal{C}_j(t)$ merge at rate
$$f_i(t)\frac{f_j(t)}{n\bar{s}_1(t)} + f_j(t)\frac{f_i(t)}{n\bar{s}_1(t)} = \frac{2f_i(t)f_j(t)}{n\bar{s}_1(t)}.$$
- New component has size $f_i(t) + f_j(t) - 2$.
- However hard to control this graph-valued process all the way from $t = 0$.

Key principle 1: Most technical definition of talk: *Blob*

Barely subcritical regime

- Recall that we are interested in times of the form $t_c + \lambda/n^{1/3}$.
- Fix $\delta \in (1/5, 1/6)$. Define
$$t_n := t_c - \frac{1}{n^\delta}.$$
- Call a component at time t_n a **Blob**.

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Figure: Blob: From http://blue-cat00.deviantart.com/art/Mr-Ice-Cream-Blob-366286224

Switching to general methodology

Three ingredients of a maximal component at criticality

Can think of the metric structure of the component in the critical scaling window as composed of three parts.

I: Blob-level superstructure

- **Random graphs:** Viewing each blob as a single vertex this encapsulates connections between blobs formed in the interval

$$\left[t_c - \frac{1}{n^\delta}, t_c + \frac{\lambda}{n^{1/3}}\right]$$

  Can hope that as we move from barely subcritical to critical scaling window, blobs merge like the multiplicative coalescent

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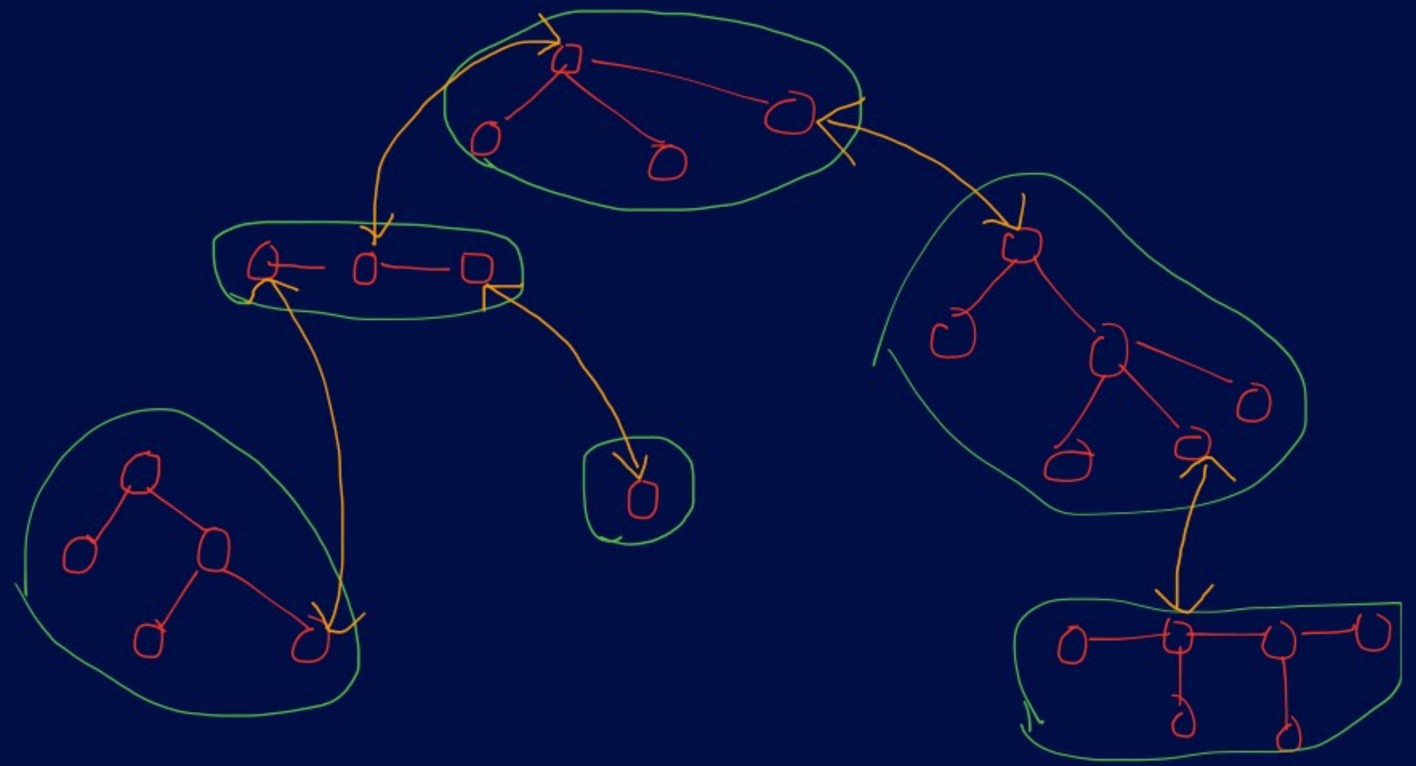
- **Abstract case:** Collection of blobs $\mathcal{V}_{\text{blob}} := [m]$ with weights $\mathbf{x}$ and parameter q. $\mathcal{G}(\mathbf{x}, q)$ random graph formed using connection probability

$$p_{ij} = 1 - \exp(-q x_i x_j)$$



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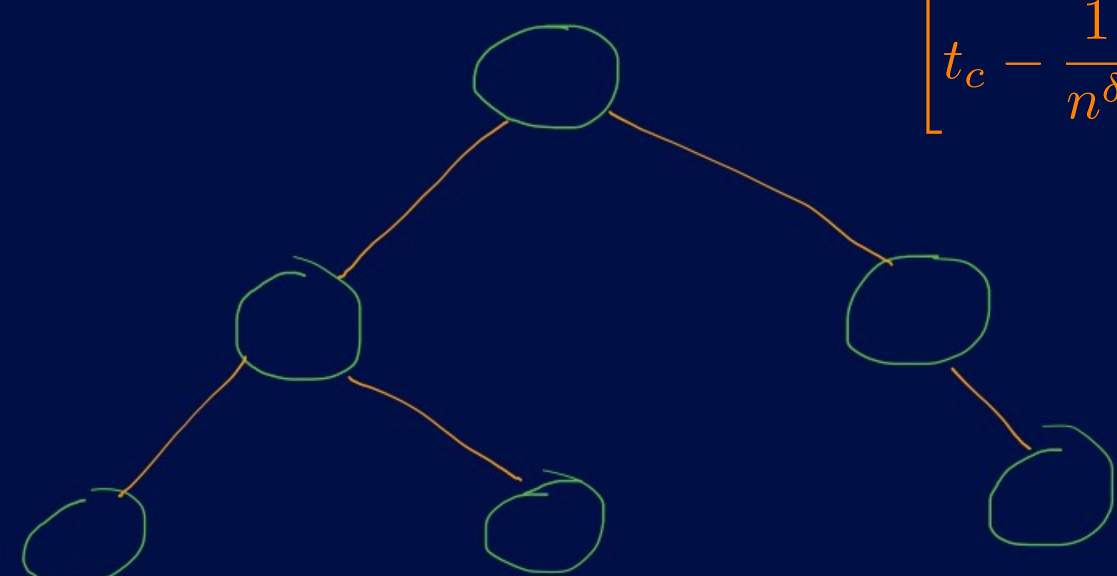

Blob-level superstructure

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II: Blobs

- **Random graphs:** Components at time t_n. Note that when we connect two vertices in blobs we do **not** choose these vertices uniformly in CM_n but with probability proportional to **number of live edges at time t_n.**

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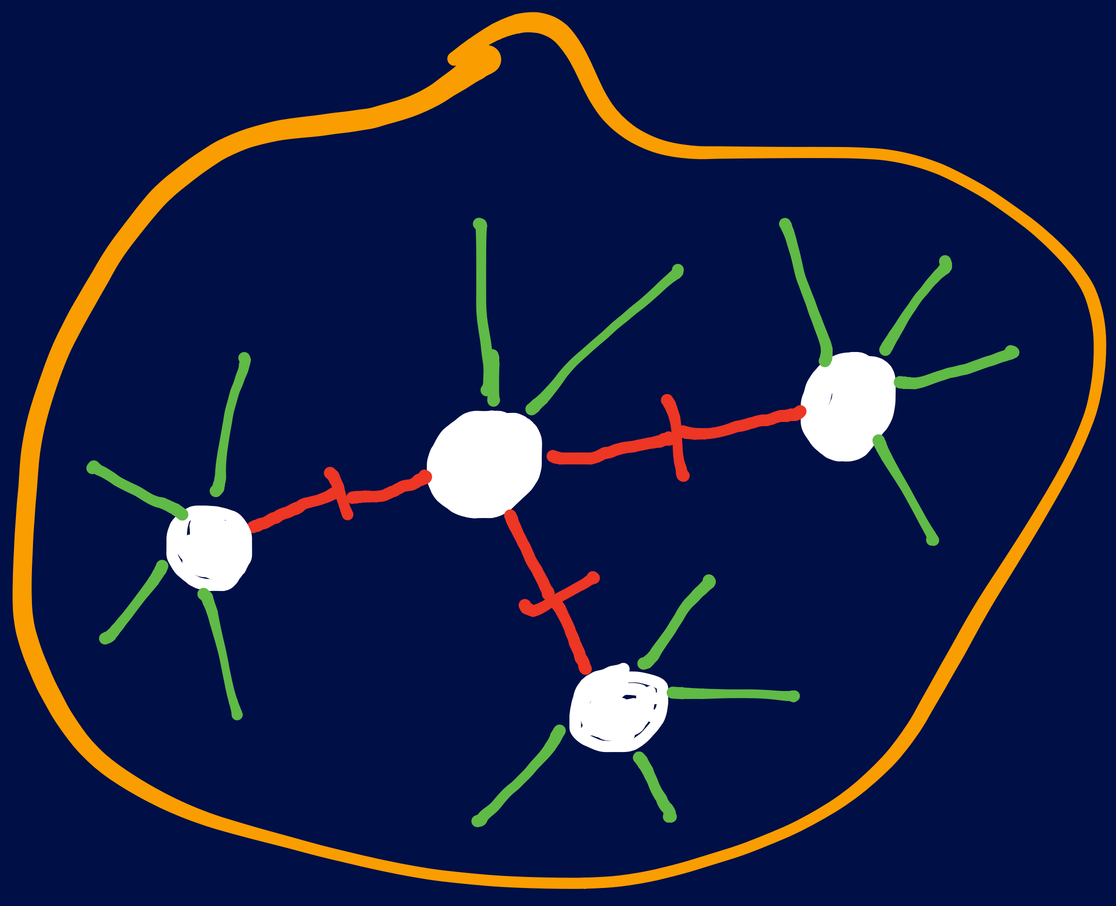

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- **Abstract case:** A family of compact connected measured metric spaces $\mathbf{M} := \{(M_i, d_i, \mu_i) : i \in \mathcal{V}\}$, one for each blob in $\mathcal{G}(\mathbf{x}, q)$. Further assume that for all $i \in \mathcal{V}$, μ_i is a probability measure namely $\mu_i(M_i) = 1$.

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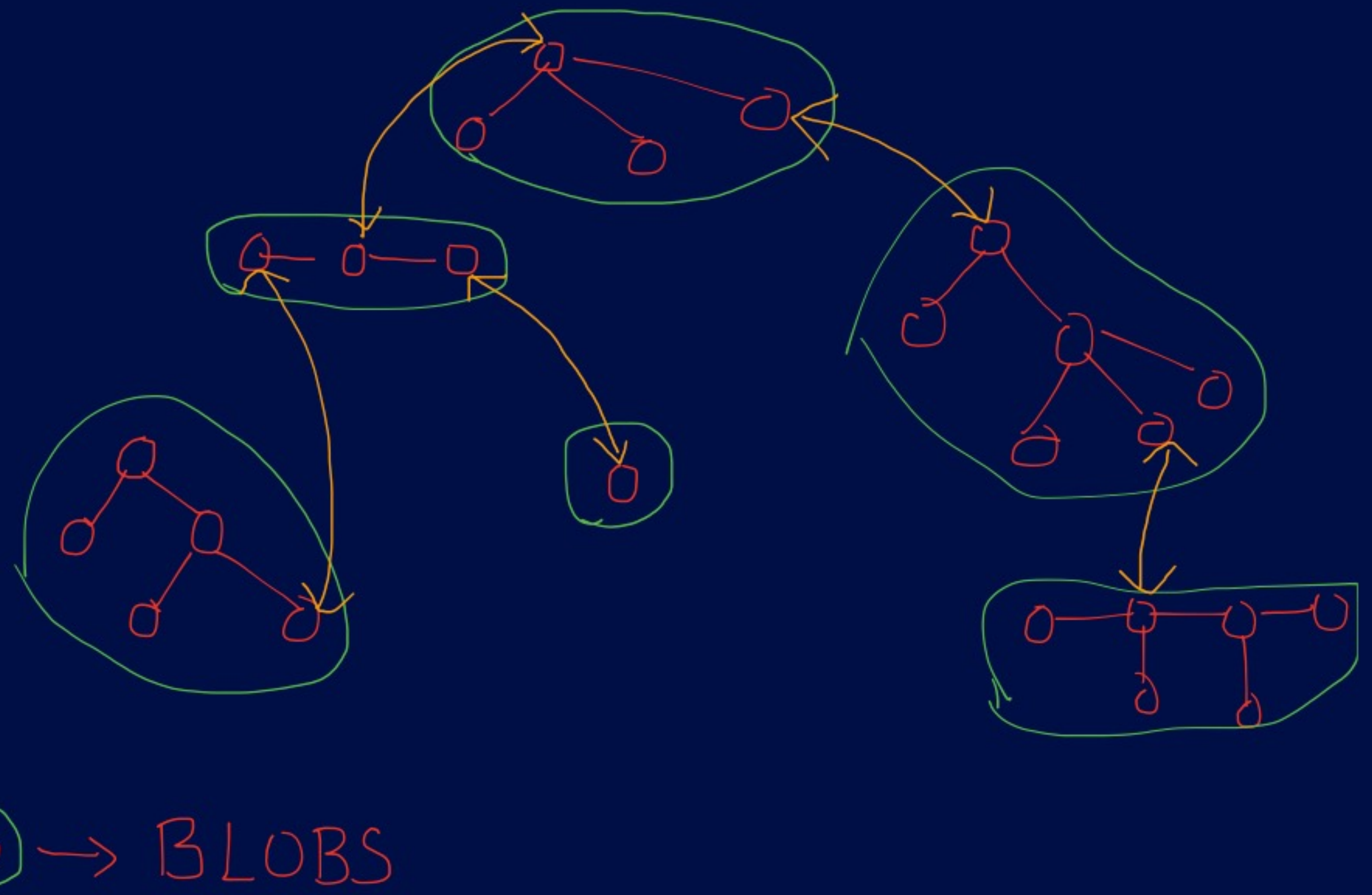
III: Blob-blob junction points

- **Random graphs:** e.g. configuration model, choose vertices with probability proportional to number of live edges at time $t_n = t_c - n^{-\delta}$.
- **Abstract case:** This is a collection of points $\mathbf{X} := (X_{i,j} : i \in \mathcal{V}, j \in \mathcal{V}_{\mathrm{blob}})$ such that $X_{i,j} \sim \mu_i \in M_i$ iid for all i, j.

Metric space with the three ingredients

- Given above 3 ingredients form metric space $\bar{M} := \bigsqcup_{i \in [n]} M_i$ in the obvious manner.
- For $x, y \in \bar{M}$

$$\bar{d}(x,y) = \inf_{k; i_0, \ldots, i_k} \left\{ k + d_{i_0}(x, X_{i_0, i_1}) + \sum_{\ell=1}^{k-1} d_{i_\ell}(X_{i_\ell, i_{\ell-1}}, X_{i_\ell, i_{\ell+1}}) + d_{i_k}(X_{i_k, i_{k-1}}, y) \right\},$$



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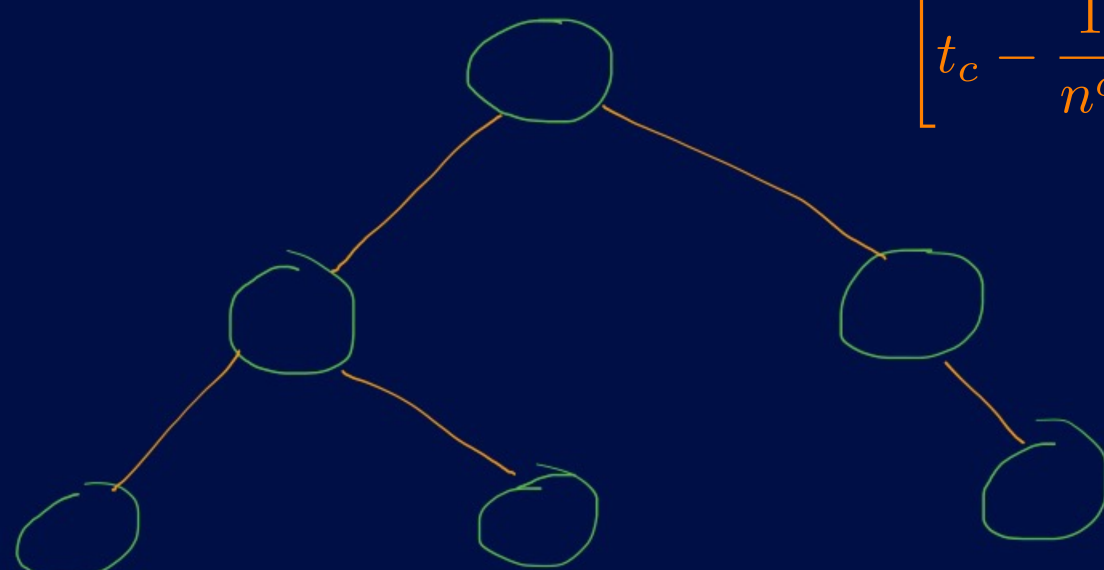

Blob-level superstructure

Key principle 2: Blob-level picture and universality

Aim: study $\mathcal{G}(\mathbf{x}, q)$.

Negligibility Assumptions

- **Aldous's assumptions** for multiplicative coalescent. $\sigma_k = \sum_{i\in[m]} x_i^k$

$$\frac{\sigma_3}{(\sigma_2)^3} \to 1, \quad q - \frac{1}{\sigma_2} \to \lambda, \quad \frac{x_{\max}}{\sigma_2} \to 0,$$

- **Additional assumptions:** There exist $\eta_0 \in (0, 1/2)$ and $r_0 \in (0, \infty)$ as $n \to \infty$, we have

$$\frac{x_{\max}}{\sigma_2^{3/2+\eta_0}} \to 0, \quad \frac{\sigma_2^{r_0}}{x_{\min}} \to 0.$$

Theorem: Blob-level scaling

Treat $(\mathcal{C}_i : i \geqslant 1)$ as measured metric spaces using graph distance and weighted measure where each blob $i \in [m]$ has weight x_i. Under above Assumptions, for maximal components in $\mathcal{G}(\mathbf{x}, q)$

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$$(\mathrm{scl}(\sigma_2, 1)\mathcal{C}_i : i \geqslant 1) \xrightarrow{\mathrm{w}} \mathbf{Crit}_\infty(\lambda)$$

What does this imply for Random graphs?

Intuitive calculation

- For wide variety of models (e.g. Janson, Janson+Riordan, Janson+Luczak, Janson + Spencer) one can show that susceptibility

$$s_2(t) = \frac{1}{n}\sum_i |\mathcal{C}_i(t)|^2 \sim \frac{\alpha}{t_c - t}$$

- Note $t_n = t_c - n^{-\delta}$. Pick a vertex V_n at random, expect $\mathbb{E}(\mathcal{C}_{V_n}(t_n)) \sim \alpha n^{\delta}$.
- Our techniques imply that at $t_c + \lambda/n^{1/3}$, $\#$ of blobs in $\mathcal{C}_1(\lambda)$ is $n^{2/3-\delta}$.
- So expect **Blob-level-superstructure** should scale like $\sqrt{n^{2/3-\delta}} = n^{1/3-\delta/2}$. Typical blob should look like a critical random tree of size n^{δ} so distance within blob $n^{\delta/2}$.
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Theorem

In critical random graphs, for the blob-level superstructure one has

$$\frac{1}{n^{1/3-\delta}}\tilde{\mathcal{C}}_1(\lambda) \xrightarrow{\text{w}} \mathrm{Crit}_1(\lambda).$$

One key idea behind blob-level scaling result: My original talk

$\mathbf{p}$-trees

Fix pmf $\mathbf{p} = \{p_1, p_2, \dots p_m\}$. A rooted random planar tree $\mathcal{T}^{\mathbf{p}}$ with vertex set $[m]$ is called a $\mathbf{p}$-tree if it has probability distribution

$$\mathbb{P}_{\text{ord}}(\mathcal{T}^{\mathbf{p}} = \mathbf{t}) = \prod_{v \in [m]} \frac{p_v^{d_v(\mathbf{t})}}{(d_v(\mathbf{t}))!}, \qquad \mathbf{t} \in \mathbb{T}_m^{\text{ord}}.$$

Tilted $\mathbf{p}$-trees

- Any rooted planar tree $\mathbf{t}$ defines a natural depth first exploration. Start with root and use order associated $\mathbf{t}$.
- $\mathscr{P}(\mathbf{t})$: collection of permitted edges (pairs of vertices both belong to stack of active vertices during exploration process).
- Define function $L : \mathbb{T}_m^{\text{ord}} \to \mathbb{R}$

$$L(\mathbf{t}) := \prod_{(i,j) \in E(\mathbf{t})} \left[\frac{\exp(ap_i p_j) - 1}{ap_i p_j} \right] \exp\left(\sum_{(i,j) \in \mathscr{P}(\mathbf{t})} ap_i p_j \right), \qquad \mathbf{t} \in \mathbb{T}_m^{\text{ord}}.$$

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$$\frac{d\tilde{\mathbb{P}}_{\mathrm{ord}}}{d\mathbb{P}_{\mathrm{ord}}}(\mathbf{t}) = \frac{L(\mathbf{t})}{\mathbb{E}_{\mathrm{ord}}[L(\mathcal{T}^{\mathbf{p}})]}, \text{ for } \mathbf{t} \in \mathbb{T}_m,$$

Connected components of $\mathcal{G}(\mathbf{x}, q)$

- $q_{u,v} = 1 - \exp(-ap_ip_j)$.
- Consider distribution on space of connected simple graphs with vertex set m

$$\mathbb{P}_{\text{con}}(G; \mathbf{p}, a) := \frac{1}{Z(\mathbf{p}, a)} \prod_{(u,v)\in E(G)} q_{uv} \prod_{(u,v)\notin E(G)} (1 - q_{uv}), \text{ for } G \in \mathbb{G}_{\mathcal{V}}^{\text{con}},$$

Major technical tool in establishing universality:

Theorem (SB, Sanchayan Sen, Xuan Wang)

A random graph $\mathcal{G}_m \sim \mathbb{P}_{\text{con}}$ with distribution as above can be constructed as follows:

1. Generate tilted $\mathbf{p}$-tree $\tilde{\mathcal{T}}$.
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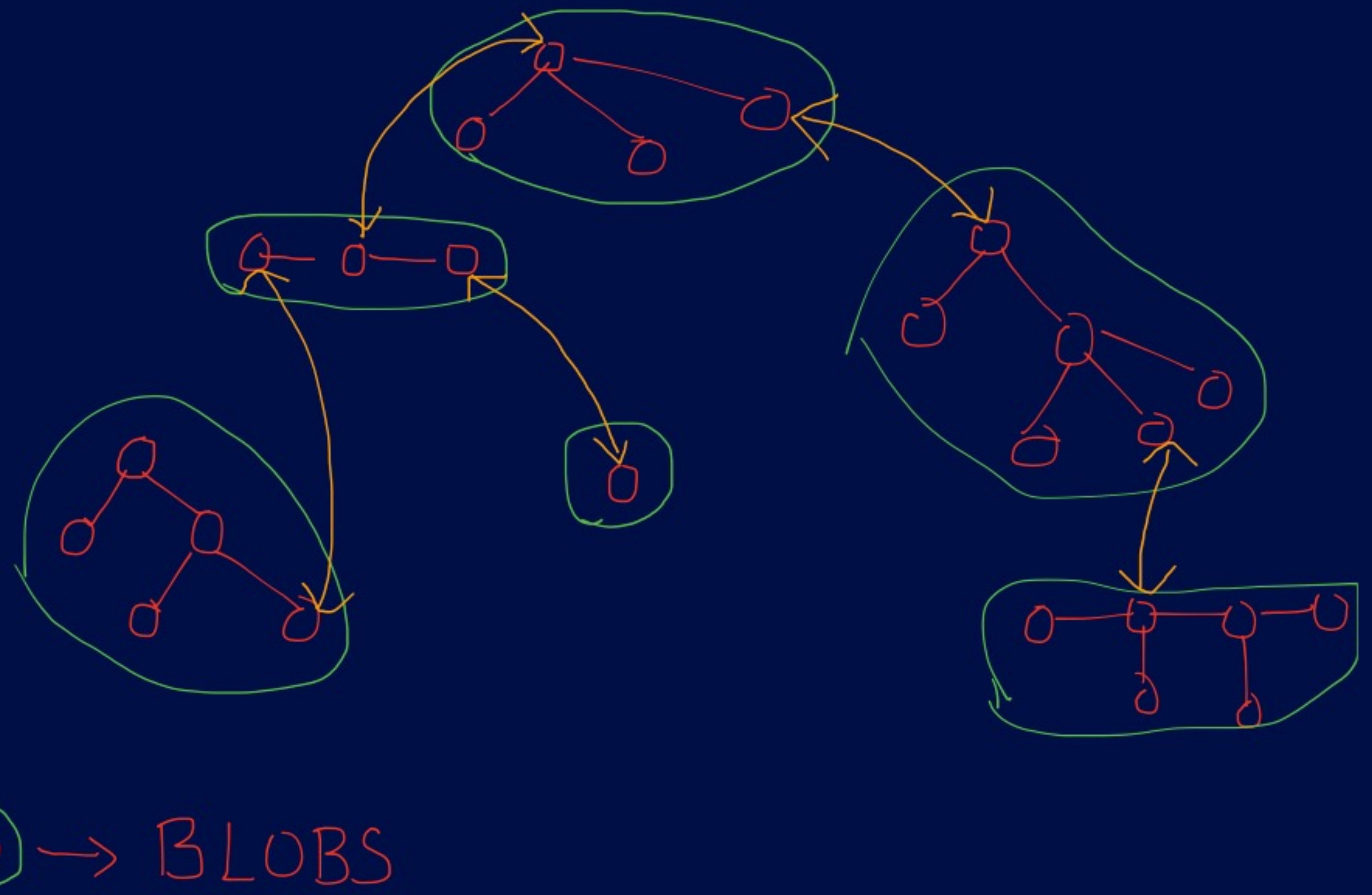
Used to show continuum scaling limits of rank-one/Norros-Reittu/Britton-Deijfen/Chung-Lu model.

Key principle 3: Incorporating blob-level information

Setting and assumptions

- Recall $X_{i,1}$: junction point in M_i picked using measure μ_i. Let $u_{i,1} = \mathbb{E}(d_i(X_{i,1}, X_{i,2}))$.
- $d_{\max} = \max_i(\mathrm{diam}(M_i))$.
- **Assumptions:** In addition to previous assumptions, assume

$$\frac{x_{\max}}{\sigma_2^{3/2+\eta_0}} \to 0, \quad \frac{\sigma_2^{r_0}}{x_{\min}} \to 0, \quad \frac{d_{\max}\sigma_2^{3/2-\eta_0}}{\sum_{i=1}^{\infty} x_i^2 u_{i,1} + \sigma_2} \to 0, \quad \frac{\sigma_2 x_{\max} d_{\max}}{\sum_{i\in[n]} x_i^2 u_{i,1}} \to 0.$$



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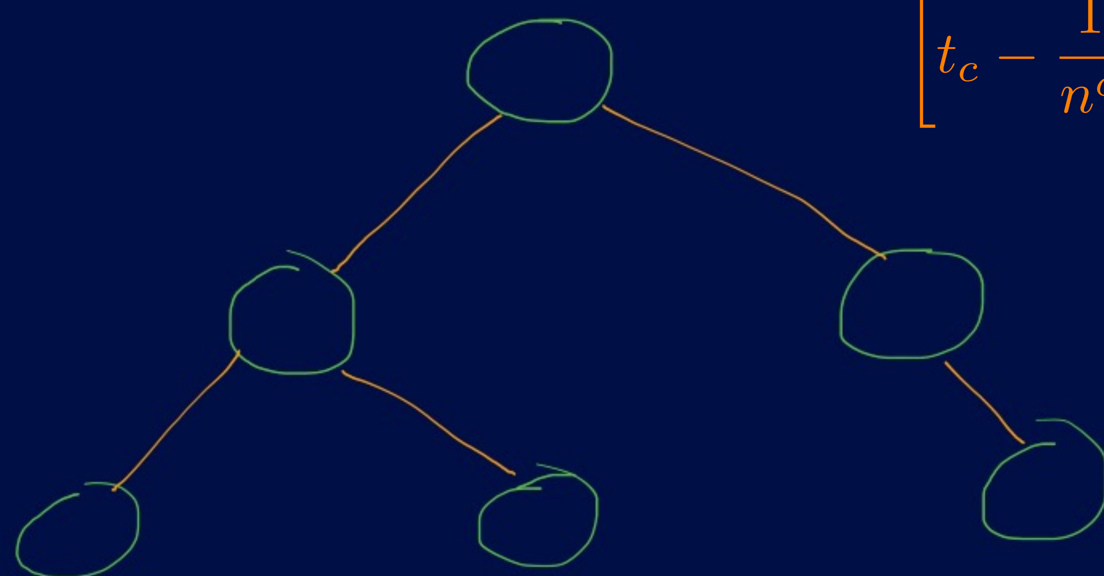

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Theorem: Complete metric space scaling

Under above assumptions

$$\left(\mathrm{scl}\left(\frac{(\sigma_2)^2}{\sigma_2 + \sum_{i\in[n]} x_i^2 u_{i,1}}, 1\right)\bar{\mathcal{C}}_i : i \geqslant 1\right) \xrightarrow{\mathrm{w}} \mathbf{Crit}_\infty(\lambda).$$

Case Study: Configuration model

Glimpse of how to carry out this program in a particular example. Assume $\lambda = 0$ for notational convenience.

What is needed?

- Show that mergers in $[t_c - n^{-\delta}, t_c]$ can be approximated via Multiplicative coalescent

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Merging dynamics

- Recall components merge at rate

$$\frac{2 f_i(t) f_j(t)}{n \bar{s}_1(t)} \approx \frac{2\nu f_i(t) f_j(t)}{n(\mu(\nu-1))}, \qquad t \in \left[t_c - \frac{1}{n^\delta}, t_c\right].$$

- Modified process $\mathcal{G}_n^{\mathrm{modi}}$: Start at time t_n with $\mathrm{CM}_n(t_n)$. For all

$$\mathbf{e} = (u, v) \in \mathrm{FR}(t_n) \times \mathrm{FR}(t_n),$$

  $\mathcal{P}_{\mathbf{e}}$ rate $\nu/(n\mu(\nu-1))$ Poisson process. When one of these ring, complete full edge but continue to consider (u, v) as "alive".

Case Study: Configuration model

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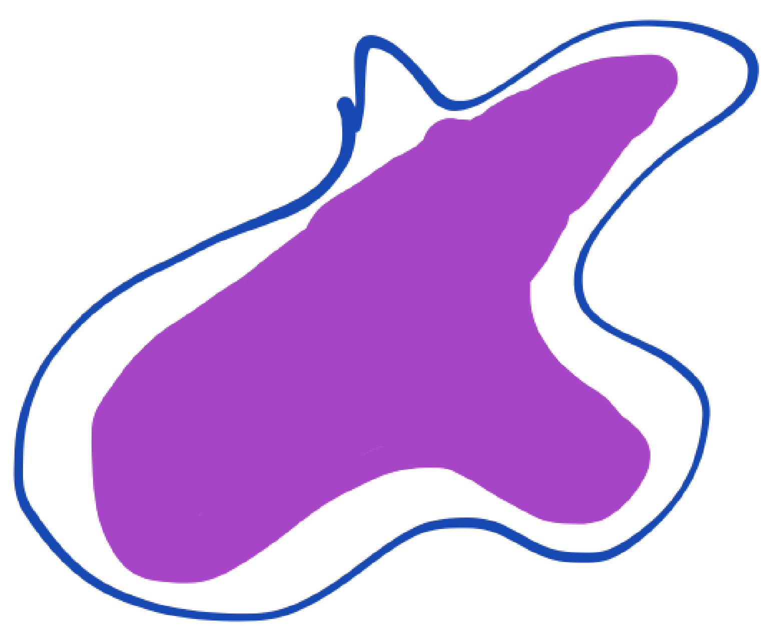
Tricky Technical argument 2 in picture form



Case Study: Configuration model

Technical argument 2

- Show that $n^{-2/3}|\mathcal{C}_i| \approx n^{-2/3}|\mathscr{C}_i^{\mathrm{modi}}|$.
- Properties of limit random metric space implies "white-space" vanishes in the limit.

Punchline

Assuming $\mathrm{CM}_n(t_n)$ (**barely subcritical regime**) has good properties, using modified process allows us to prove asserted limit for maximal components.

Barely subcritical $\mathrm{CM}_n(\cdot)$

Theorem: Bounds on maximal component and diameter

Given $\delta < 1/4$ and $\alpha > 0$, there exists $C = C(\delta, \alpha) > 0$ such that

$$\mathbb{P}\left(\mathcal{C}_1(t_c - t) \leqslant \frac{C(\log n)^2}{(t_c - t)^2},\right.$$

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Why important

Recall universality result:

$$\left(\mathrm{scl}\left(\frac{(\sigma_2)^2}{\sigma_2 + \sum_{i\in[n]} x_i^2 u_{i,1}}, 1\right) \bar{\mathcal{C}}_i : i \geqslant 1\right) \xrightarrow{\mathrm{w}} \mathbf{Crit}_\infty(\lambda).$$

Barely subcritical $\{\mathrm{CM}_n(t) : 0 \leqslant t \leqslant t_n\}$

Definitions

- **Susceptibility functions:** $s_l(t) := \frac{1}{n}\sum_i f_i^l(t)$, $\quad g(t) := \frac{1}{n}\sum_i f_i(t)|\mathcal{C}_i(t)|$.
- **Distance based susceptibility:** $\mathcal{D}_1(\mathcal{C}(t)) = \sum_{e,f\in\mathcal{C}(t), e,f \text{ free}} d(e,f)$.

$$\bar{\mathcal{D}}(t) := \frac{1}{n}\sum_i \mathcal{D}_1(\mathcal{C}_i(t)).$$

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Fix $\delta \in (1/6, 1/5)$ and $t_n = t_c - n^{-\delta}$. Then

$$\left|\frac{n^{1/3}}{s_2(t_n)} - \frac{\nu^2 n^{1/3-\delta}}{\mu(\nu-1)^2}\right| \xrightarrow{\mathrm{P}} 0,$$

$$\frac{s_3(t_n)}{s_2^3(t_n)} \xrightarrow{\mathrm{P}} \frac{\beta}{\mu^3(\nu-1)^3}.$$

and further

$$\frac{\bar{\mathcal{D}}(t_n)}{n^{2\delta}}$$

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and further

$$\frac{\bar{\mathcal{D}}(t_n)}{n^{2\delta}} \xrightarrow{\mathrm{P}} \frac{\mu(\nu-1)^2}{\nu^3}, \qquad \frac{g(t_n)}{n^\delta} \xrightarrow{\mathrm{P}} \frac{(\nu-1)\mu}{\nu^2}.$$

Barely subcritical $\{\mathrm{CM}_n(t) : 0 \leqslant t \leqslant t_n\}$: Proof idea

- **Idea 1:** Use couplings to barely subcritical branching processes. Used in our analysis of IRG.

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- **Idea 2:** *Make dynamics work for us.*
- **Example:** At rate $f_i(t)f_j(t)/n\bar{s}_1(t)$ components $\mathcal{C}_i(t), \mathcal{C}_j(t)$ merge. Assume this happens due to merging $e_0 \in \mathcal{C}_i$ and $f_0 \in \mathcal{C}_j$. Change

$$n(\Delta\bar{\mathcal{D}}(t)) = 2\sum_{\substack{e\in\mathcal{C}_i,\\ e\neq e_0}}\sum_{\substack{f\in\mathcal{C}_j,\\ f\neq f_0}}(d(e,e_0)+d(f,f_0)+1) - 2\sum_{e\in\mathcal{C}_i} d(e_0,e) - 2\sum_{f\in\mathcal{C}_j} d(f_0,f)$$

$$= 2\left[\sum_{e\in\mathcal{C}_i}\sum_{f\in\mathcal{C}_j}(d(e_0,e)+d(f,f_0)+1) - \sum_{e\in\mathcal{C}_i}(d(e,e_0)+1) - \sum_{f\in\mathcal{C}_j}(d(f,f_0)+1)+1\right]$$

$$- 2\mathcal{D}(u) - 2\mathcal{D}(v)$$

$$=2\left[\mathcal{D}(u)f_j + f_i\mathcal{D}(v) + f_if_j - \mathcal{D}(u) - f_i - \mathcal{D}(v) - f_j + 1\right] - 2\mathcal{D}(u) - 2\mathcal{D}(v).$$

Suggests that $\bar{\mathcal{D}}(t) \to d(t)$ where limit function d satisfies differential equation:

$$d'(t) = \frac{1}{\mathfrak{S}_1}\left[4d\mathfrak{S}_2 + 2\mathfrak{S}_2^2 - 4d\mathfrak{S}_1 - 4\mathfrak{S}_2\mathfrak{S}_1 + 2\mathfrak{S}_1^2 - 4d\mathfrak{S}_1\right],$$

where $\mathfrak{S}_1, \mathfrak{S}_2$ limits of s_2, s_1. Similar simpler analysis for s_2, s_3.

Barely subcritical $\{\mathrm{CM}_n(t) : 0 \leqslant t \leqslant t_n\}$: Proof idea

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where

$$\begin{aligned} e_3(t) = &- 4\nu^3\mu - 9\nu^2\mu e^{2t} + 9\nu^3\mu e^{2t} - 6\nu\mu e^{4t} + 12\nu^2\mu e^{4t} \\ &- 6\nu^3\mu e^{4t} - \mu e^{6t} + 3\mu\nu e^{6t} - 3\nu^2\mu e^{6t} + \nu^3\mu e^{6t}. \end{aligned}$$

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Differential equation method

- See nice work of Tom Kurtz and Nick Wormald's beautiful survey.
- **Here:** Limiting functions explode at t_c.
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as $n \to \infty$.

The differential equation approximation required $\delta < 1/5$

Conclusion

- Described methodology to understand metric level structure of random graph models at criticality.
- One key point: dynamics.
- Works when (a) from the barely subcritical regime to the critical scaling window, components (“blobs”) merge approximately like the multiplicative coalescent; (b) Good properties of the blobs at the entrance boundary.
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Extensions specific to models in talk

- **Configuration model:** Assumed exponential tails. Just a technical assumption to keep the paper to below 100 pages. Arises to get easy bounds in the subcritical regime. Can/should be easily reducible to finite moment conditions. *Finite third moment?*
- **IRG:** Again assume finite state space and strict positivity of the kernel κ to ignore issues such as reducibility of the associated multi-type BP. [BJR 05] derive conditions for general IRG when scaling exponents (barely supercritical regime) match those of Erdos-Renyi. *Extend results to this regime?*

The end

Thank you for your attention!