Lecture 2: Dynamic network models
Probabilistic and statistical methods for networks
Berlin Bath summer school for young researchers

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August, 2017

Motivation

Preferential attachment

- Last few years: enormous interest in formulating models to "explain" real-world networks (e.g. network of webpages, the Internet, social networks, gene regulatory networks etc).

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- *How to analyze variants such as limited choice or non-local preferential attachment. Analysis? Performance in practice?*

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5. Variants I: Superstar model (Mike Steel, Tauhid Zaman, SB)
6. Variants II: Preferential attachment with choice (Omer Angel, Robin Pemantle, SB)

Basic model of growing trees

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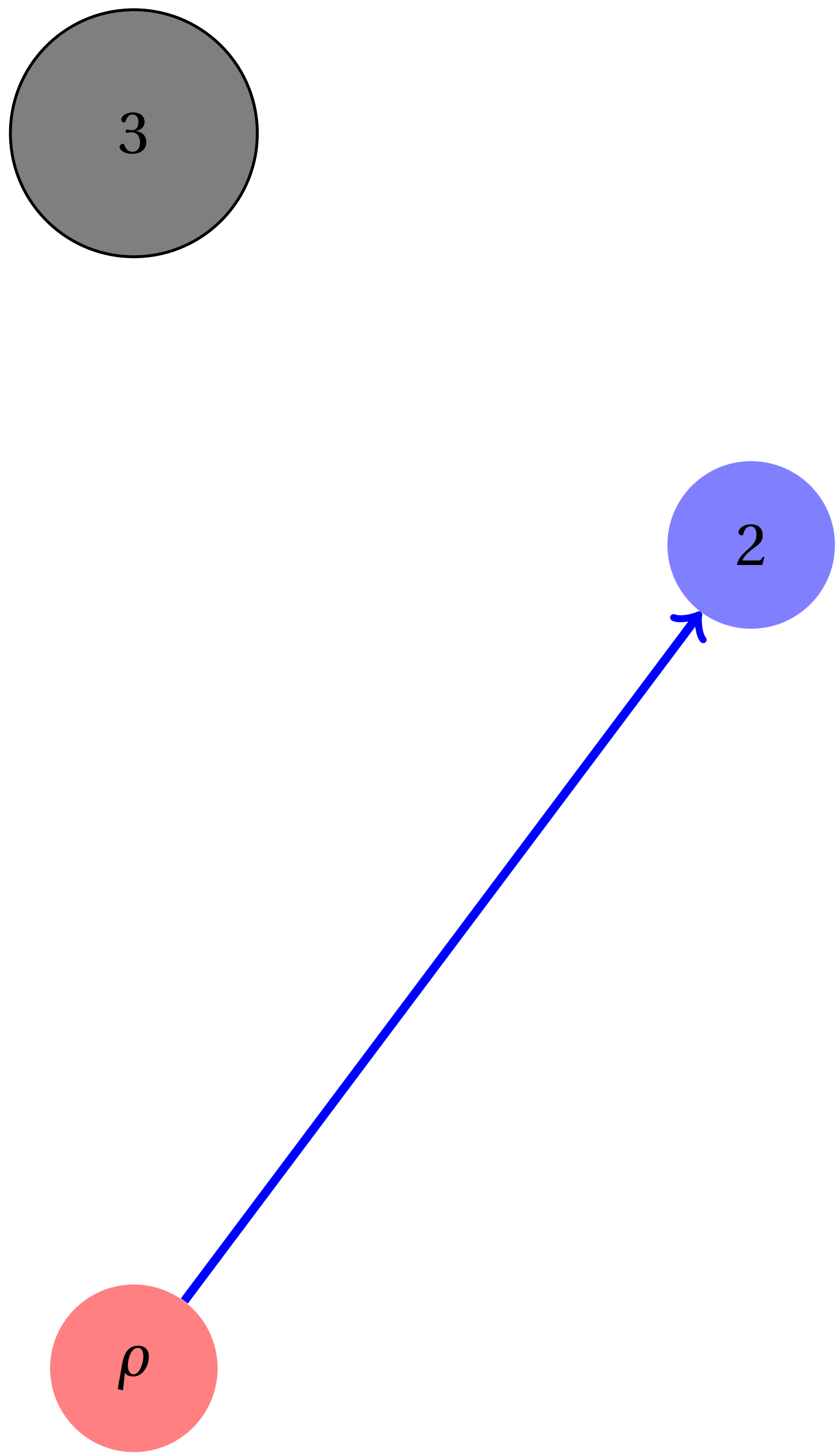
- $D(v,n) =$ *out-degree* of node v at time n.

Preferential attachment: Example $\mathscr{T}_3$



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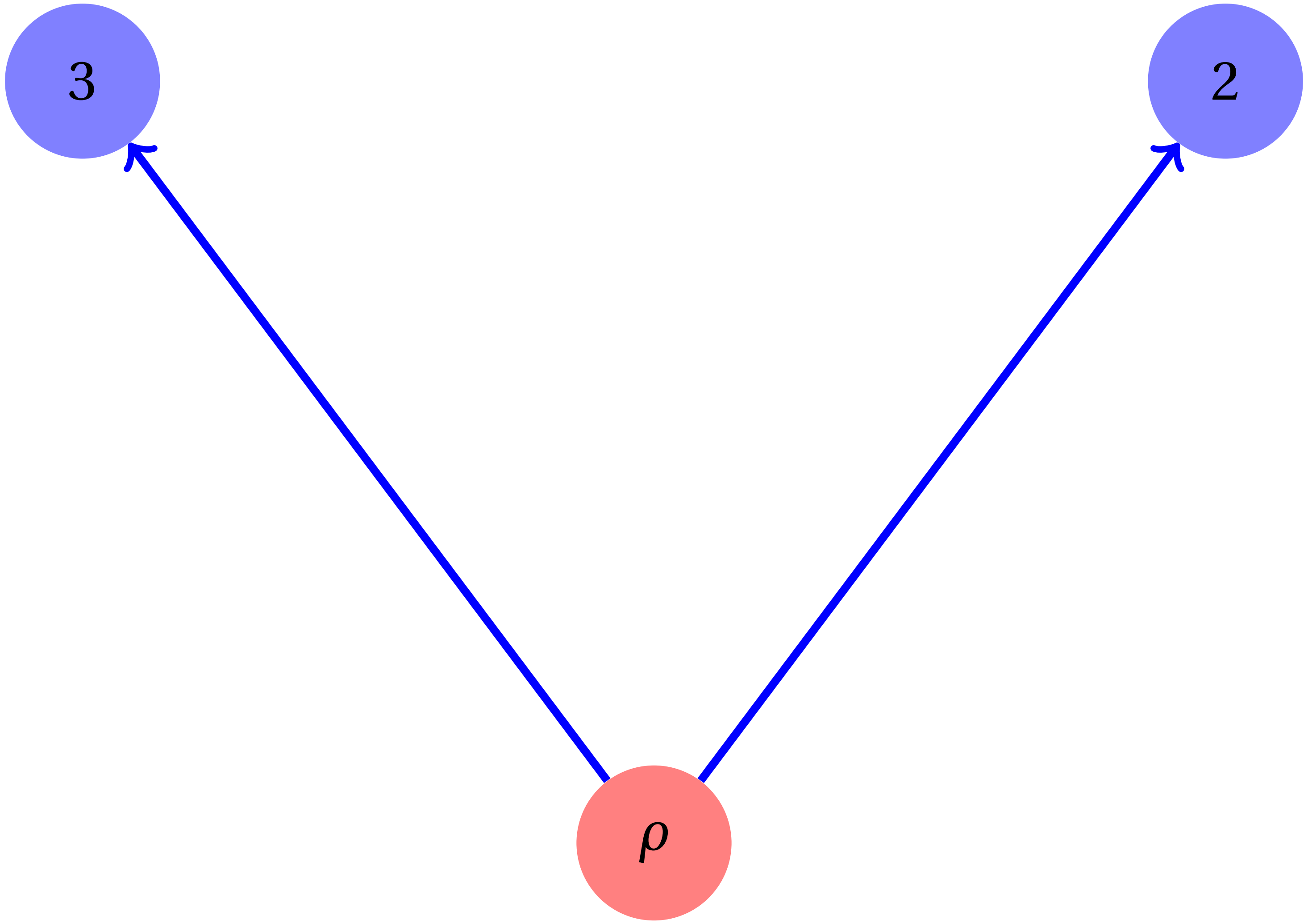

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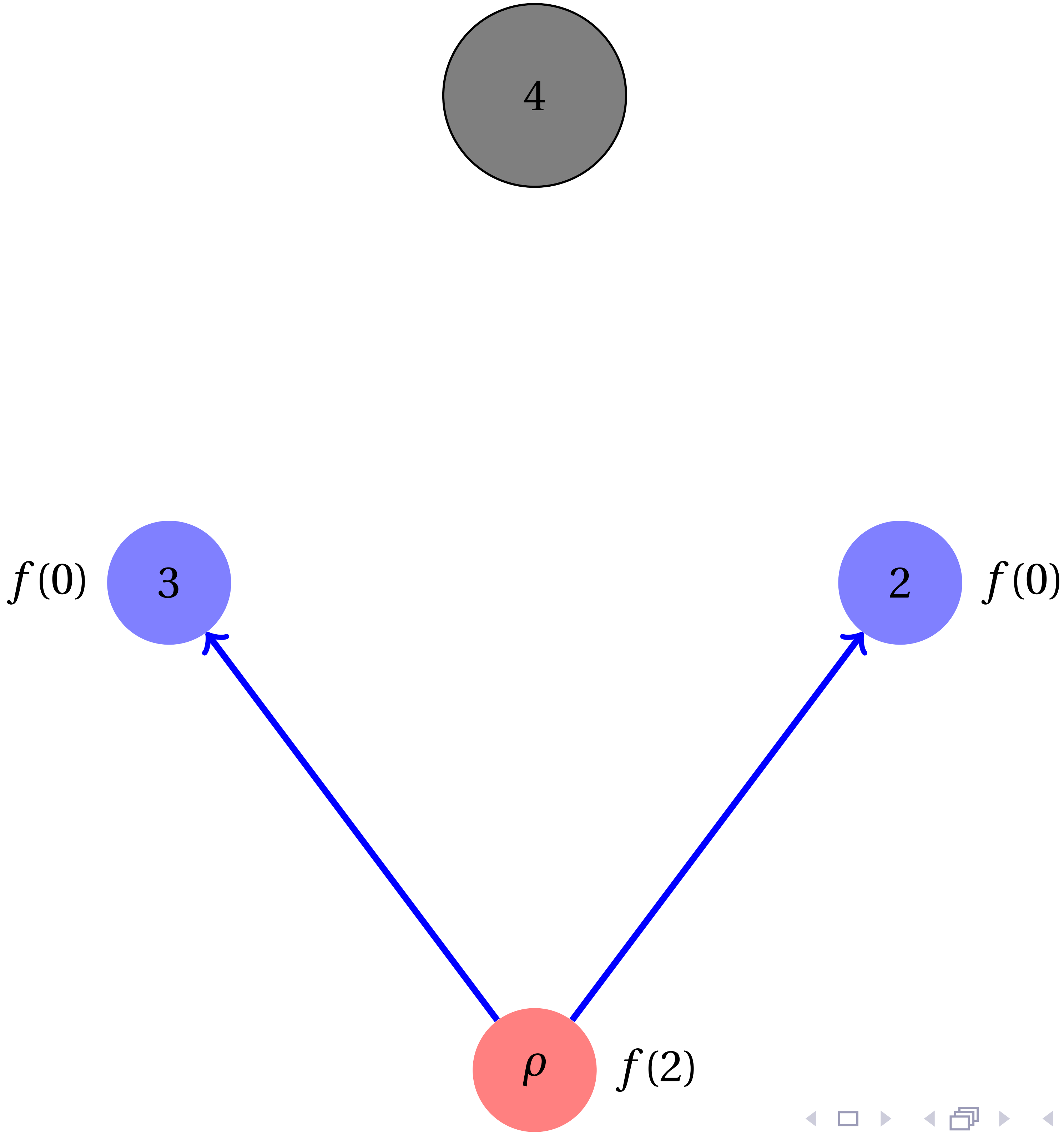
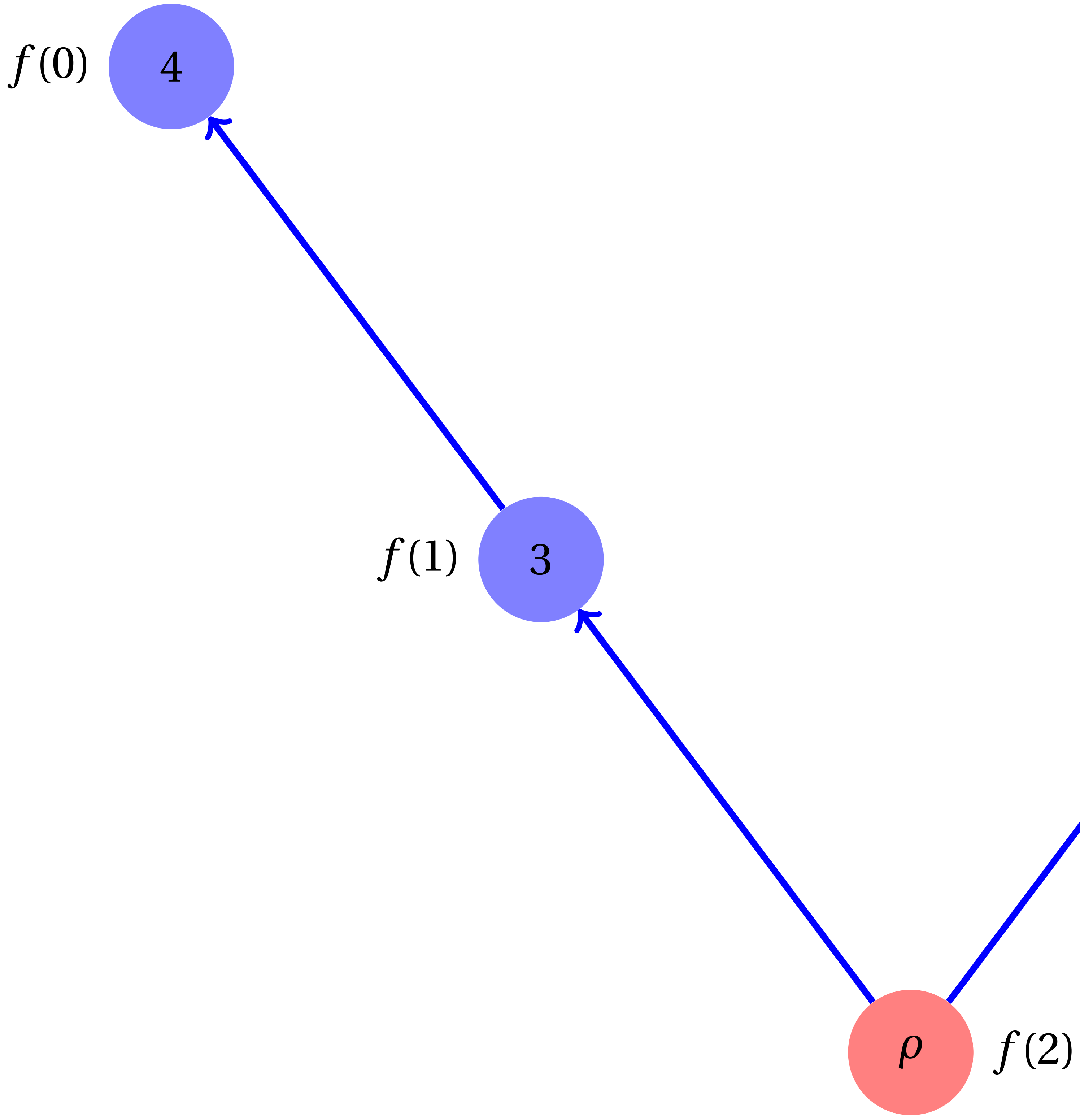


Preferential attachment: Example $\mathscr{T}_4$



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4. **Sublinear Pref Attachment:** $f(k) = (k+1)^{\alpha},\ 0 < \alpha < 1$

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- *Then* $\mathbb{P}(J = i) \propto d_i$.

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Continuous time construction

Point process corresponding to attractiveness function f

- $\mathscr{P}$ is Markov pure birth process with rate description

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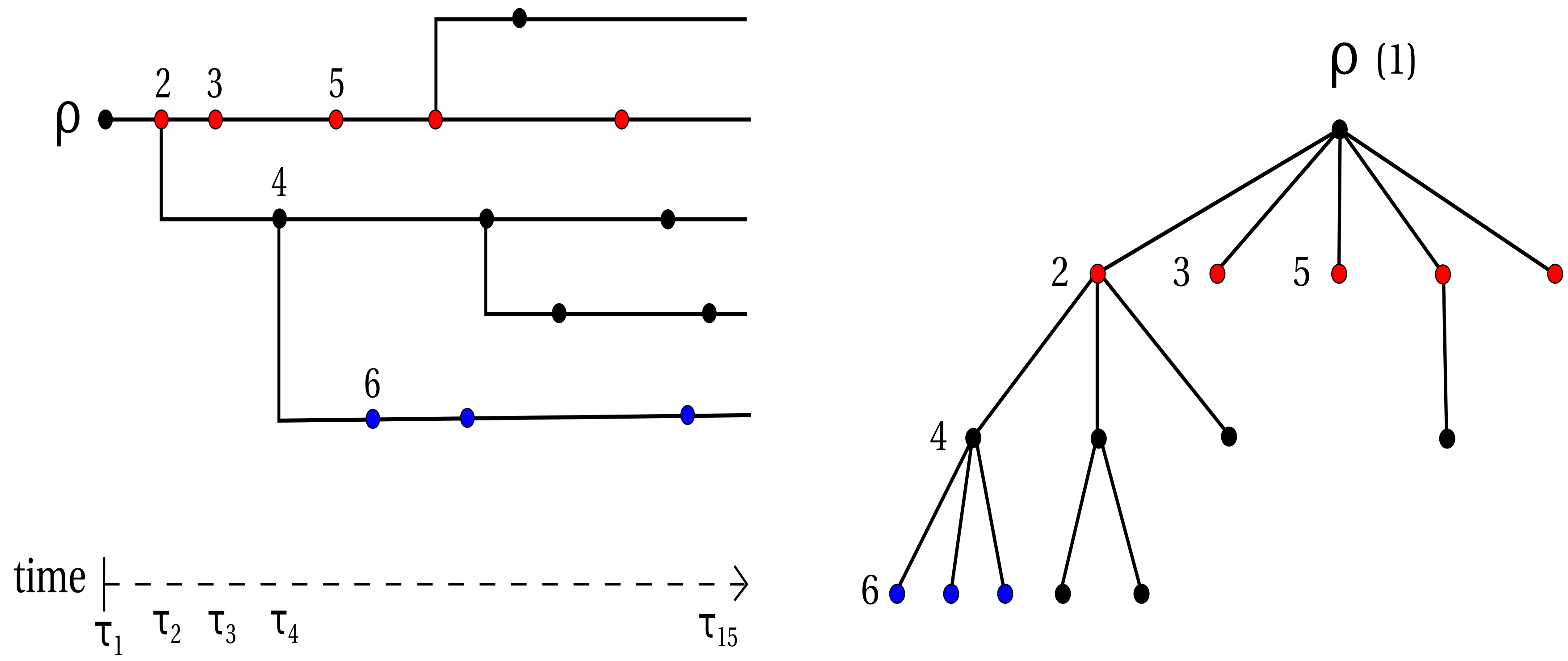
Key connection

$\tau_n = \inf\{t : \mathscr{F}(t) = n\}$ *then* $\mathscr{F}(\tau_n) \stackrel{d}{=} \mathcal{T}_n^f$.

Continuous time and discrete time in pictures



Branching process theory

Asymptotics

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- Processes grow exponentially: $|\mathscr{F}(t)| \sim e^{\lambda t}$
- Here λ is a very important characteristic : called the *Malthusian rate of growth*
- Given by the formula:

$$\mathbb{E}(\mathscr{P}(T_\lambda)) = 1,$$

  where $T_\lambda \sim \exp(\lambda)$ independent of $\mathscr{P}$.

Exact result

Under technical conditions $(\mathbb{E}(\mathscr{P}(T_\lambda), \log^+ \mathscr{P}(T_\lambda)) < \infty)$:

- $|\mathscr{F}(t)|e^{-\lambda t} \xrightarrow{a.s.} W$.
- In our settings: $W > 0$ a.s.
- Bottom line:

$$\tau_n \sim \frac{1}{\lambda} \log n \pm O_P(1)$$

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- More refined analysis (Mori/Pekoz+Rollin+Ross) gives

$$\frac{\deg_n(\rho)}{\sqrt{n}} \xrightarrow{a.s.} Z \qquad Z \text{ has explicit recursive construction.}$$

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- With a bit more work, possible to deduce distributional convergence for the maximal degree.
- Example of interesting results: Sublinear pref attachment $f(k) = (k+1)^\alpha$

$$\frac{deg_n(\rho)}{(\log n)^{\frac{1}{1-\alpha}}} \xrightarrow{P} \left(\frac{1}{\theta(\alpha)}\right)^{\frac{1}{1-\alpha}}$$

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Thus

$$\frac{B_{h_n}}{h_n} \leq \frac{\tau_n}{h_n} \leq \frac{B_{h_n+1}}{h_n}$$

Now use the fact [Pittel's argument] that

$$\frac{\tau_n}{\frac{1}{\lambda}\log n} \xrightarrow{a.s.} 1 \Rightarrow \frac{h_n}{\log n} \xrightarrow{P} C_{\texttt{model}}$$

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1. Preferential attachment: Base model
   - Continuous time construction
   - Local weak convergence
2. Twitter event networks and the superstar model
   - Retweet Graph and Superstar Model
   - Main Results
   - Comparison with Preferential Attachment Model
   - Superstar Model: Tools for Analysis
3. Power of choice and random trees
4. Conclusion

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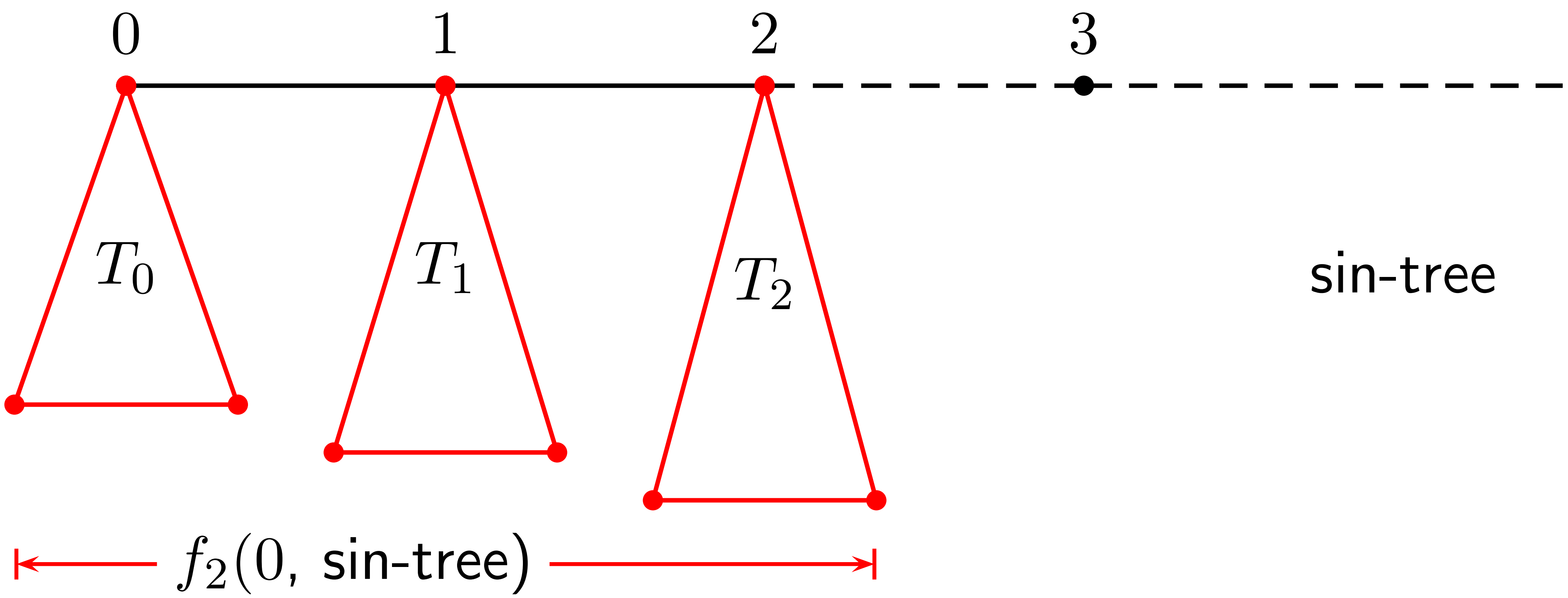
Suggests tree "below" random node looks like $\mathscr{F}(T_\lambda)$ i.e. branching process run for random exponential amount of time.

Sin-tree



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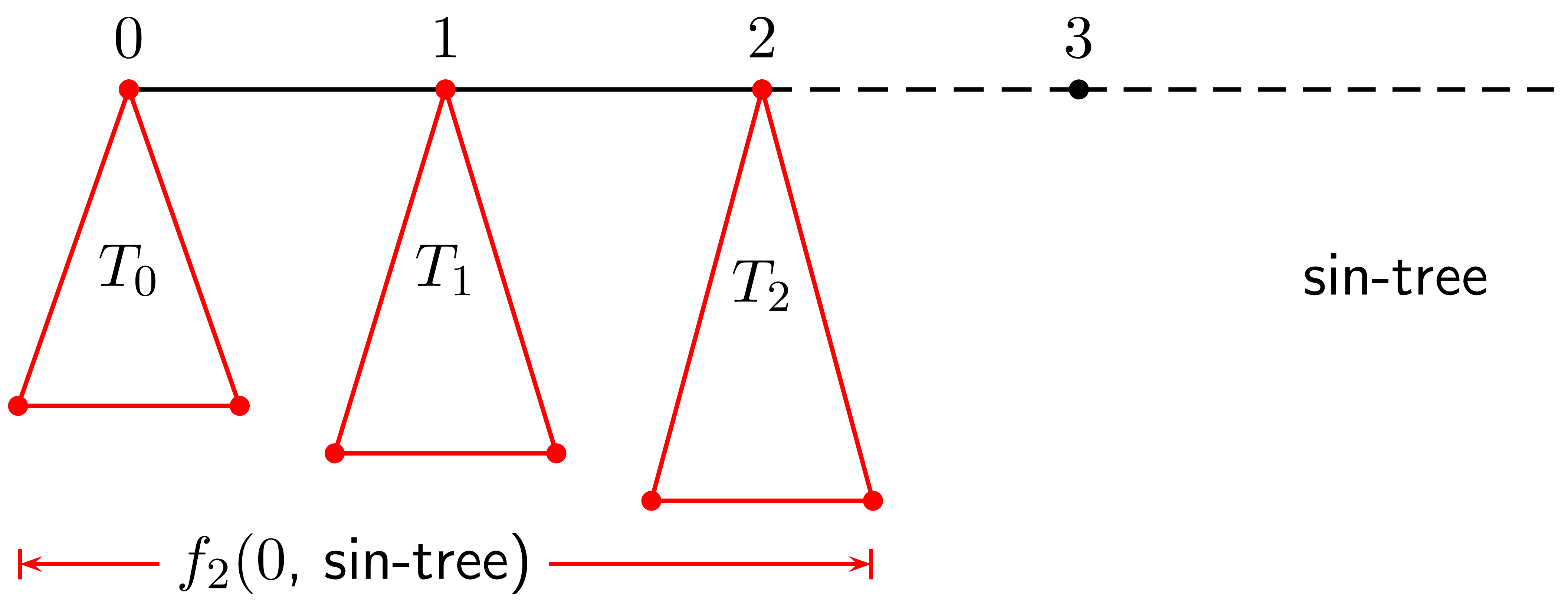
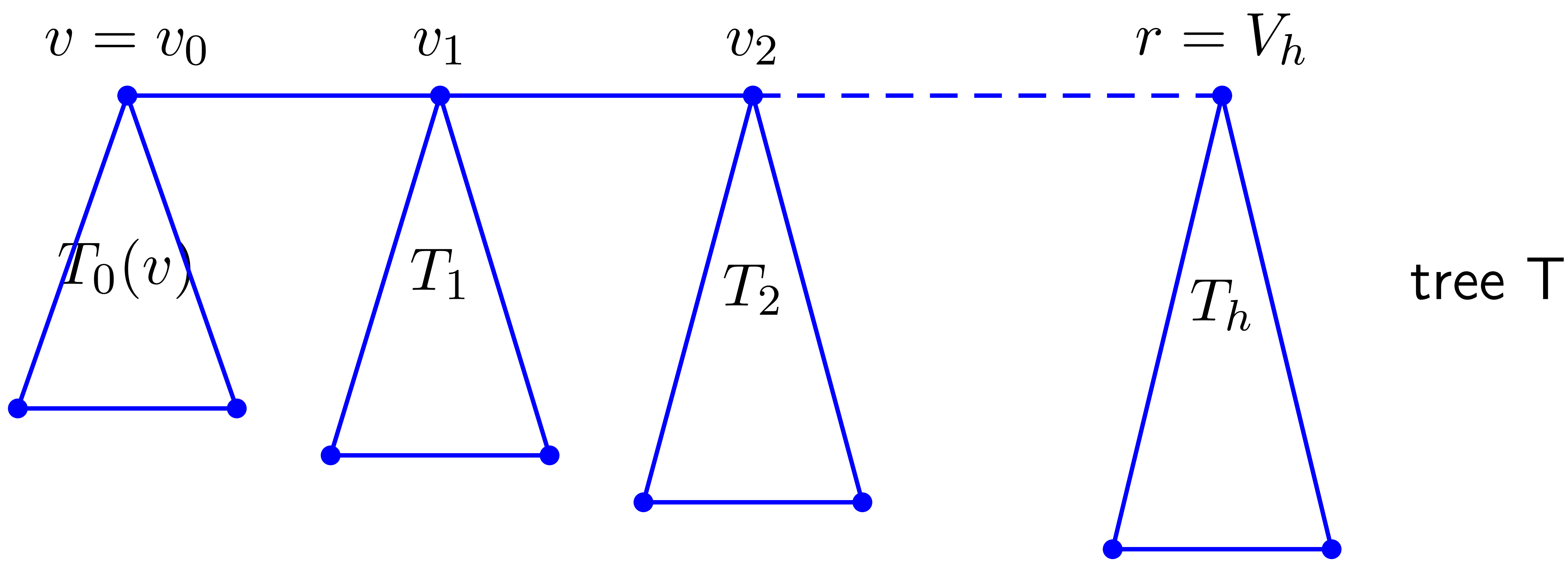
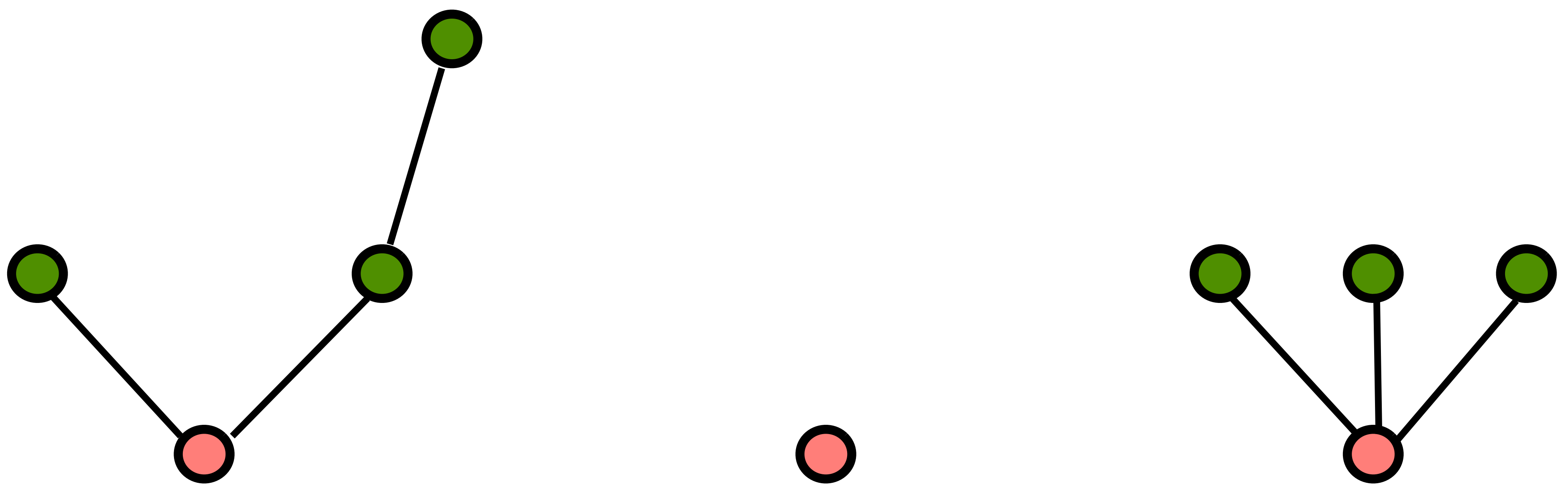


Can think of random sin-trees

T is a tree with root r. Given a vertex v, there exists a unique path $v_0 = v, v_1, ..., v_h = r$ from v to the root.





t_0 t_1 t_2

Decompose tree into a sequence of finite rooted subtrees or fringes $(T_0(v), T_1(v), T_2(v), \ldots)$. For each $k \geq 1$,

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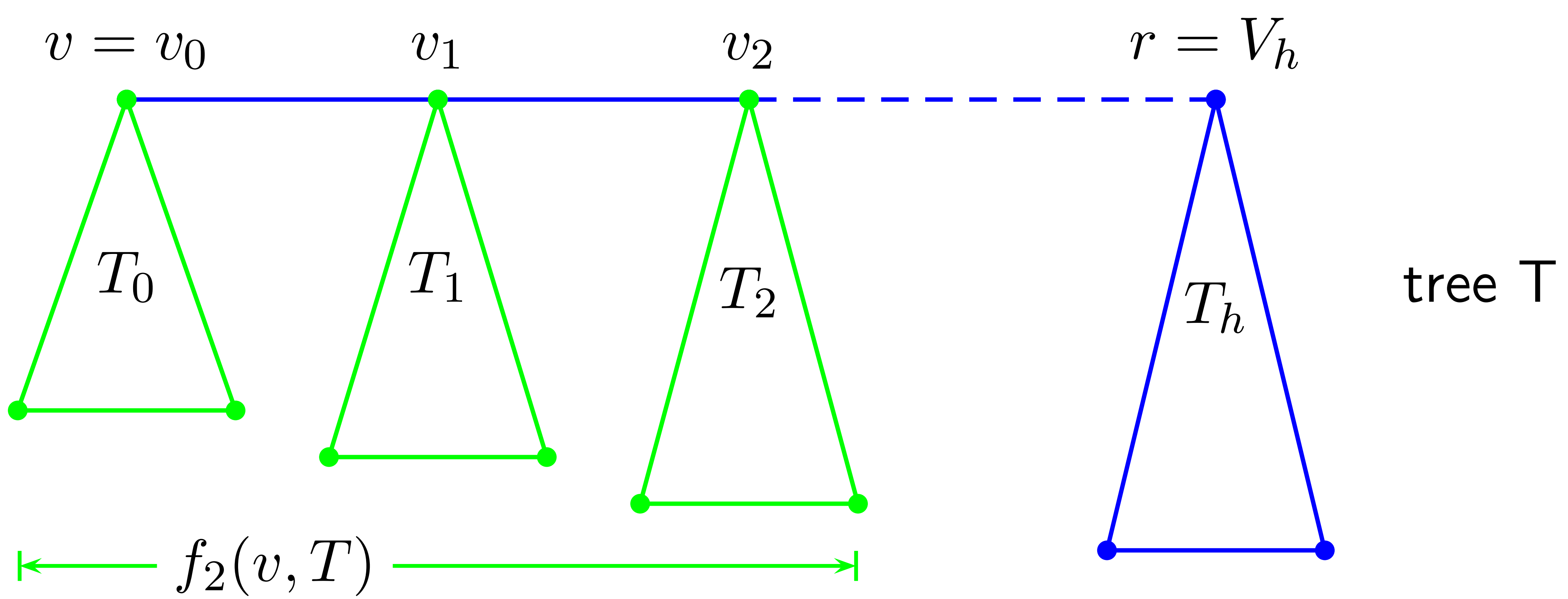

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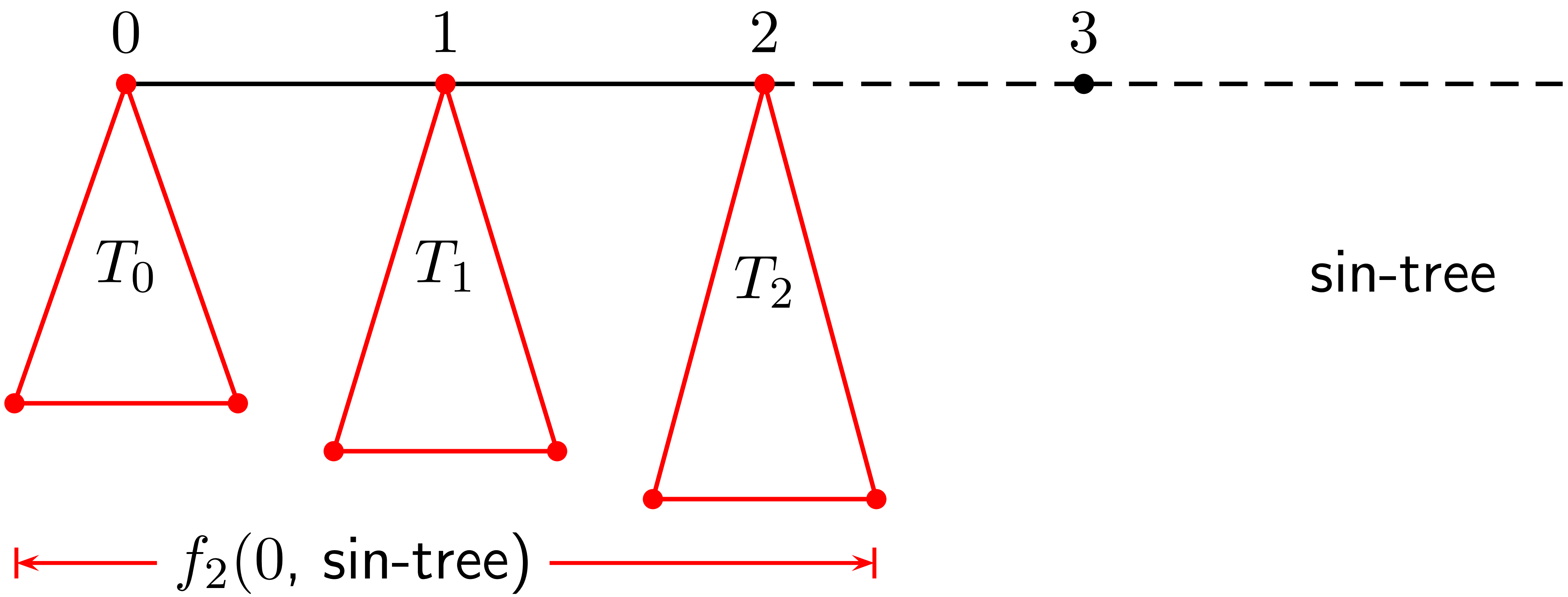

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- $X_0 \sim \exp(\alpha)$ and for $i \geq 1$, $X_i \sim \mu$. $S_n = \sum_0^n X_i$.
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  0 designated as the root. $\mathscr{F}_{X_0}$ to be rooted at 0 and for $n \geq 1$ consider $\mathscr{F}_{S_n,S_{n-1}}$ to be rooted at n.
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- *Power law degree distribution!*

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What more can one do with this machinery?

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Setting

- For the convergence of spectral distribution can take general families of trees satisfying sin-tree convergence.
- For maximal eigen value convergence talking about preferential attachment with $f(v,n) = \text{Deg}(v,n) + a$.

Main result

Theorem (SB, Evans, Sen 08)

(a) Consider a sequence of trees converging in fringe since to a random infinite $\mathtt{sin}$-tree. Then there exists a model dependent probability distribution function F such that

$$d(F_n, F) \xrightarrow{P} 0, \qquad \text{as } n \to \infty.$$

(b) Let $\gamma_\alpha = \alpha + 2$. Then for the linear preferential attachment model

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Spectral distribution turns out to be a local property of random node, maximal eigen values, local property about the root.

Spectral distribution: Method of proof

Stieltjes transform

$$s(z) = \int_{\mathbb{R}} \frac{1}{x - z} \, dF_n(x)$$

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$$R_{vv}(z) = \frac{1}{-z + \sum_1^{N(v)} R_{v_i v_i}(z) + R_{A(v)}^{\mathtt{big}}(z)}$$

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- So not hard to show that there exists a fixed Stieltjes transform $s(z)$ such that,

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Properties and questions

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- Implies that for most standard models, limiting F has dense set of atoms
- **Open Question:** Does limiting F have absolutely continuous part?
- Connections to areas such as Random Schrodinger operators?

At this point

- If one can embed things in a continuous time all good things happen.
- How far can one push such embeddings? Can these continuous time branching processes arise in the limit even when no such embeddings exist?

Outline

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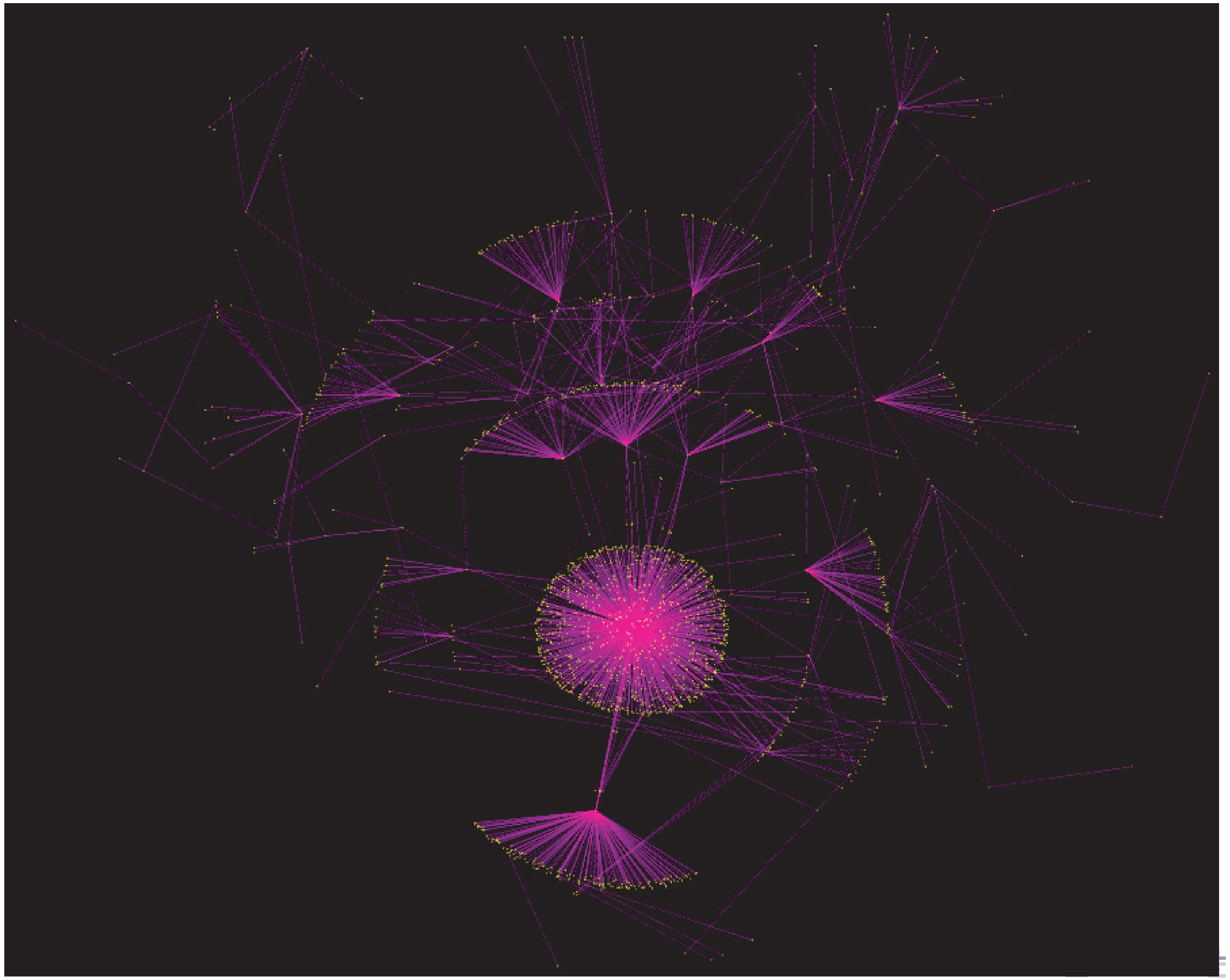

Some Empirical Retweet Graphs

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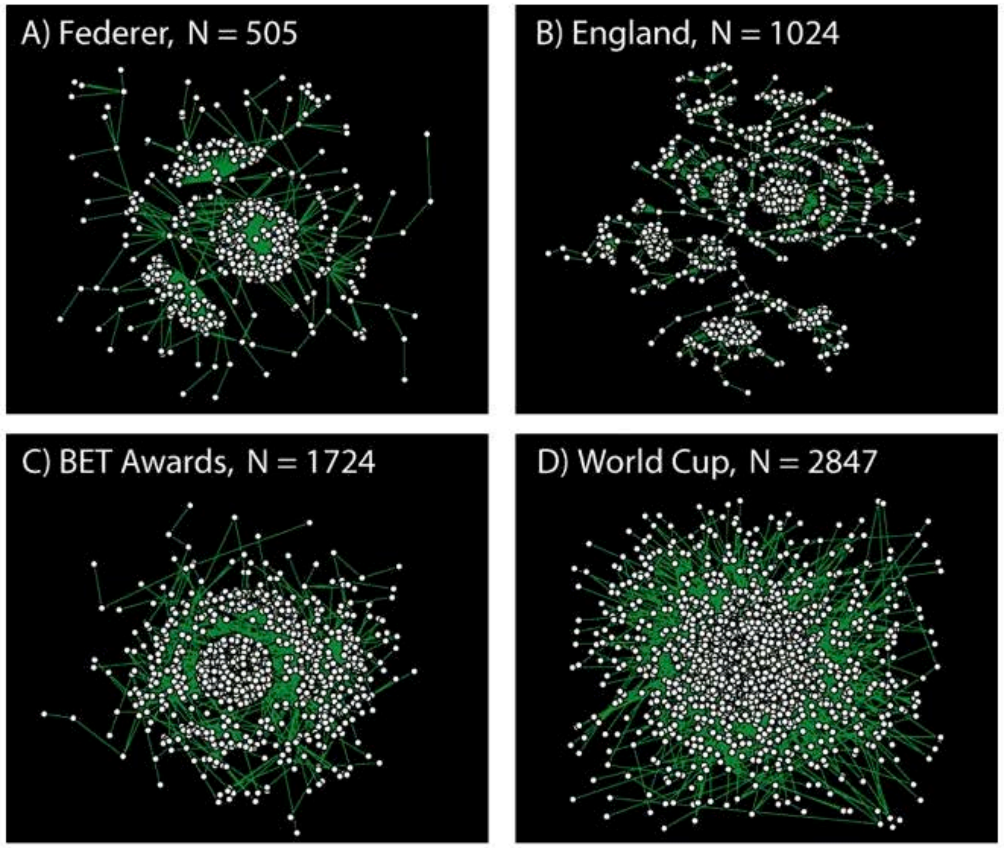

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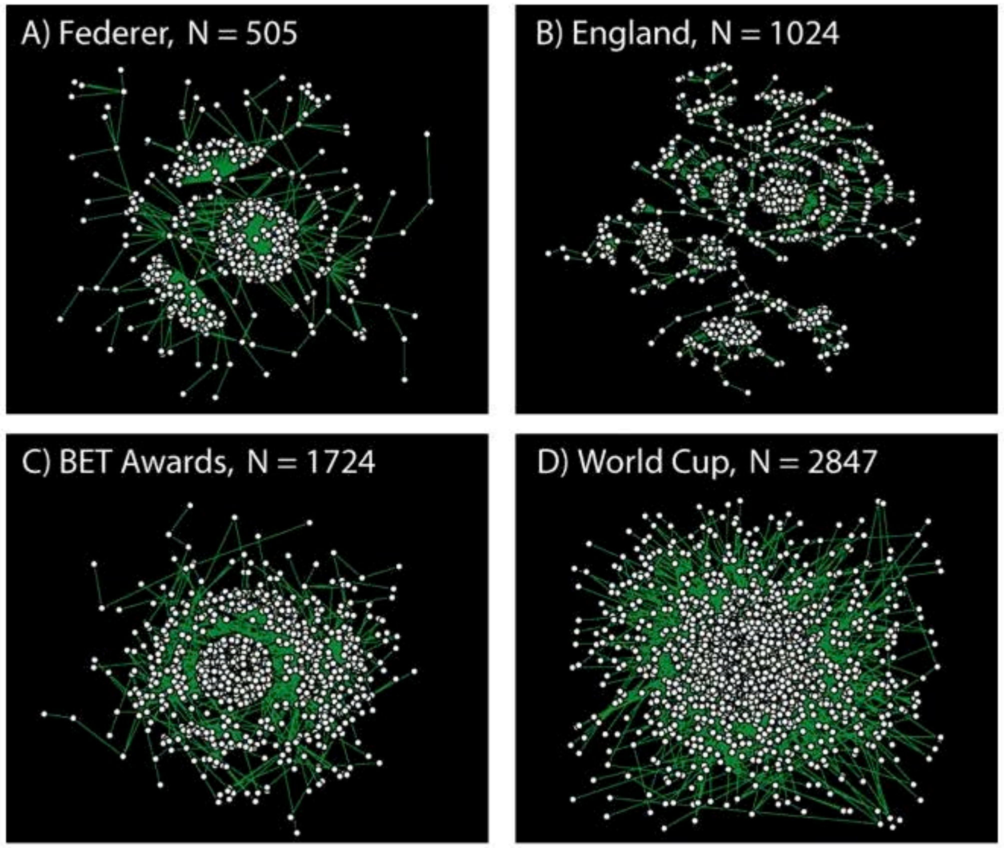

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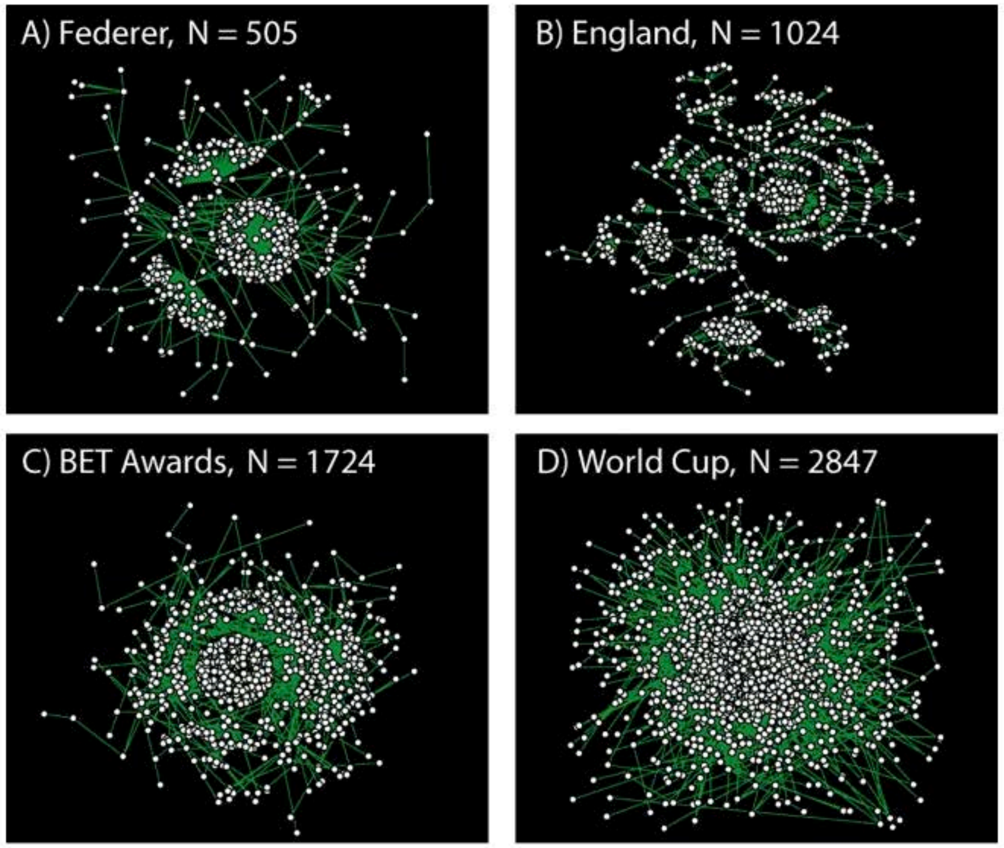

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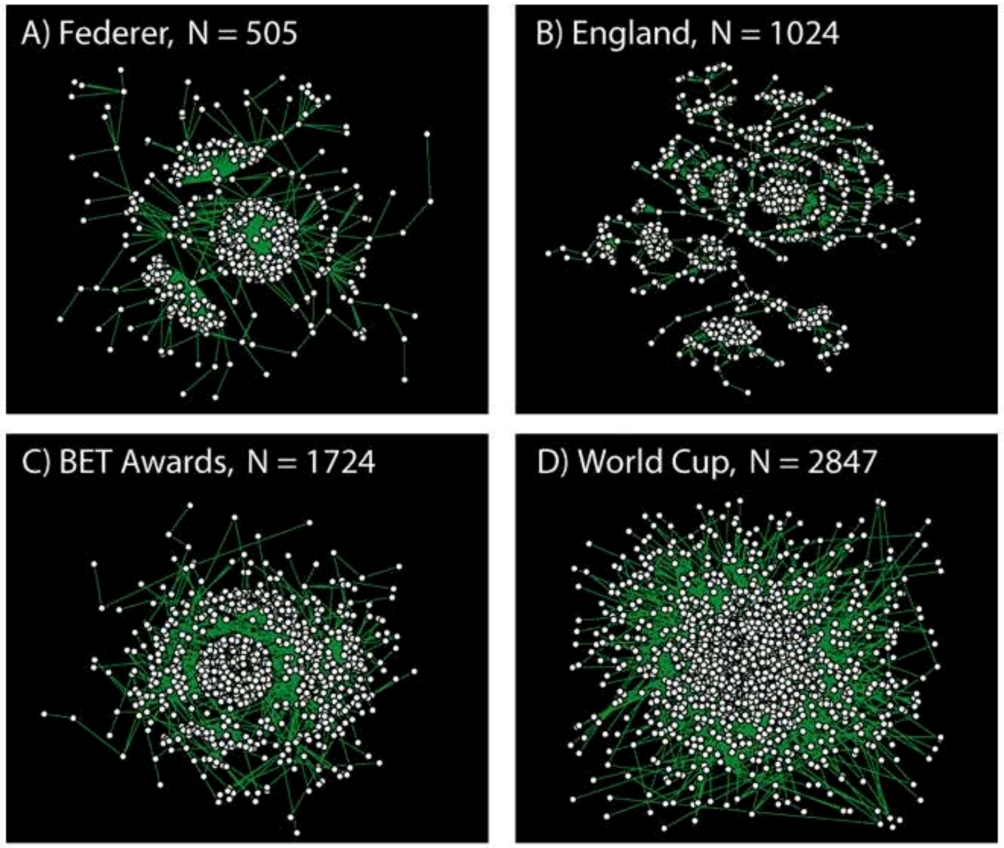

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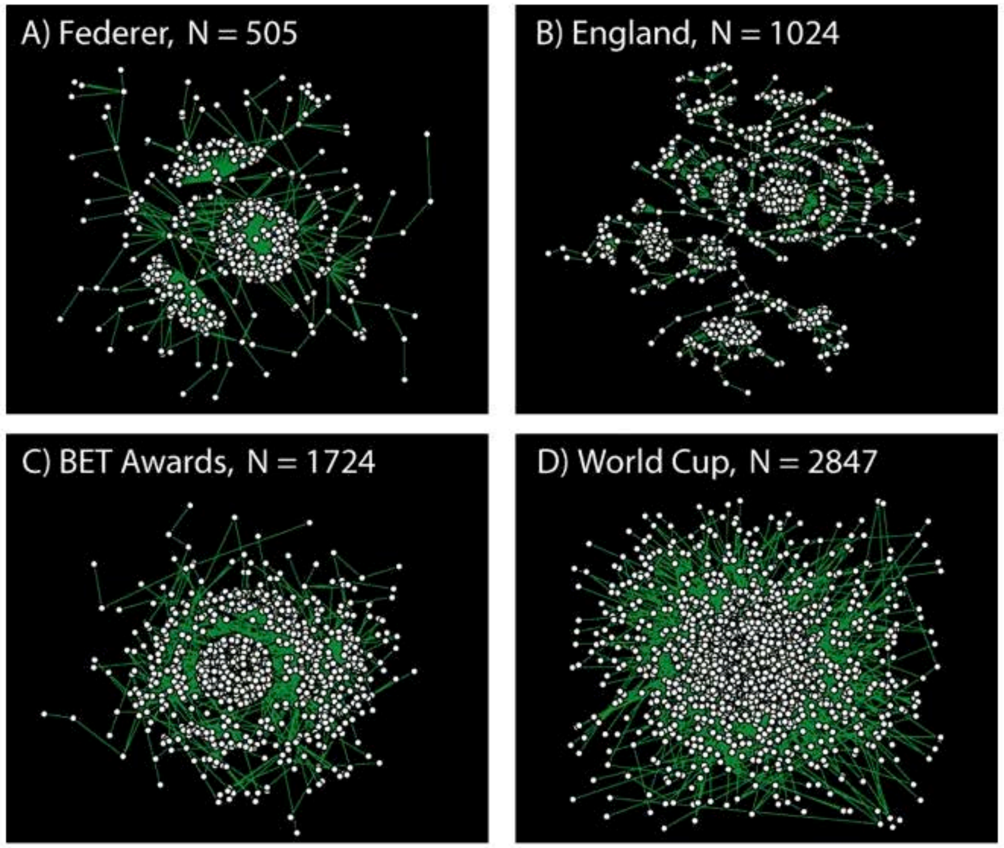

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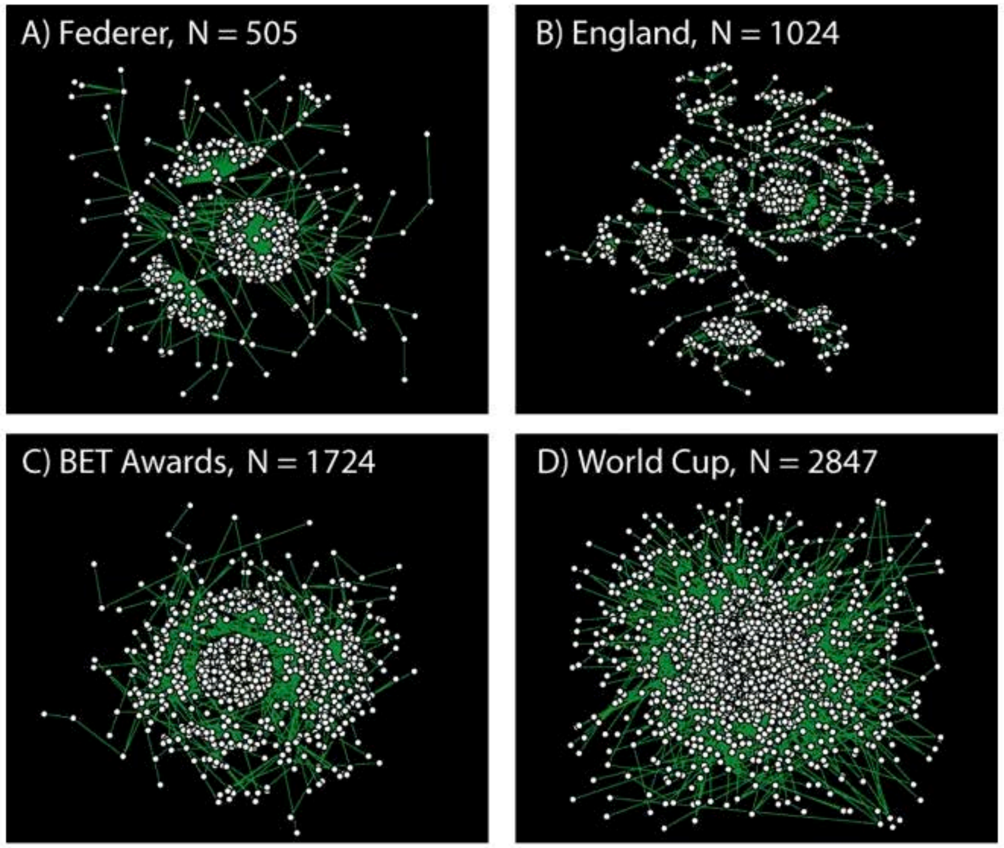

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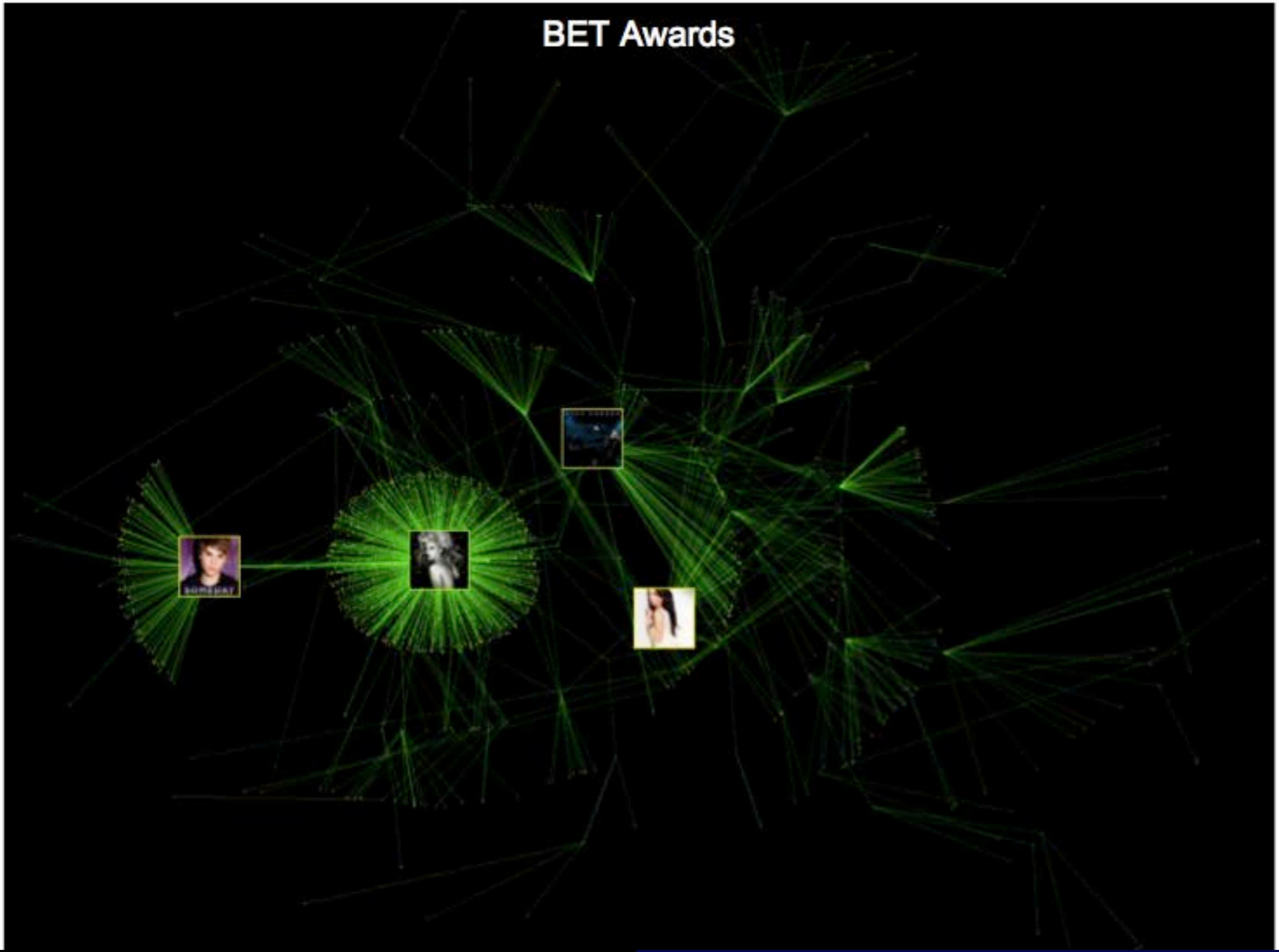

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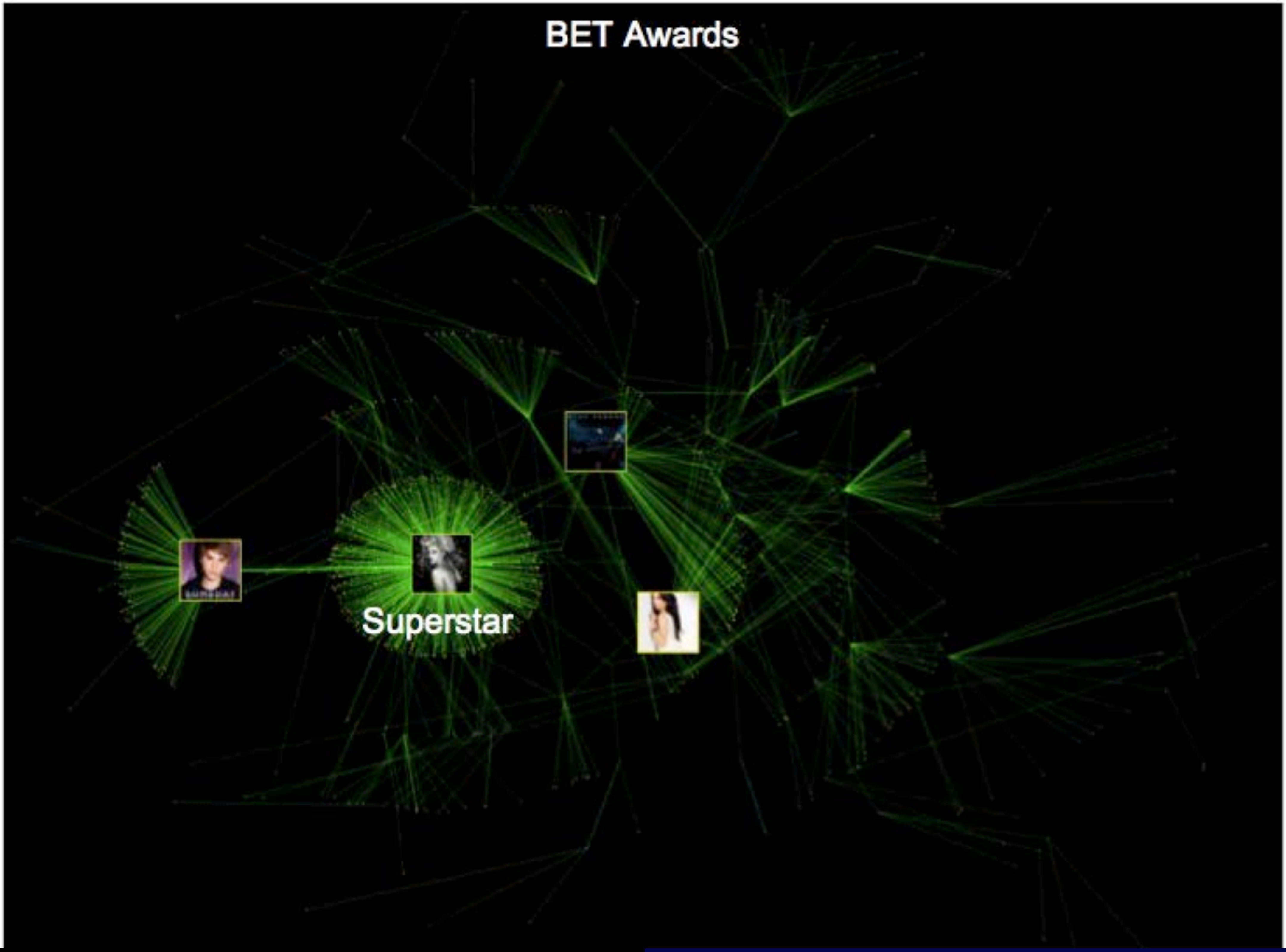
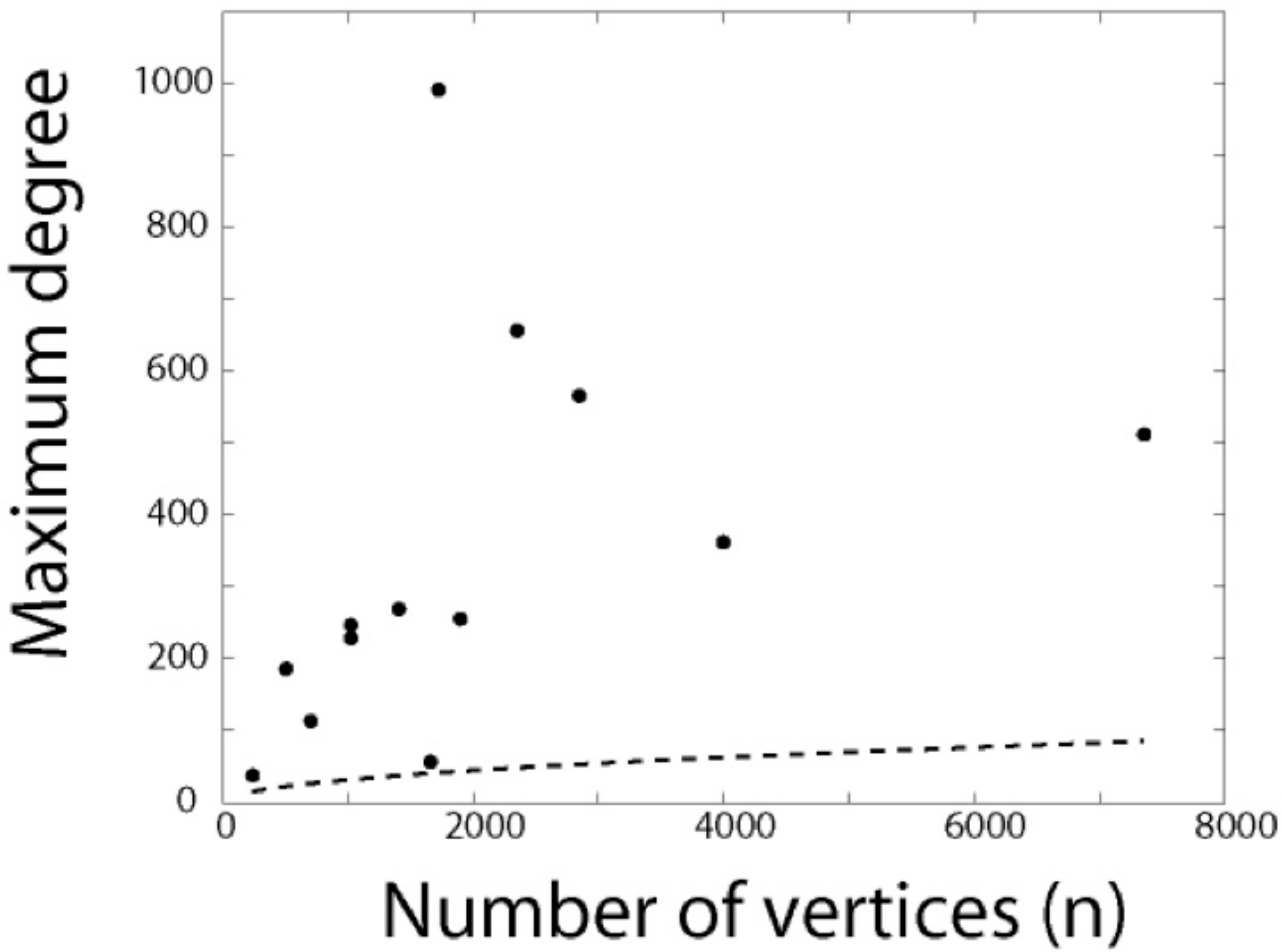


The superstar model

- Max degree in retweet graph is on the order of graph size (i.e. $M_G \sim pn$)
- Preferential attachment predicts sub-linear max degree



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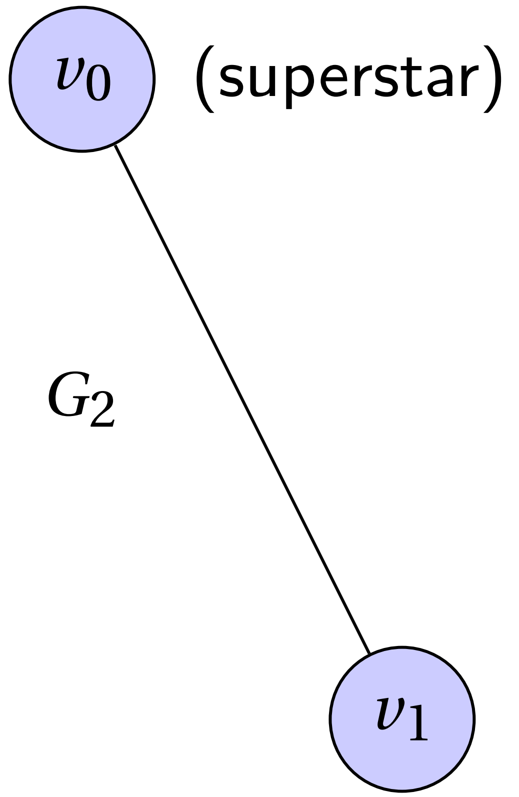

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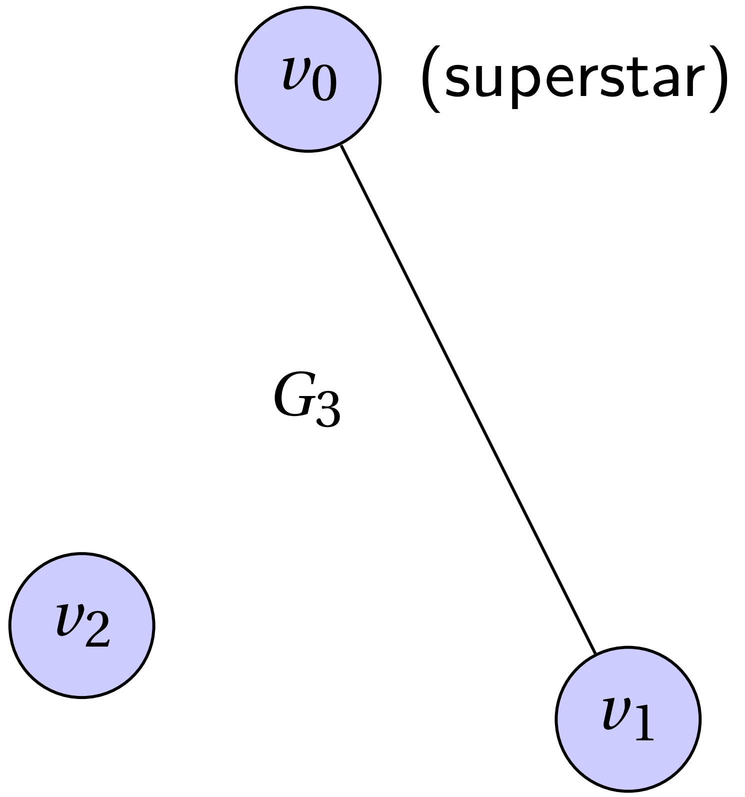

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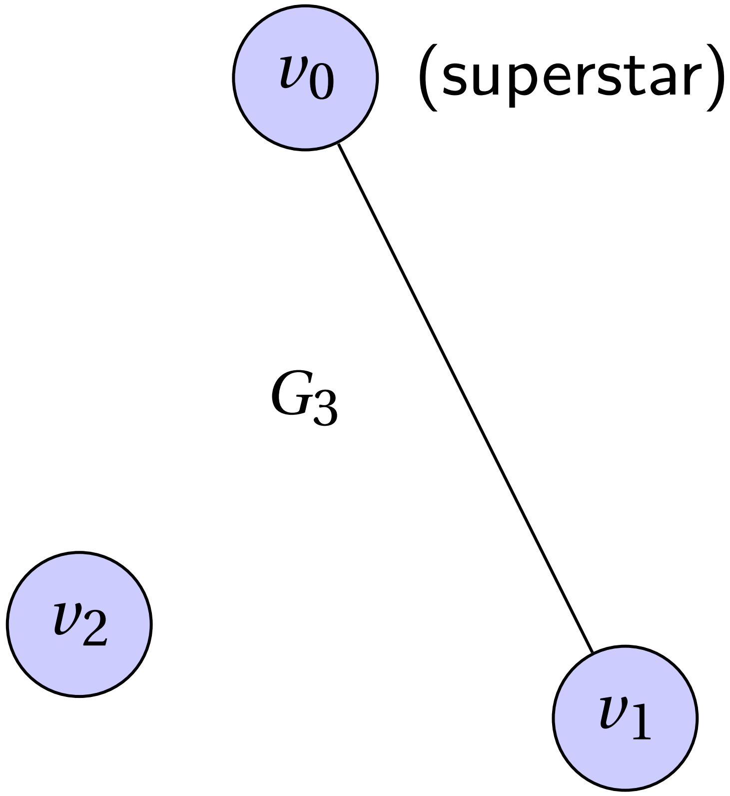

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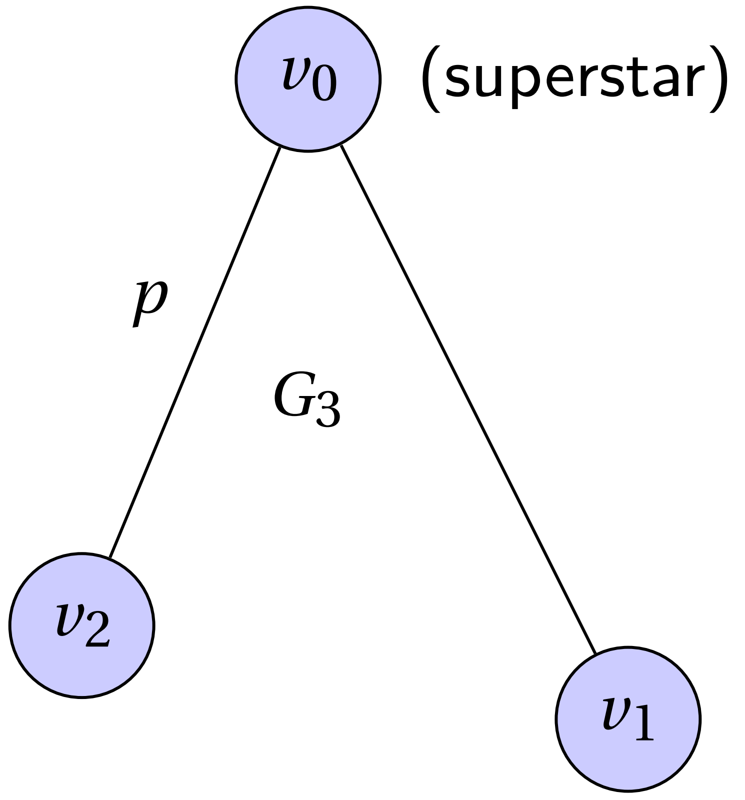

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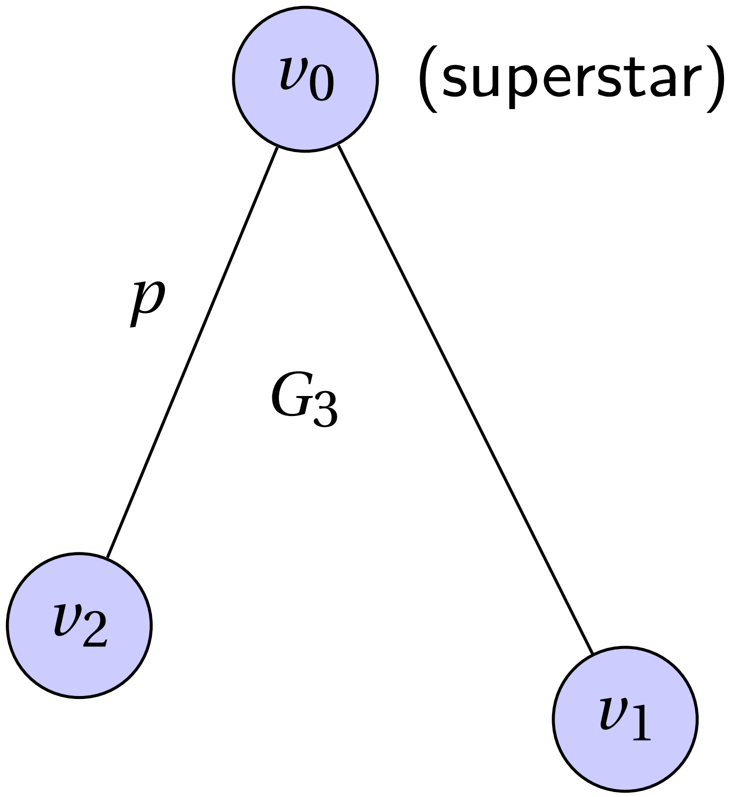

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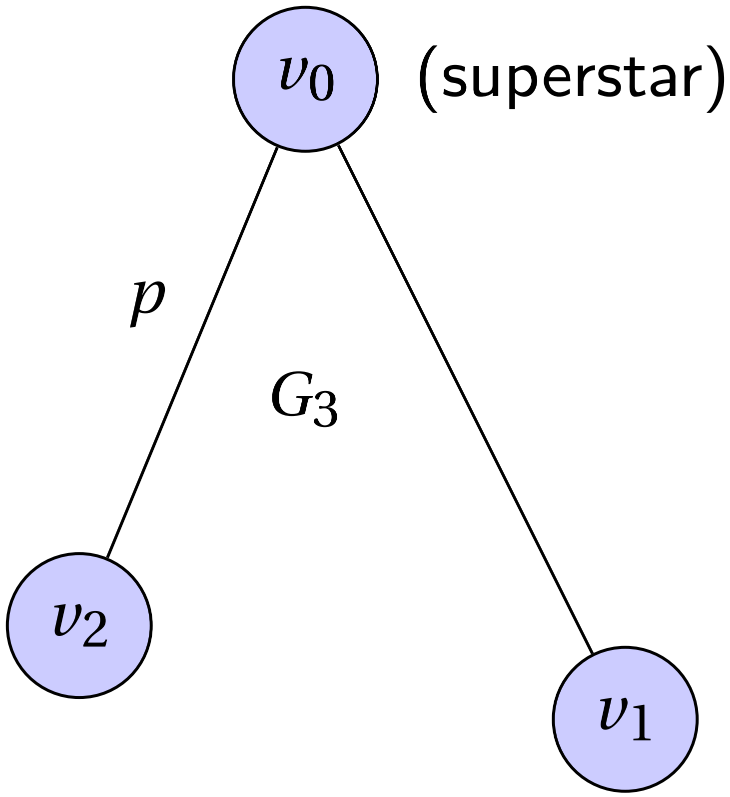

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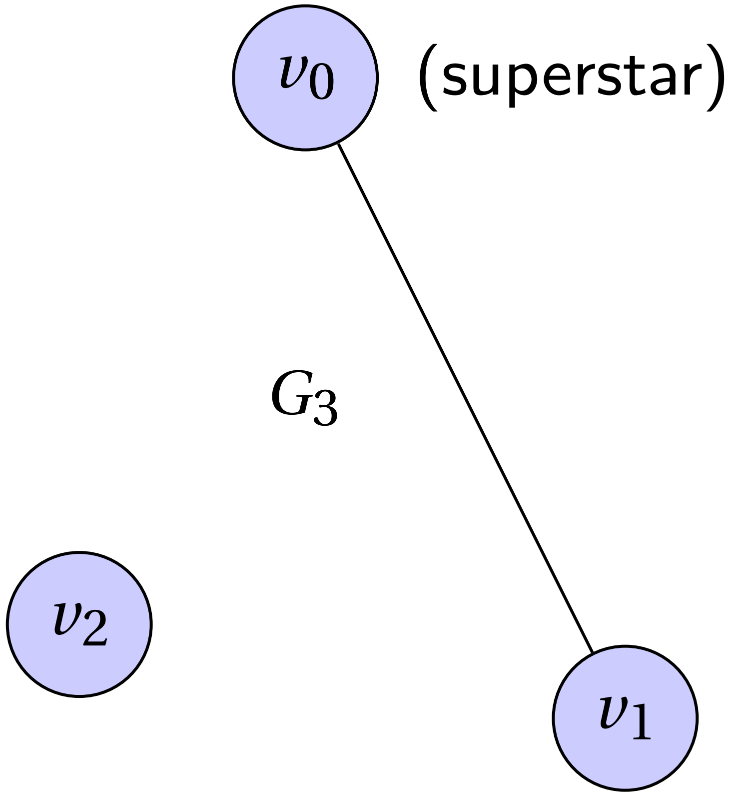

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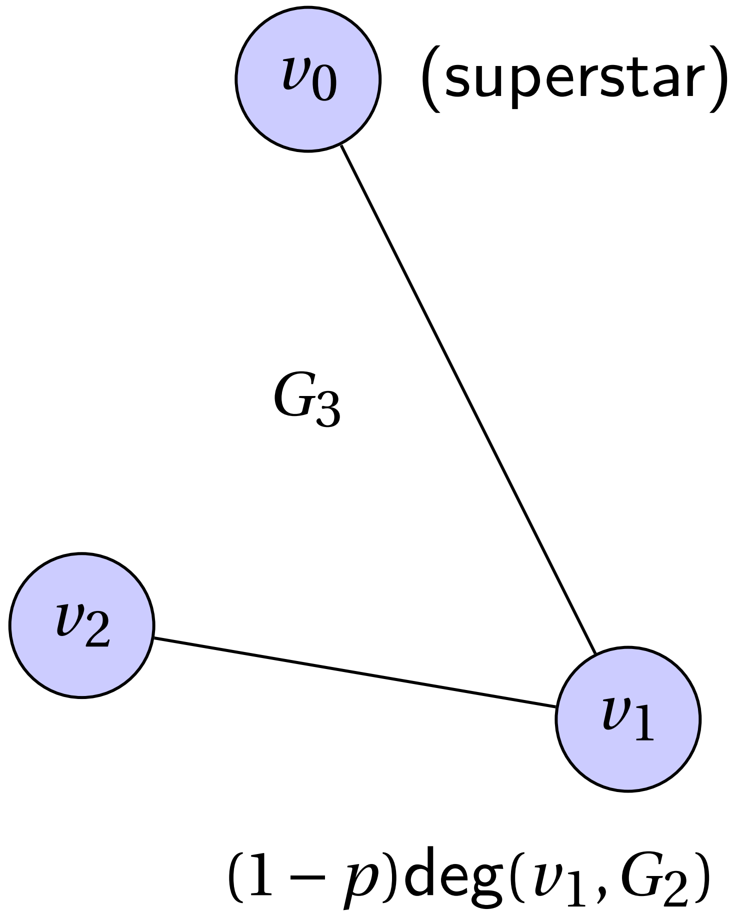

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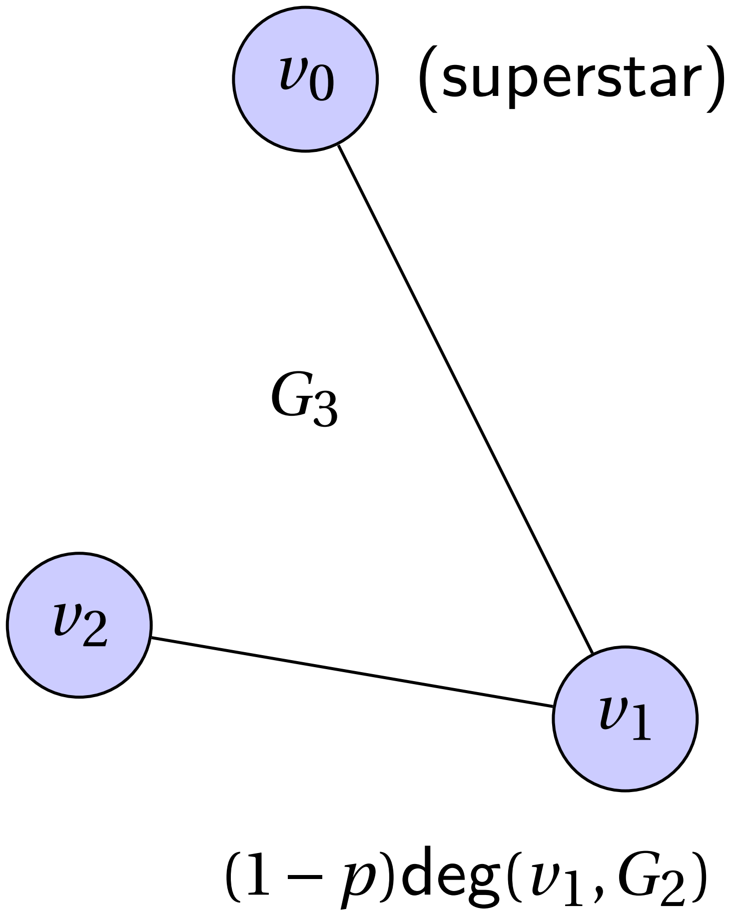

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The only model parameter is p: The superstar parameter

This is a very simple model: But (1) it has empirical benefits and (2) it is tractable — though not particularly easy.

Outline

Superstar Degree

Theorem

Let $\deg(v_0, G_n)$ *be the superstar degree. Then we have that*

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- The value of p determines other features of the graph — the Superstar Model is *testable*.

Non-Superstar Degree

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Let $\deg_{\max}(G_n)$ *be the maximal non-superstar degree:*

$$\deg_{\max}(G_n) = \max_{1\le i\le n} \deg(v_i, G_n)$$

and let

$$\gamma = \frac{1-p}{2-p}.$$

Then there exists a non-degenerate, strictly positive random variable Δ^* *such that*

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Realized Degree Distribution in the Superstar Model

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and introduce the superstar model scaling constant

$$f_{SM}(k, p) = \frac{2-p}{1-p}(k-1)! \prod_{i=1}^{k} \left(i + \frac{2-p}{1-p} \right)^{-1}.$$

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- This contrasts with the preferential attachment model which scales like k^{-3}

Height result

Theorem

Let $W(\cdot)$ be the Lambert special function with $W(1/e) \approx 0.2784$. Then with probability one we have

$$\lim_{n\to\infty} \frac{1}{\log n} \mathscr{H}(G_n) = \frac{1-p}{W(1/e)(2-p)}.$$

Outline

1. Preferential attachment: Base model
   - Continuous time construction
   - Local weak convergence
2. Twitter event networks and the superstar model
   - Retweet Graph and Superstar Model
   - Main Results
   - Comparison with Preferential Attachment Model
   - Superstar Model: Tools for Analysis
3. Power of choice and random trees
4. Conclusion

Superstar Model vs Preferential Attachment

Model	Superstar Model	Preferential Attachment
Superstar Degree		
Maximal non-superstar degree exponent		
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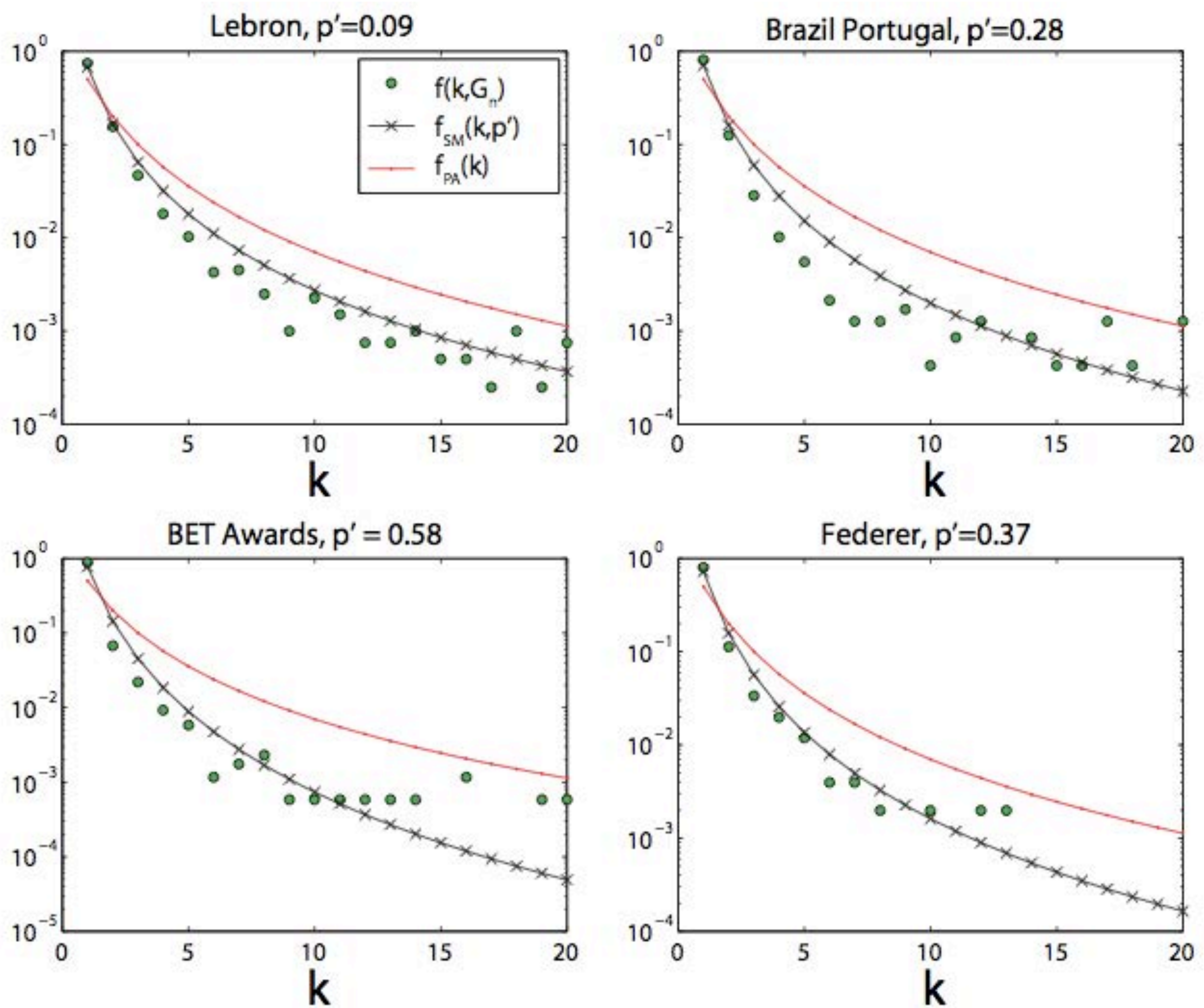
- Comparison: Preferential Attachment always predicts...

$$f_{PA}(k) = \frac{4}{k(k+1)(k+2)}$$

Degree distribution



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- Next: How Can one Analyze the Superstar Model?

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- News You Can Use? One can see the benefits of using multi-type branching processes. One can see that the connection between the Yule process and preferential attachment is natural.

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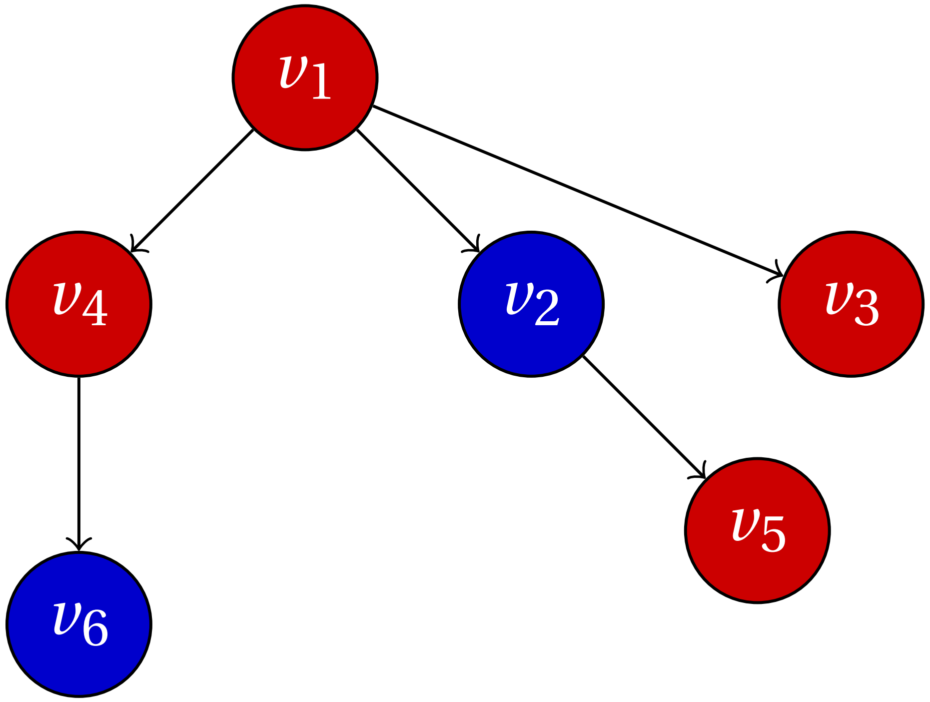

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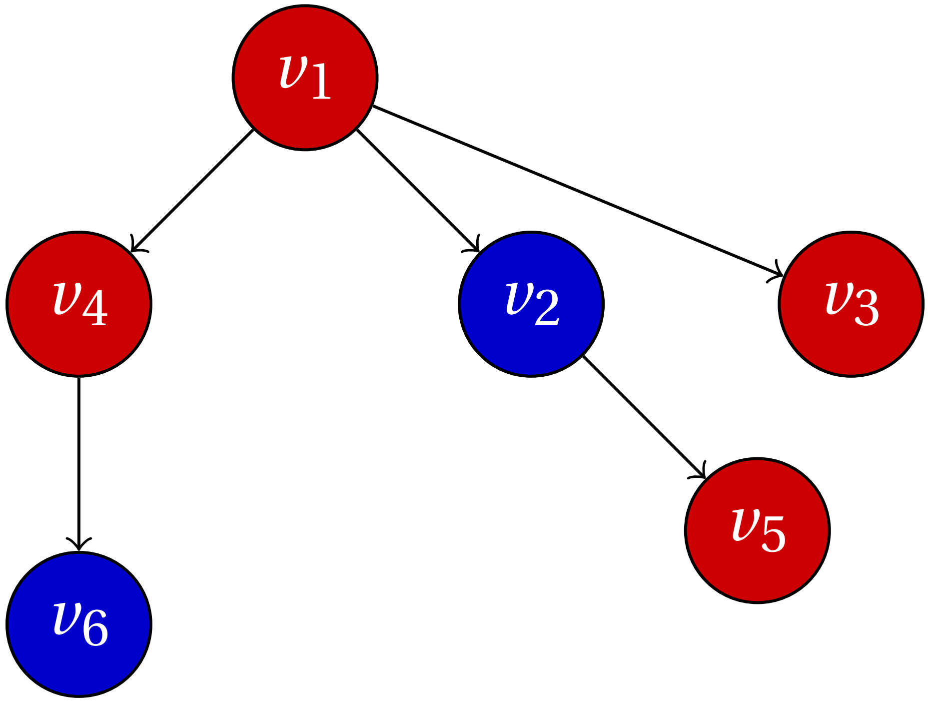

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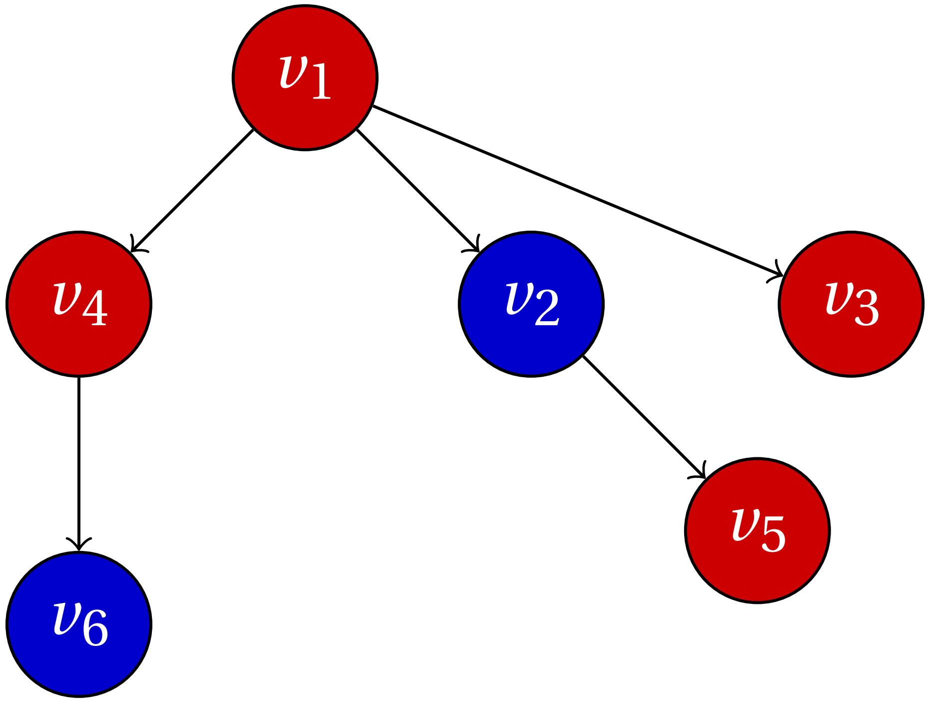

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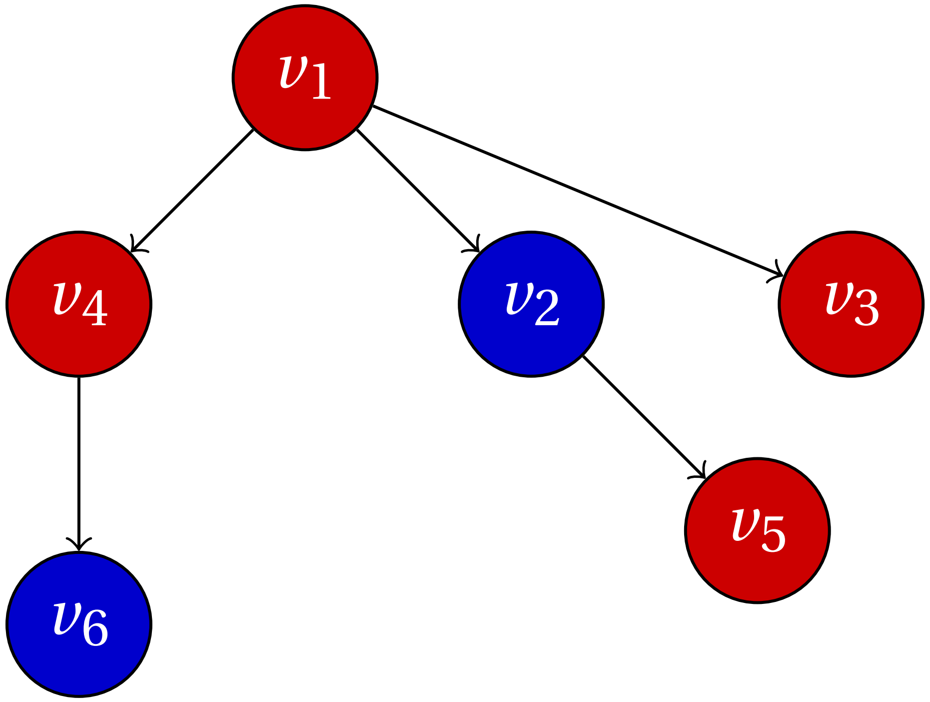

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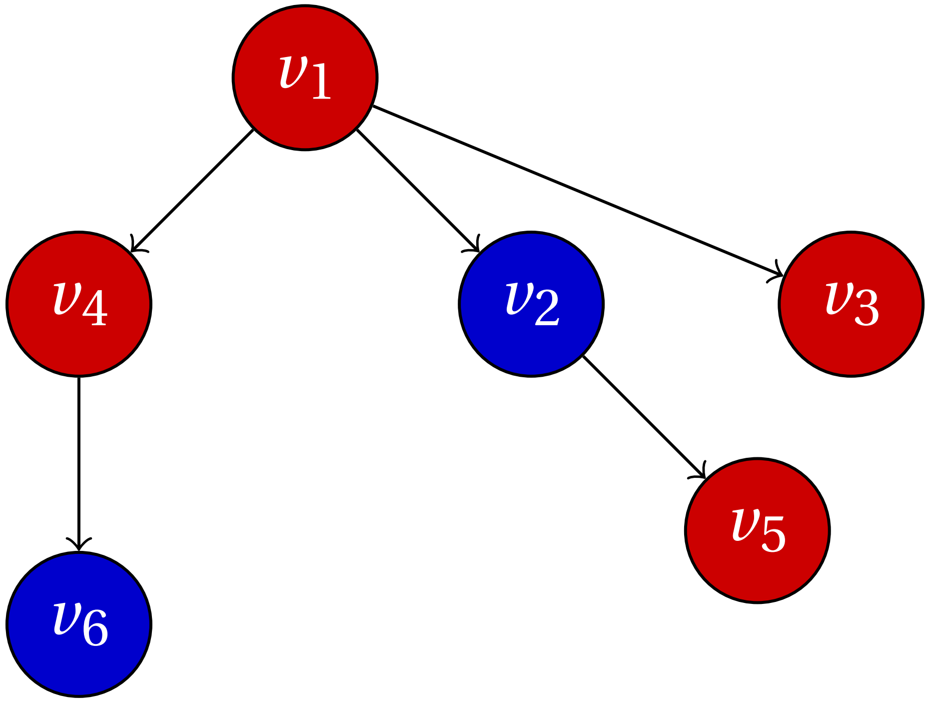

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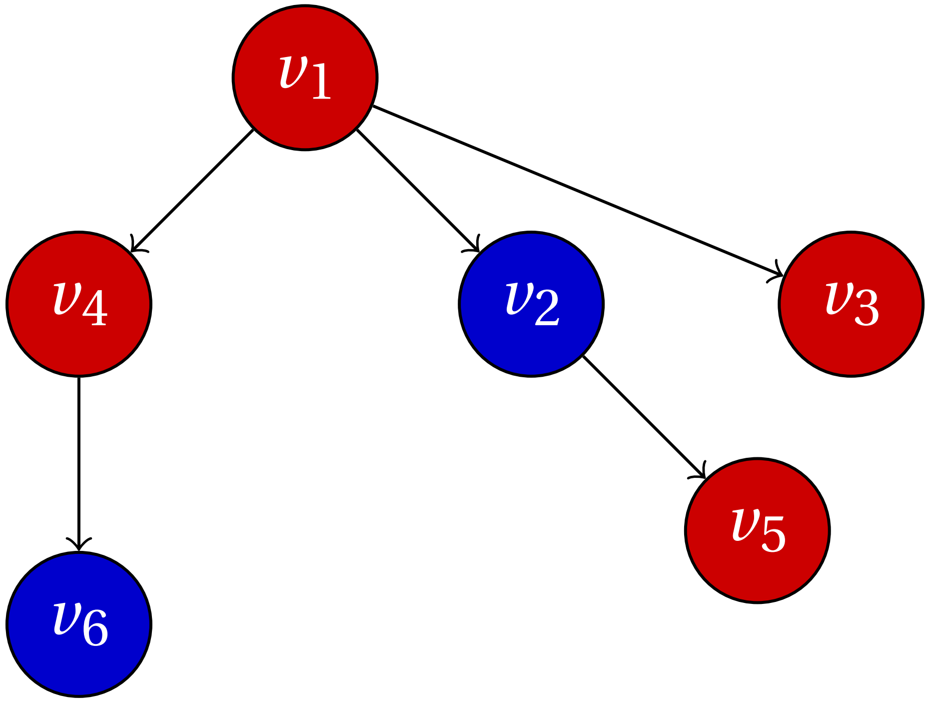

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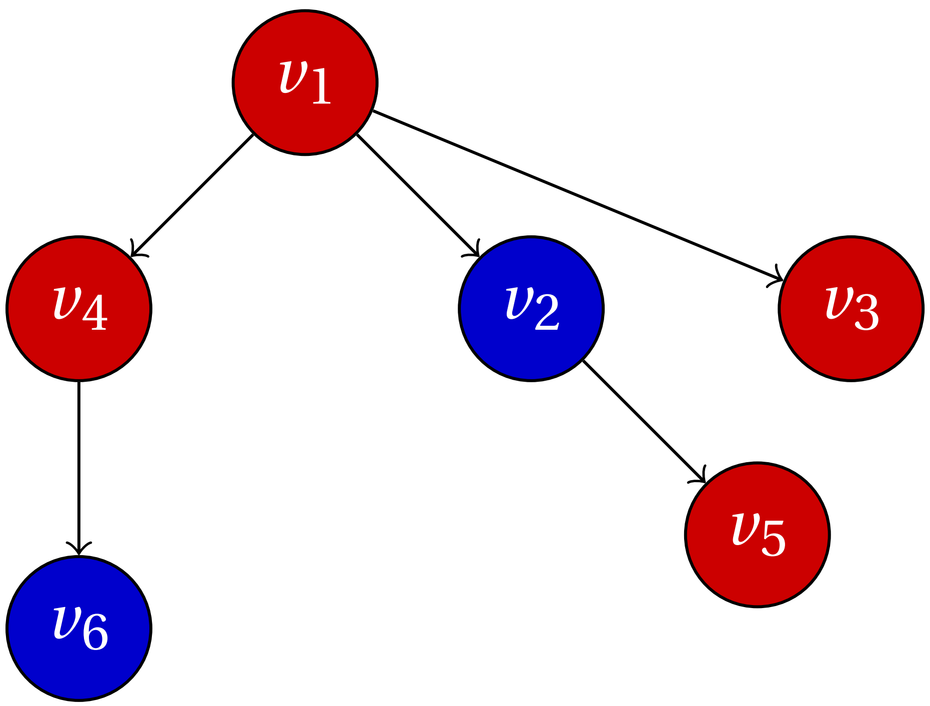

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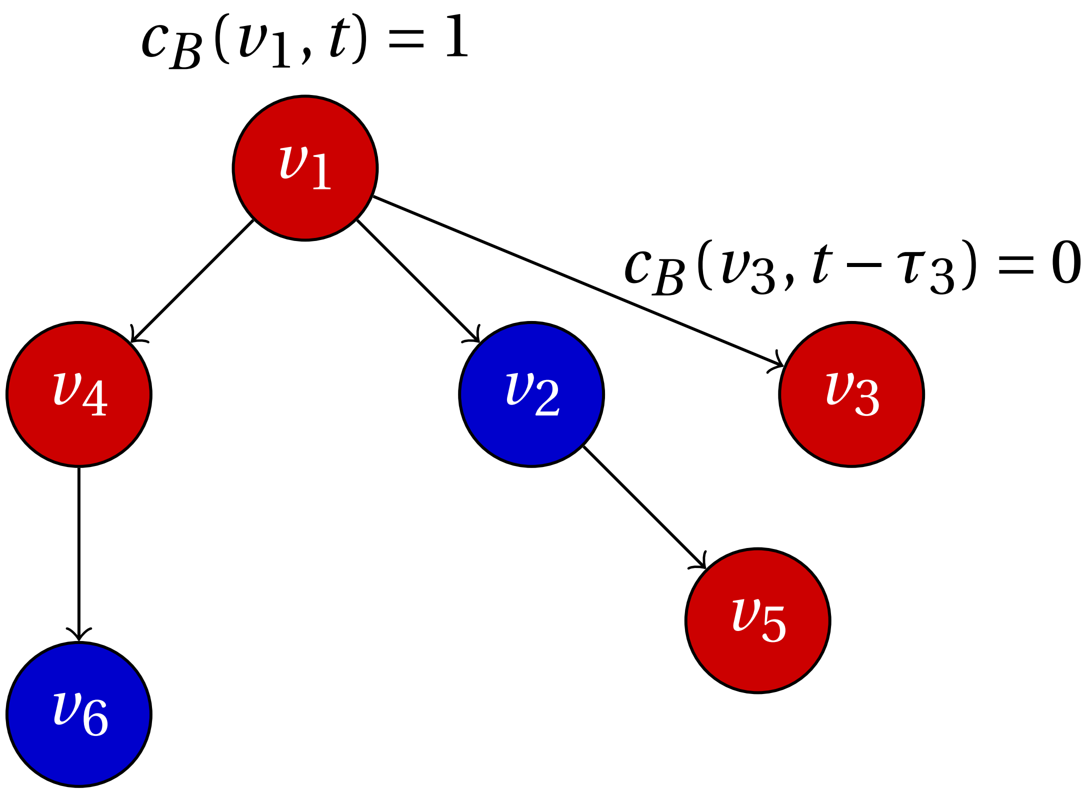

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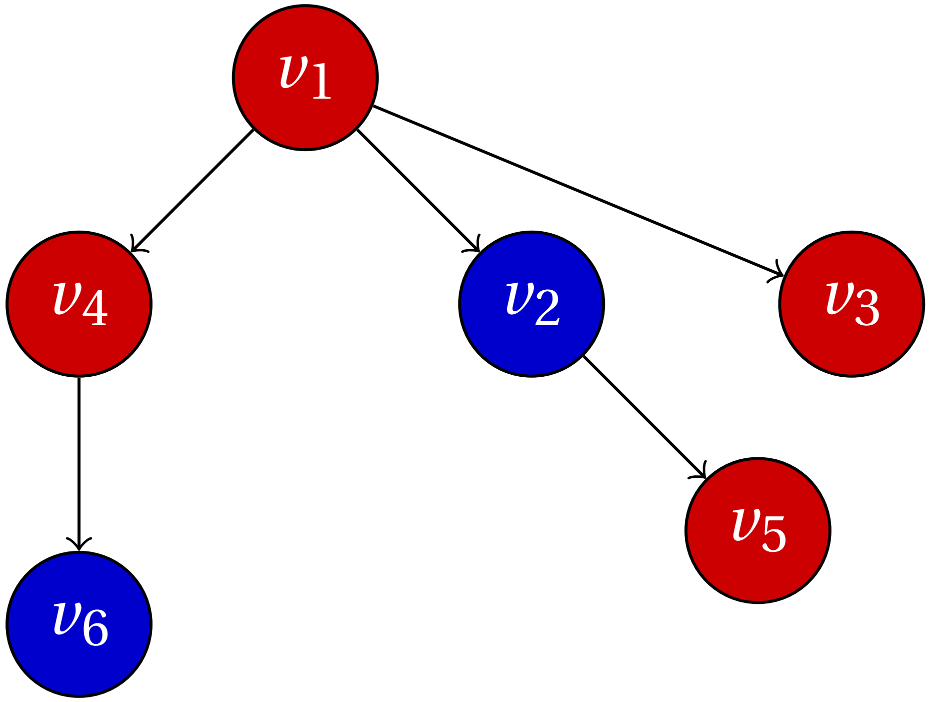

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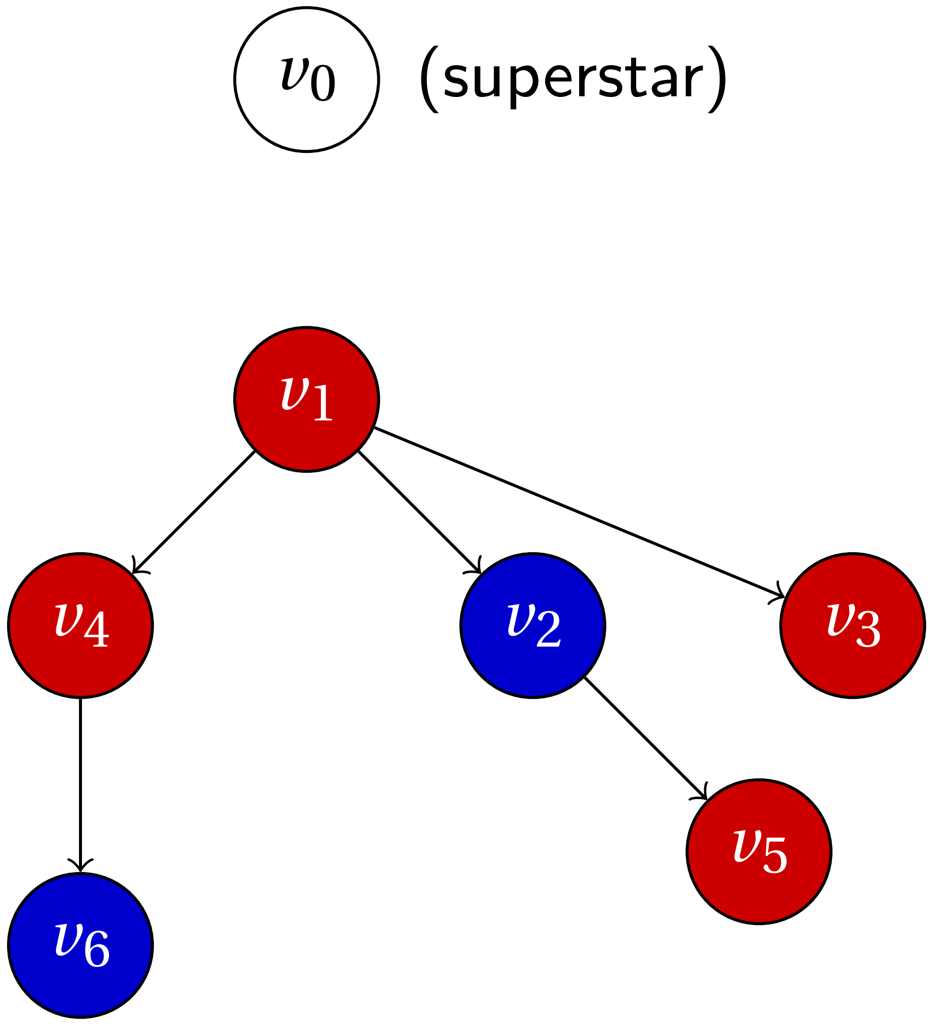

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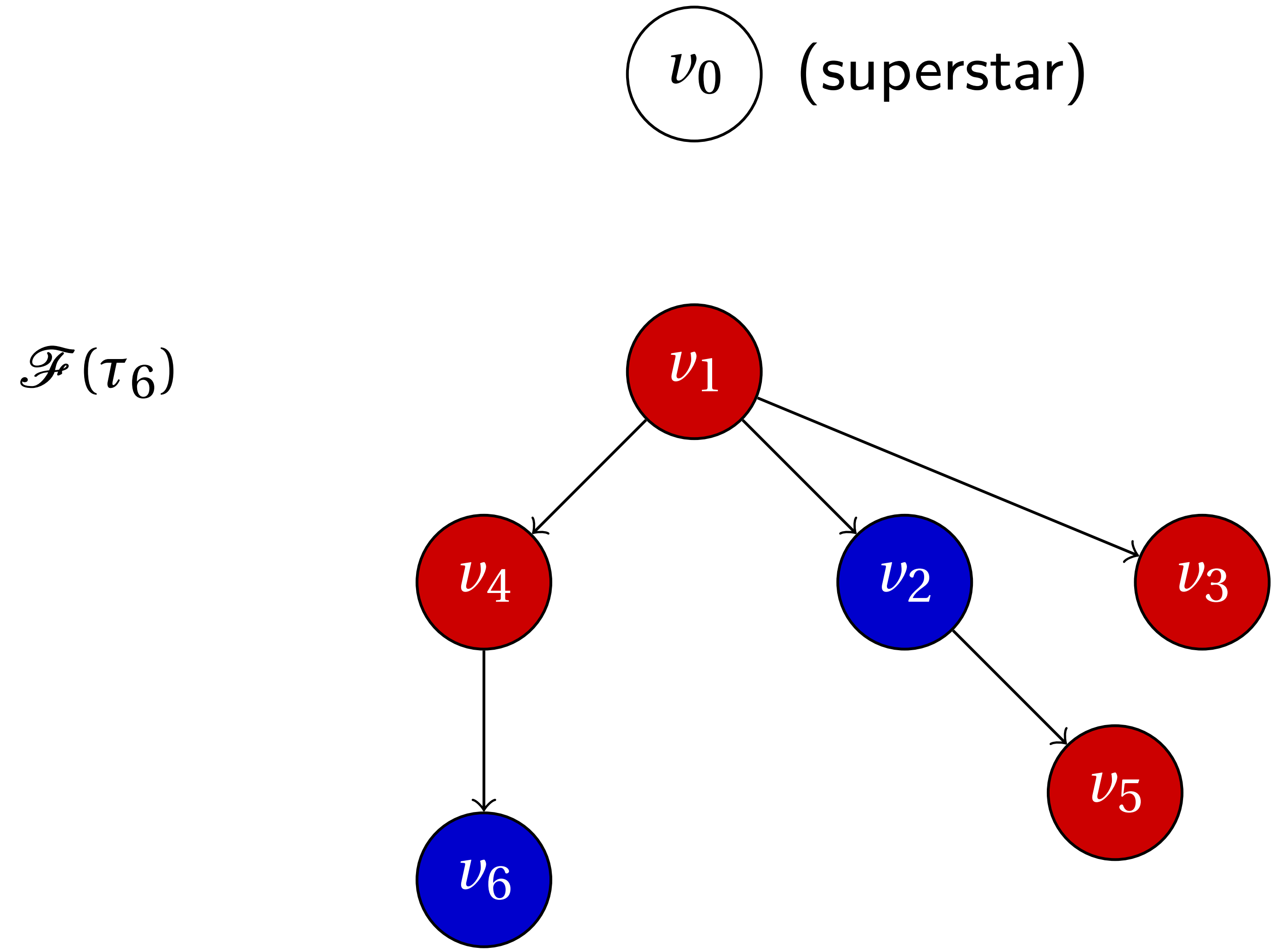

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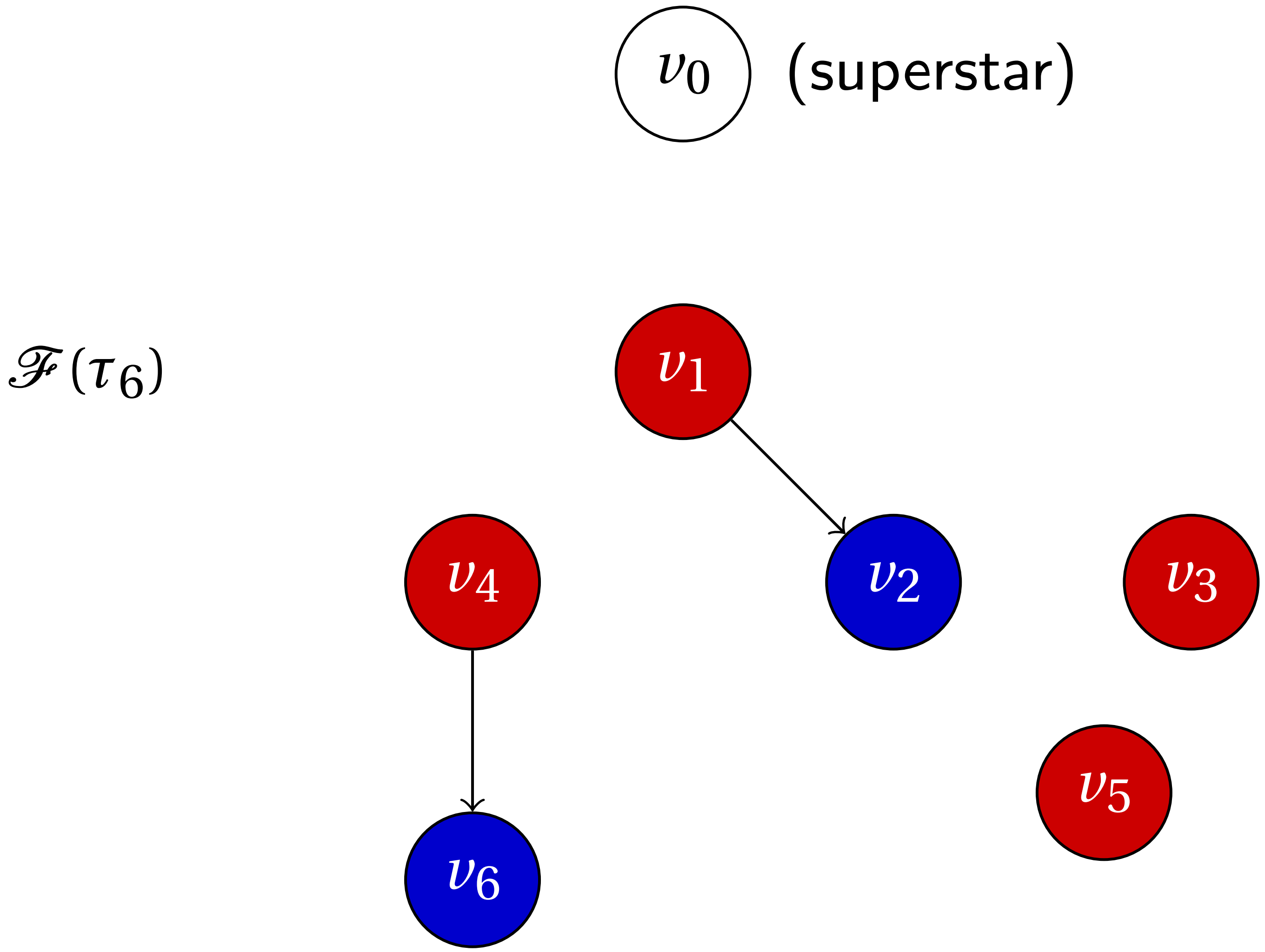

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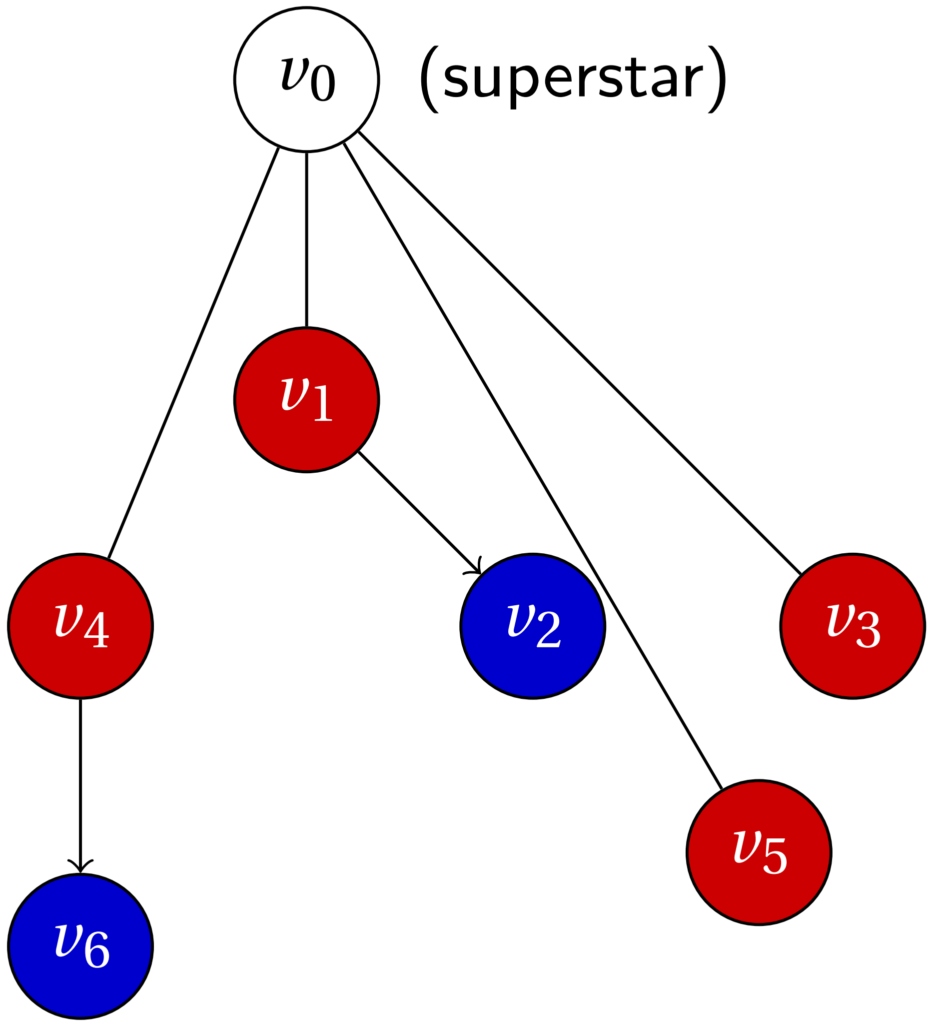

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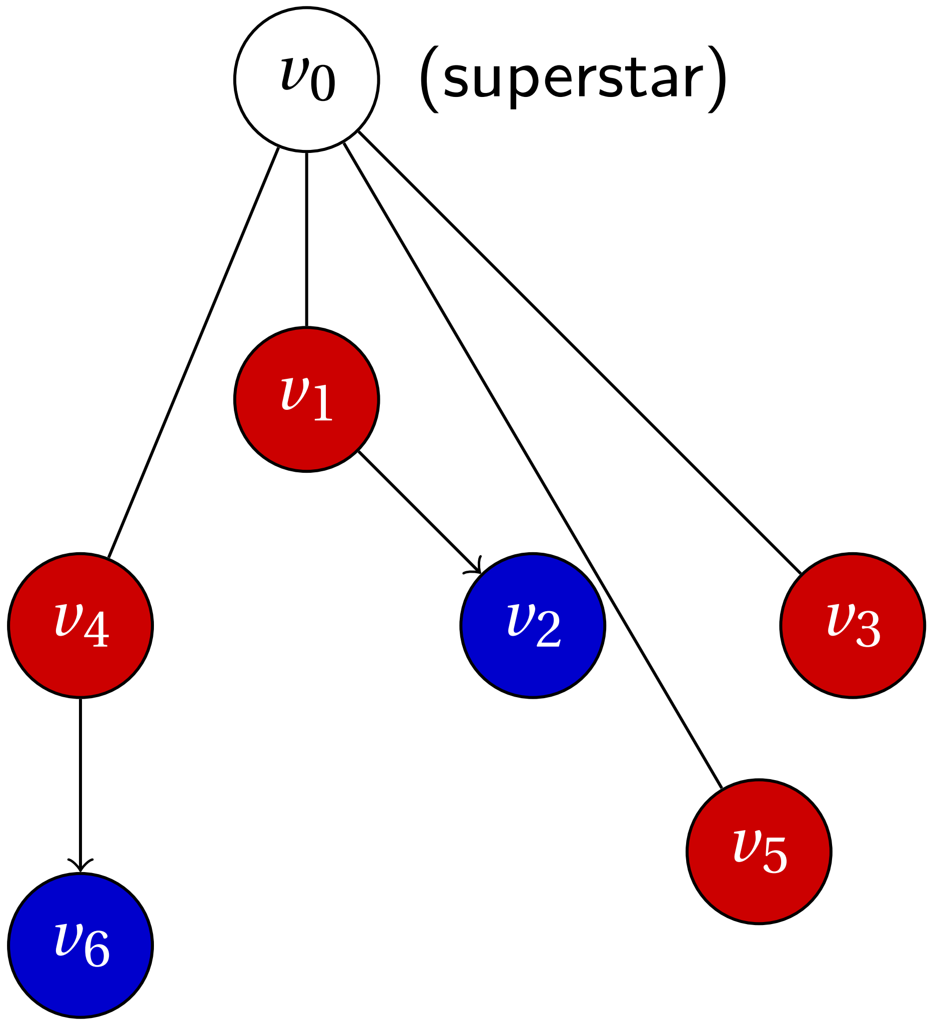

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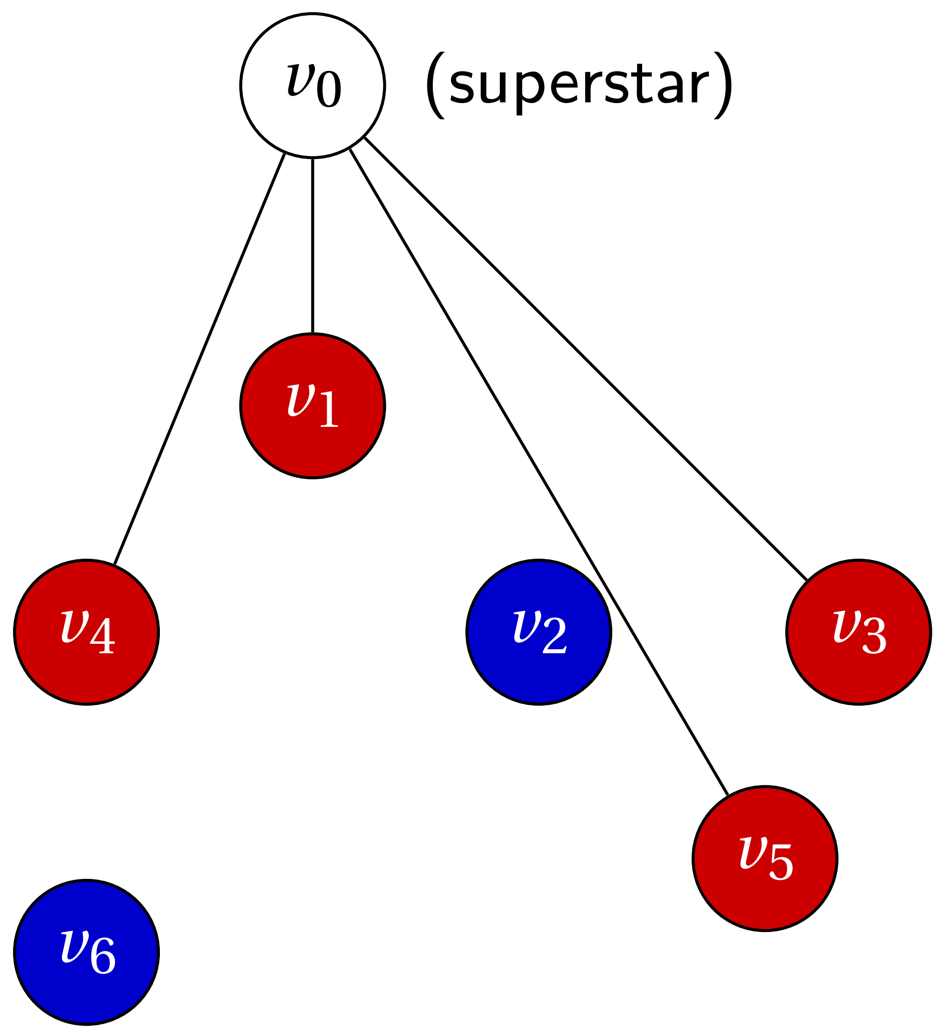

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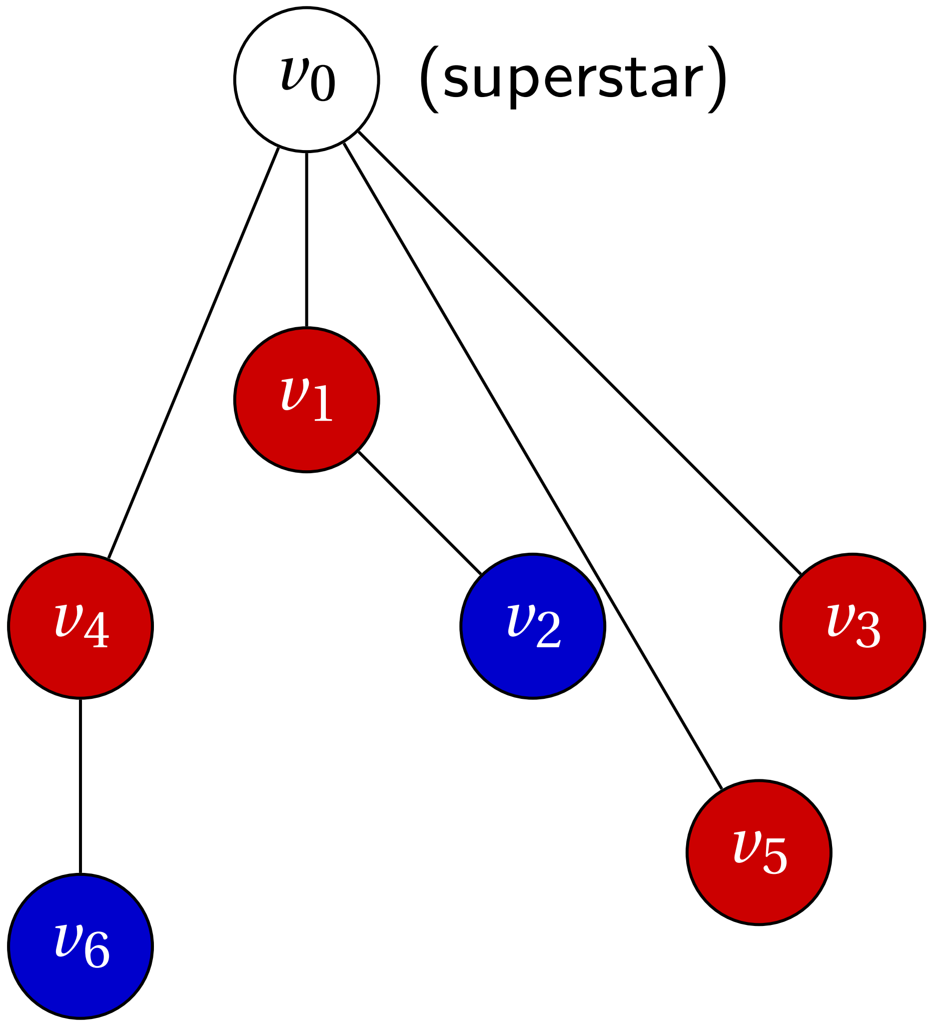

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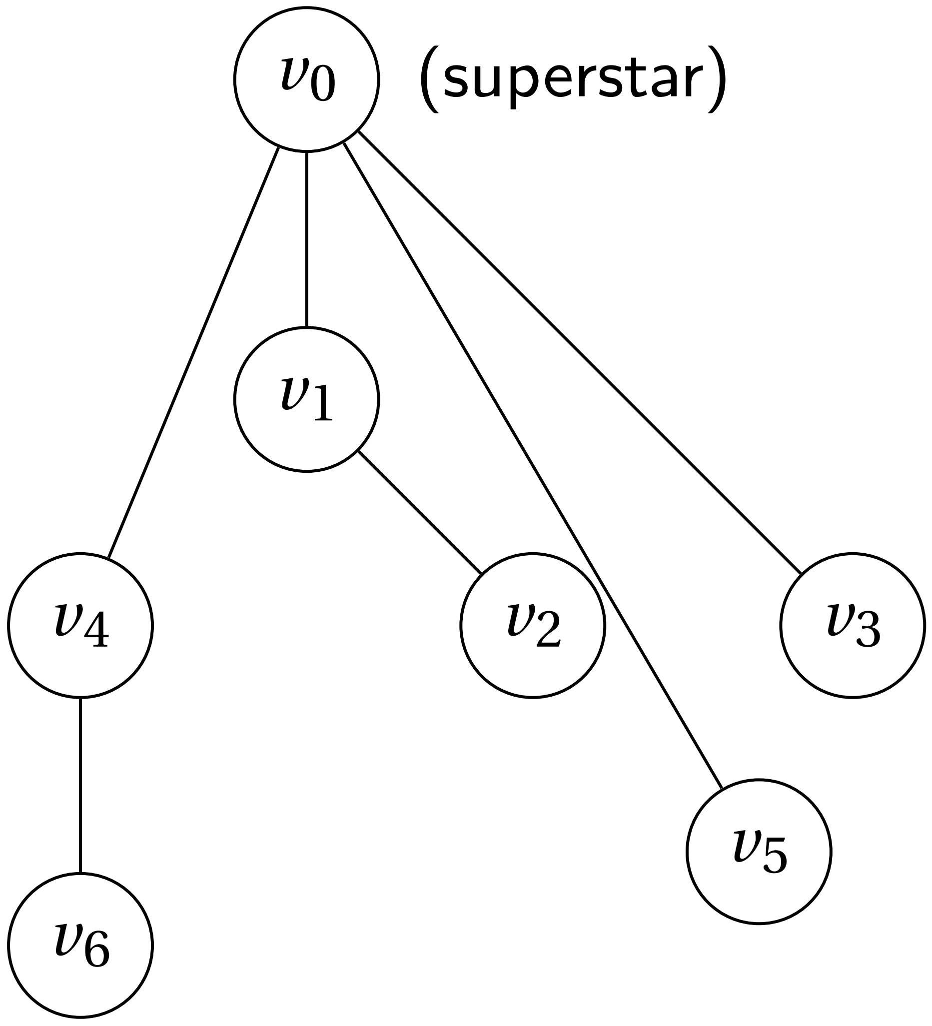

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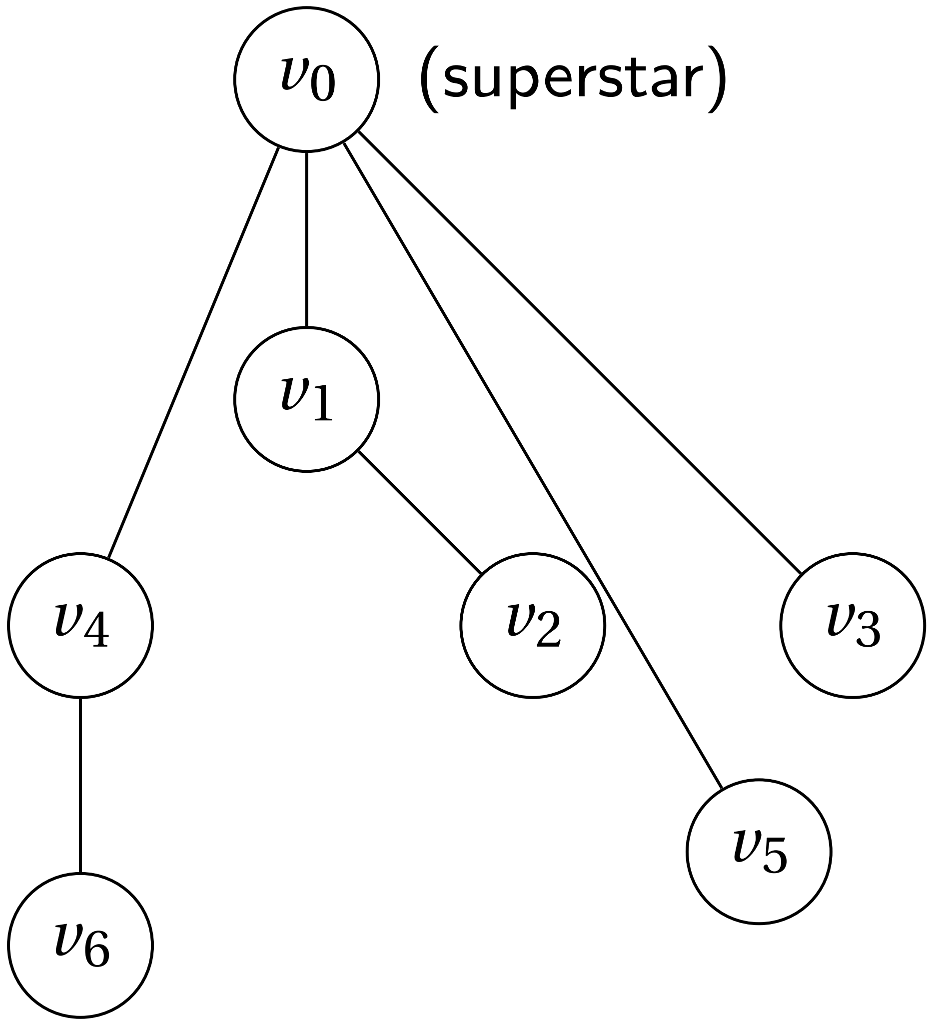

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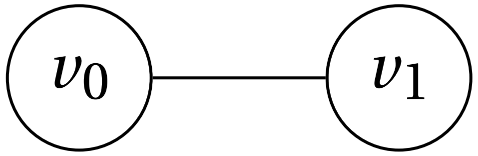

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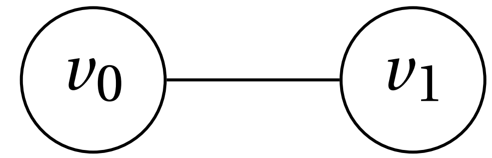

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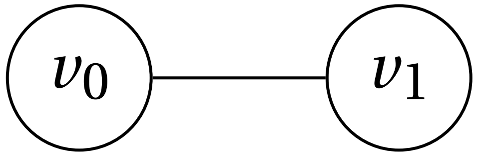

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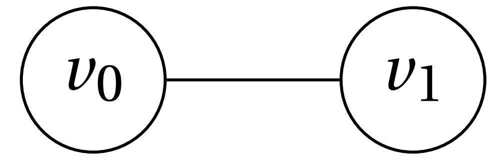

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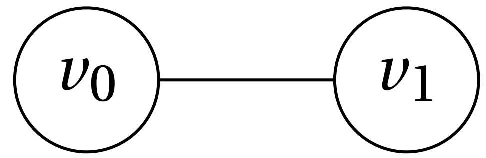

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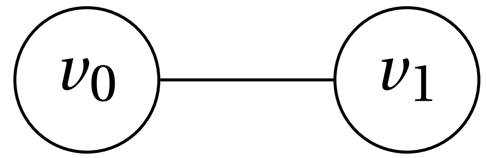

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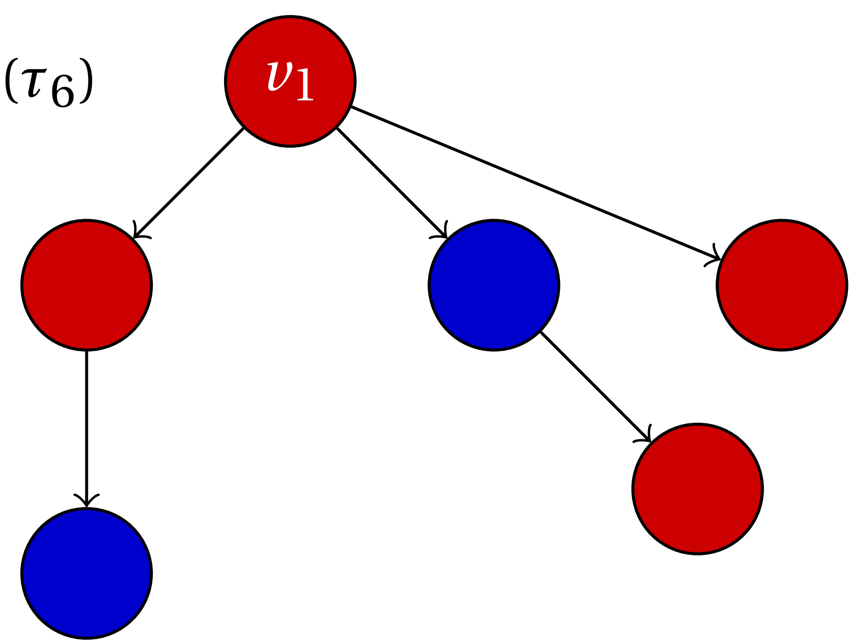
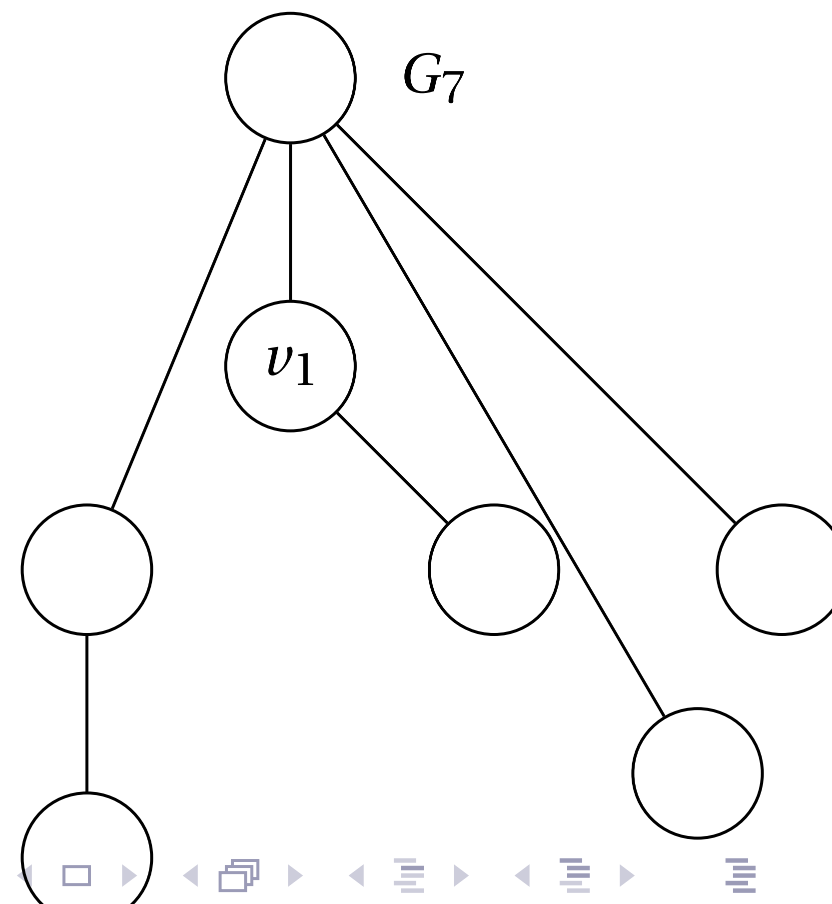
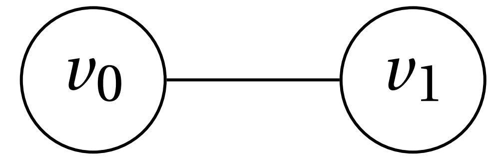




Relating the BP Construction with the Superstar Model

- Claim: $S(\tau_n)$ is "probabilistically the same" as G_{n+1}
- Base case: $S(\tau_1) = G_2$



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- Superstar: probability of joining superstar = probability of red vertex being born = p
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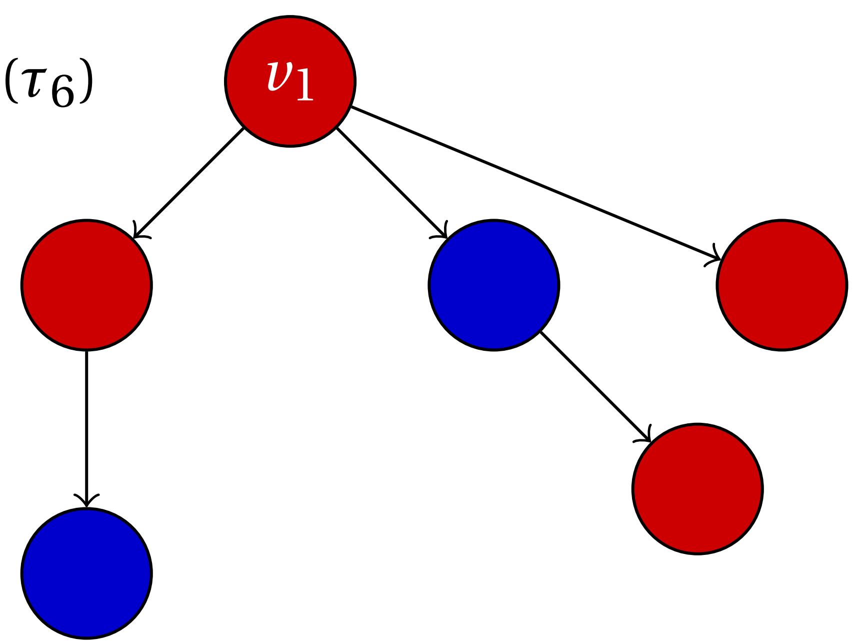
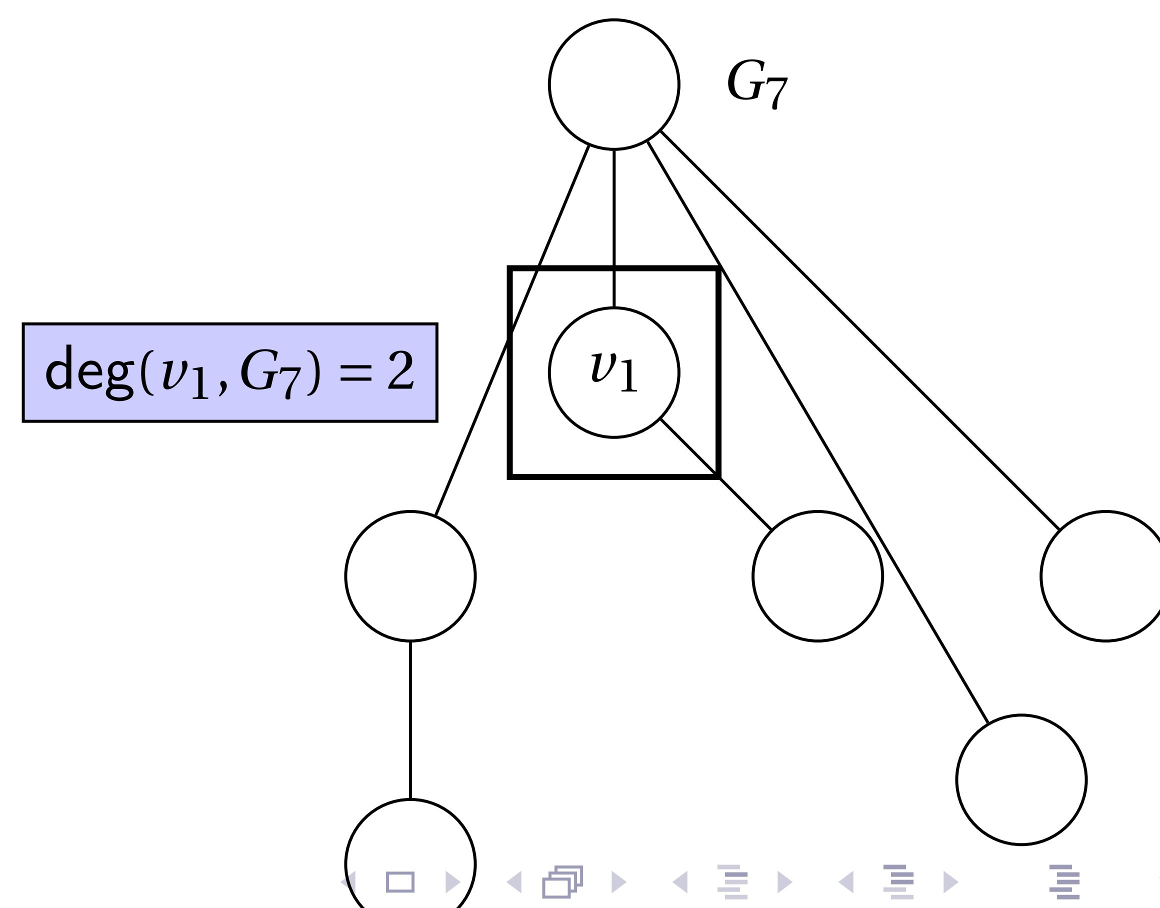
$\mathscr{F}(\tau_6)$





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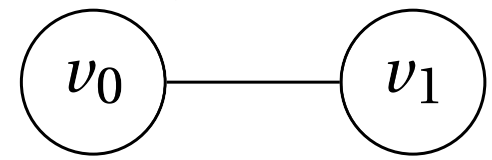

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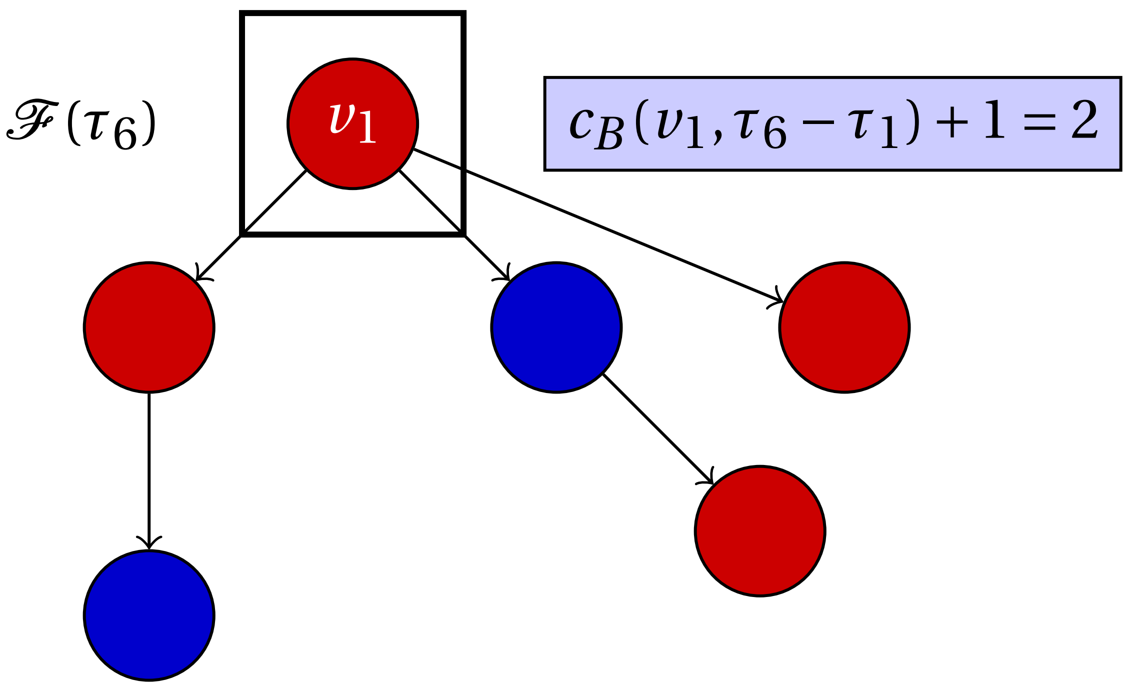

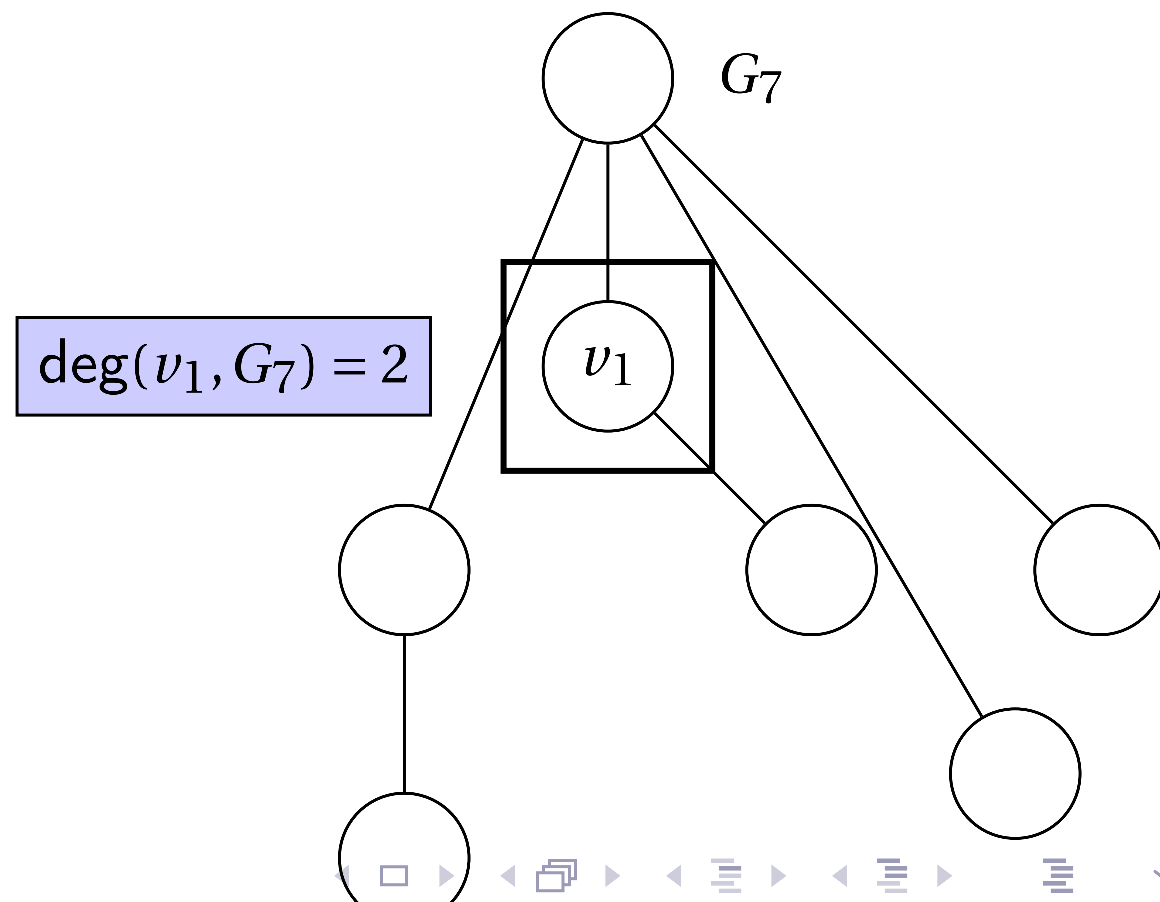

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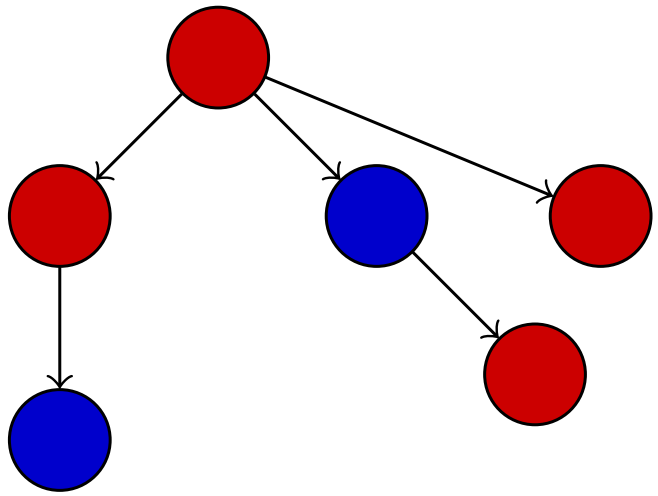

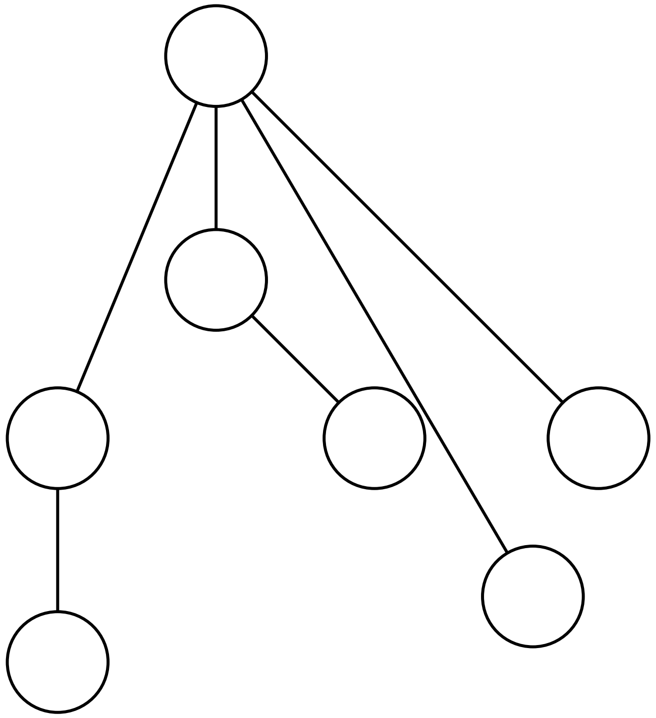

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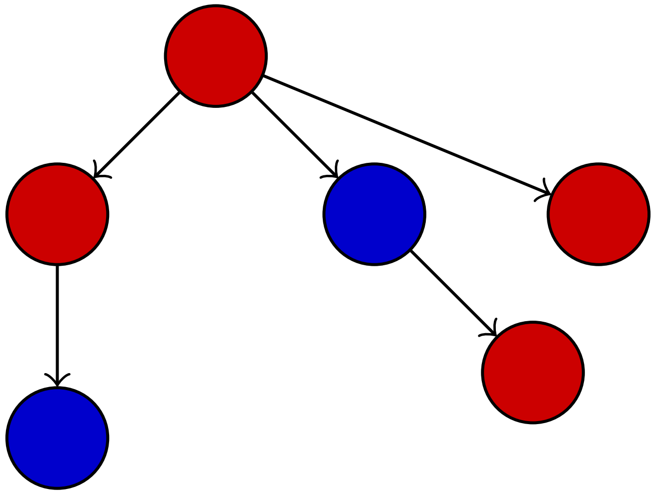

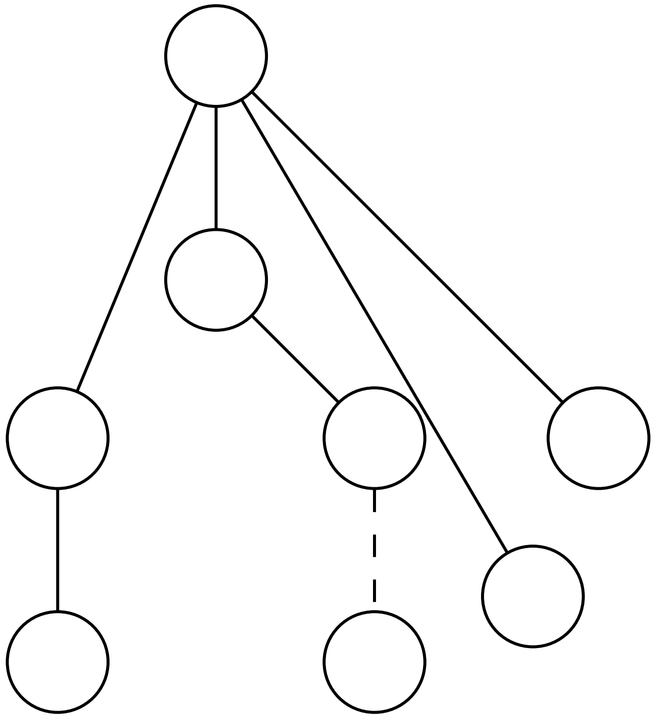

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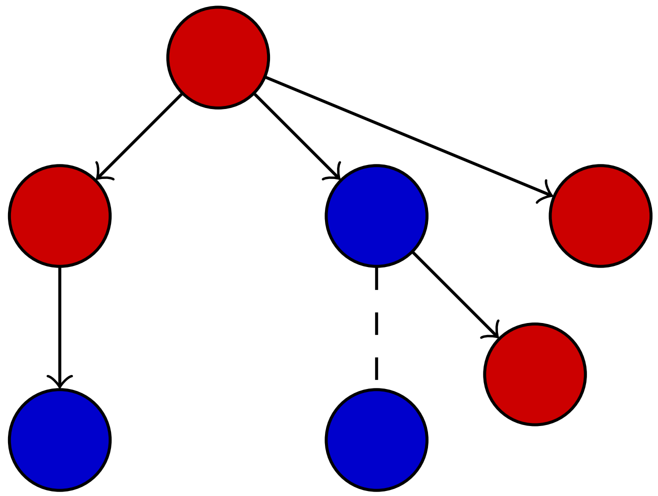

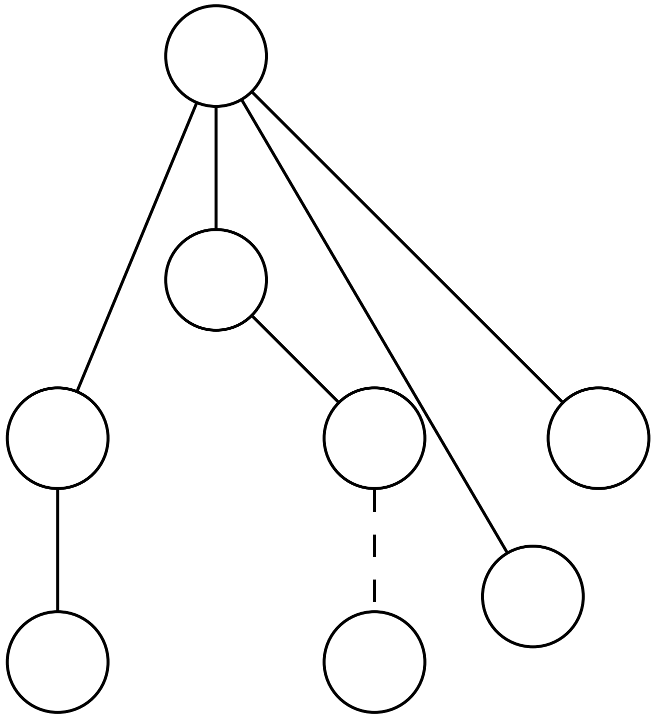

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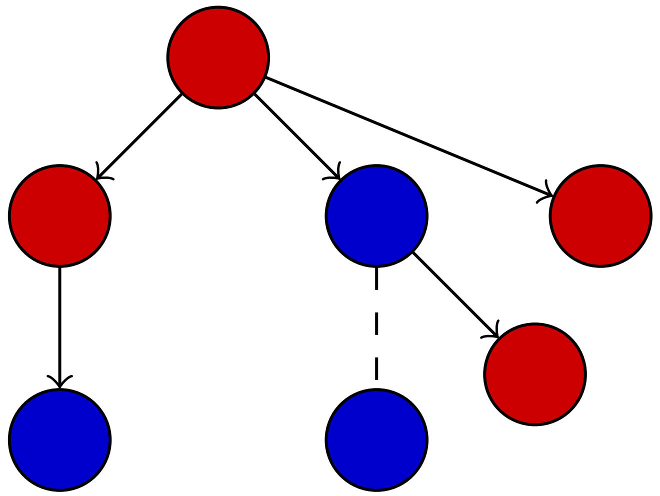

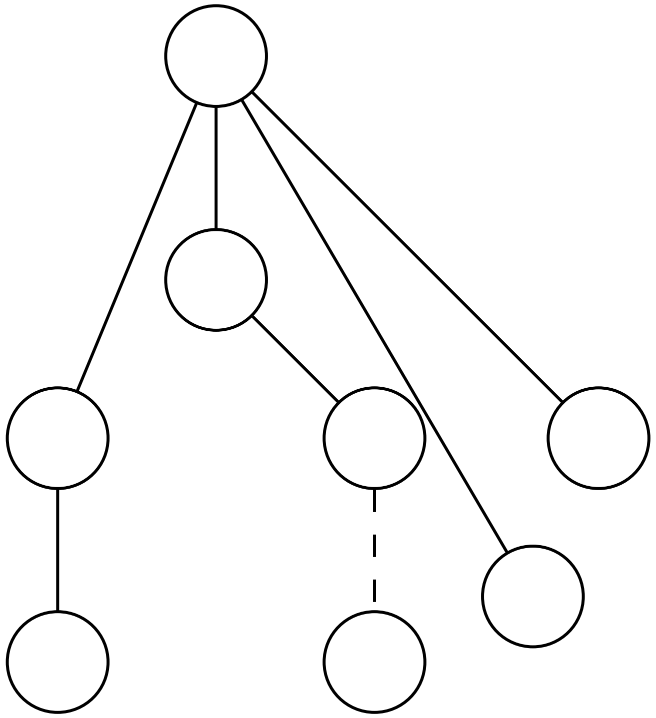

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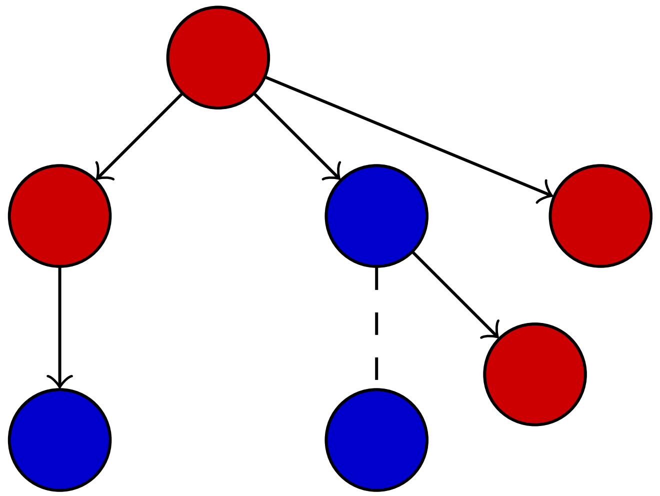

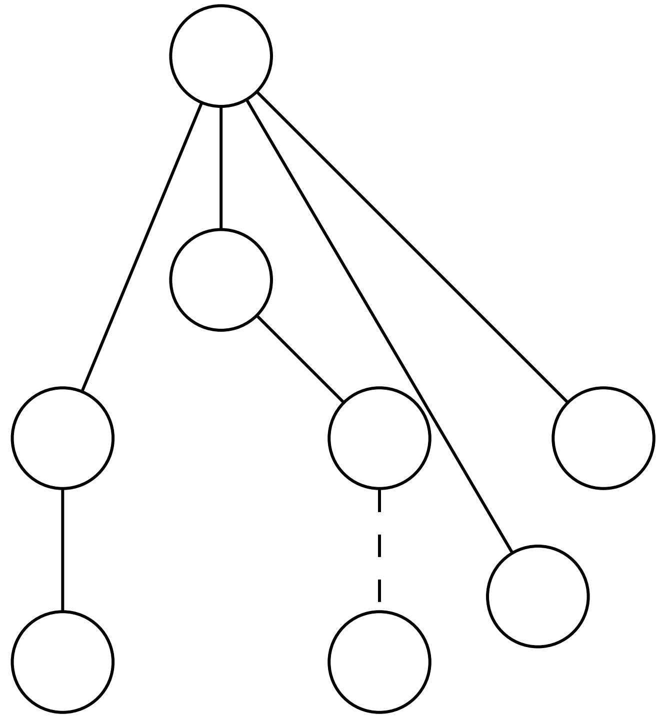

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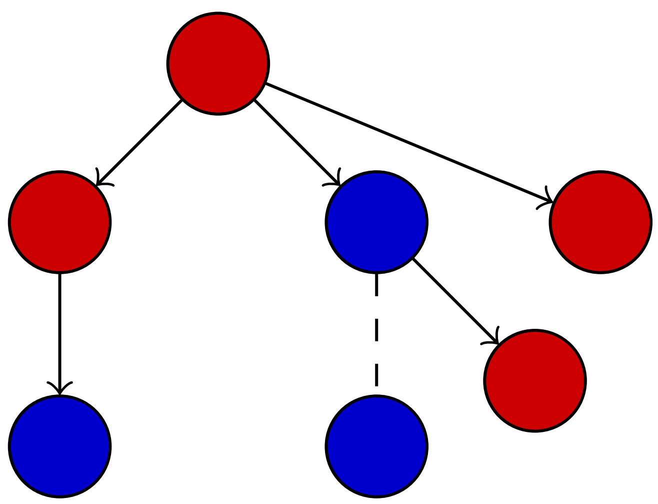

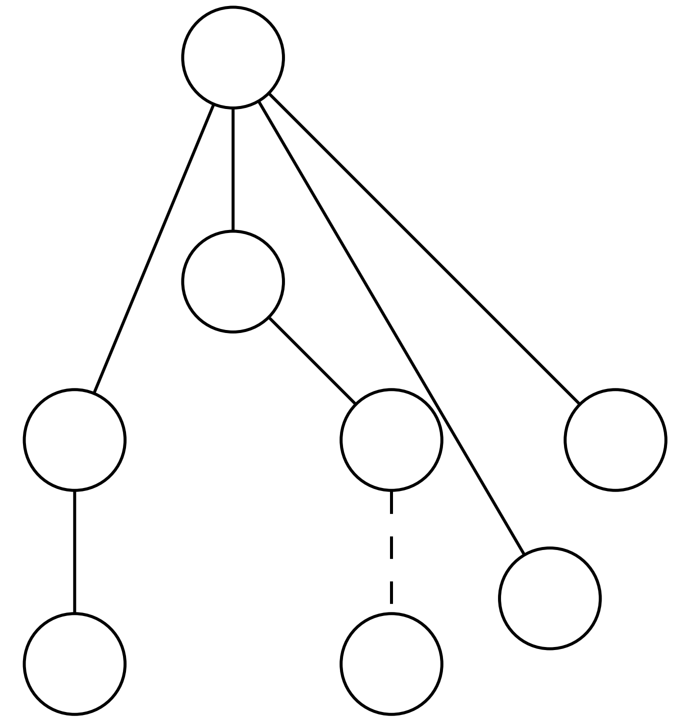

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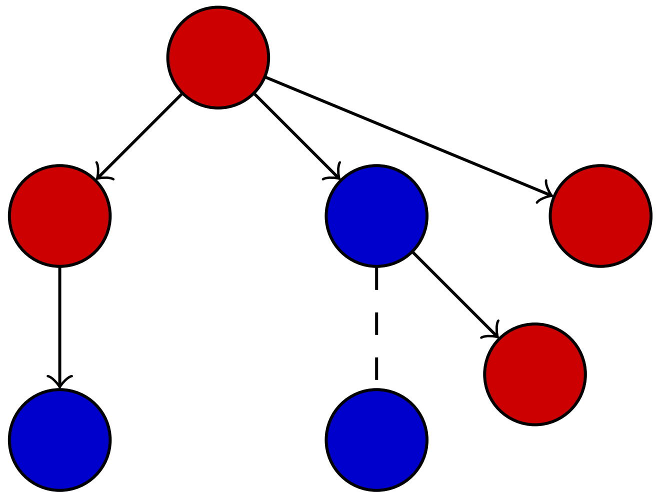

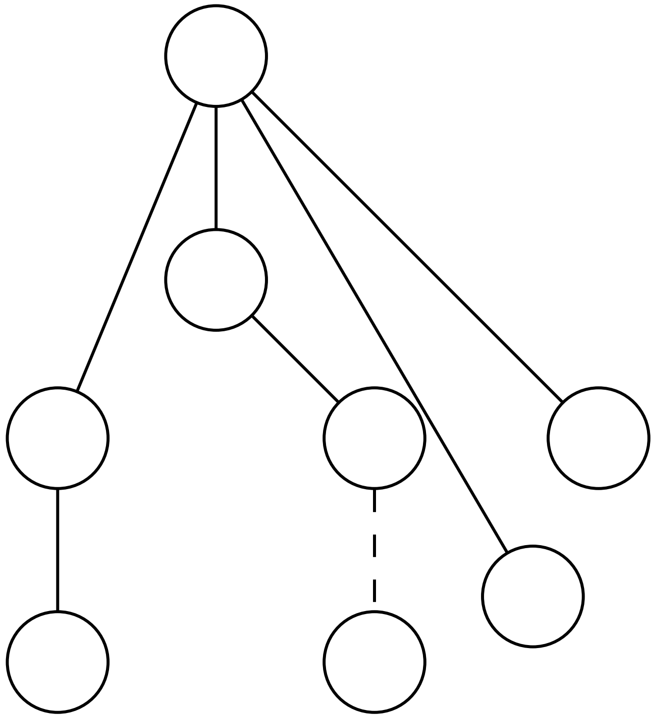

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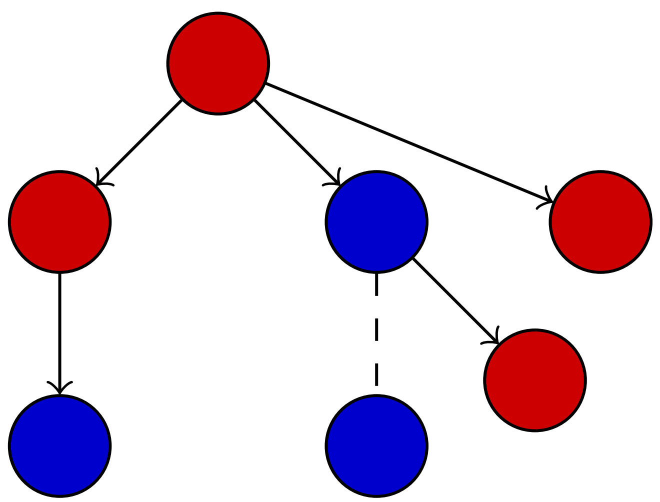

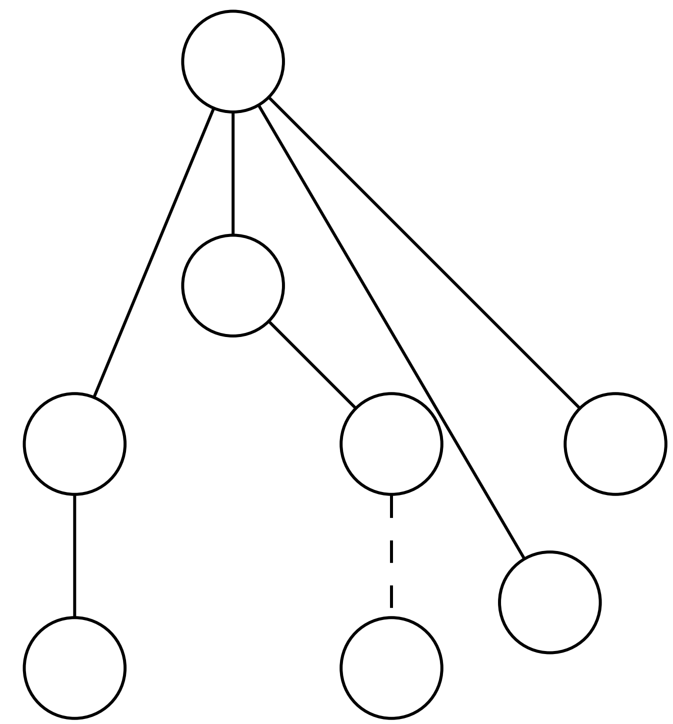

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$\deg(v_k, G_n)$

Power of choice in random trees [D'Souza, Mitzenmacher]

Model: Motivation and construction

- Usual pref. attachment: Basic assumption: every new vertex has knowledge of entire network
- Each stage new vertex chooses 2 vertices uniformly at random
- Connect to vertex with maximal degree **amongst** the ones chosen (breaking ties with probability $1/2$)
- Model which incorporates randomness as well as limited choice
- Let $\mathcal{T}_n$ denote the tree on n vertices

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Theorem (Angel, Pemantle, SB)

There exists a rooted limiting random tree $\mathscr{T}_\infty$, described by Jagers-Nerman stable age distribution theory such that such that $\mathscr{T}_n$ converges locally $\mathscr{T}_\infty$.

Main idea

- Pick a vertex uniformly at random: Chance it is a leaf in $\mathscr{T}_n$?

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- Rate one poisson process, marking each with probability $p_0/2$, time of first point: $X_0 \sim \exp(p_0/2)$
- So probability not a leaf: $1 - p_0 = \mathbb{P}(T > X_0)$

Description of the limit tree

Recursive construction of the degree

- Let p_0 limiting fraction of leaves
- Define $q_0 = p_0/2$
- Then p_0 obtained by doing the following: Let $T \sim \exp(1/2)$ and $X_0 \sim \exp(q_0)$. Then
$$1 - p_0 = \mathbb{P}(T > X_0)$$
- $$p_0 = \frac{\sqrt{5}-1}{2}$$
- General, having obtained p_k, get p_{k+1} by solving
$$1 - (p_0 + \cdots + p_{k+1}) = \mathbb{P}(X_0 + \cdots X_{k+1} > T)$$
where
$$X_{k+1} \sim \exp(p_0 + \cdots + p_k + \frac{p_{k+1}}{2})$$

Description of $\mathscr{T}_\infty$

- After having obtained p_i, let $L_i = \sum_{j=0}^{i} X_j$
- Consider the point process $\mathscr{P}_{\max} = (L_0, L_1, \ldots)$
- Define

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Theorem

- Then $\mathscr{T}_\infty$ is the Jagers-Nerman stable age distribution tree with offspring distribution $\mathscr{P}_{\max}$, age distribution $\exp(1/2)$ and time to nearest ancestor $\nu_{\max}$
- Implies convergence of global functionals as well such as the spectral distribution of adjacency matrix

Dynamic random graphs

- Lots of interesting questions
- Understanding what happens for general **unbounded size** rules such as product rule (*explosive percolation*).
- Small variants of standard models turn out to be technically much more challenging, requiring the development of new machinery.
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Next lecture

Back to critical random graphs: suppose we were interested in the metric structure of maximal components. What can we say? Why should one care.