Lecture 2: Dynamic network models
Probabilistic and statistical methods for networks
Berlin Bath summer school for young researchers

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Motivation

Preferential attachment

- Last few years: enormous interest in formulating models to “explain” real-world networks (e.g. network of webpages, the Internet, social networks, gene regulatory networks etc).
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- Main thrust: asymptotic information on the degree distribution.
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- What if we wanted asymptotics for Global characteristics e.g. spectral distribution of adjacency matrix?
- How to analyze variants such as limited choice or non-local preferential attachment. Analysis? Performance in practice?
Outline of the talk

1. Preferential attachment model
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2. Continuous time branching processes
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3. Local weak convergence
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5. Variants I: Superstar model (Mike Steel, Tauhid Zaman, SB)
6. Variants II: Preferential attachment with choice (Omer Angel, Robin Pemantle, SB)
Basic model of growing trees

Setting

- At time 2 start with two vertices labelled with $[2] := \{1, 2\}$ connected by single directed edge $1 \rightarrow 2$. 

Attractiveness function: Assume we are given a (possibly random) function $f(v, n): V_n \to \mathbb{R}^+$. 

Dynamics: At time $n+1$, new model labelled $n+1$ enters system. Node $n+1$ attaches to node in $T_n$ with probability proportional to $f(v, n)$. Most examples we consider: 

$$f(v, n) = f(D(v, n))$$ 

where $f : \{0, 1, \ldots\} \to (0, 1)$ is a fixed function. 

$D(v, n)$ is the out-degree of node $v$ at time $n$. 

Shankar Bhamidi Lecture 2
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- Let \(\mathcal{T}_n = (\mathcal{V}_n := [n], \mathcal{E}_n)\) be the tree at time \(n\).
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- At time \(n + 1\), new node labelled \(n + 1\) enters system.
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- $D(v, n) = \text{out-degree}$ of node $v$ at time $n$.  

Preferential attachment: Example $\mathcal{T}_3$
Preferential attachment: Example
Preferential attachment: Example

\[ f(0) \rightarrow 3 \rightarrow \rho \rightarrow f(2) \]

\[ 4 \rightarrow f(0) \rightarrow 2 \]

\[ \Omega_f(2) \]

\[ f(0) \]
Preferential attachment: Example $T_4$

$\rho$

$f(2)$

$\begin{array}{c}
4 \\
3 \\
2 \\
\end{array}$

$f(0)$

$f(1)$
Examples of attractiveness functions

Attachment trees

**YSBA (Yule-Simon-Barabasi-Albert) model:** $f(k) = k + 1$
Examples of attractiveness functions

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2. **Linear Preferential attachment model:** \( f(k) = k + 1 + \alpha, \ \alpha > 0 \)
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3. **Random fitness models** Every new vertex $v$ given a fitness $f_v \sim v$ (independent across vertices).
   - (a) **Multiplicative fitness:** $f(k) = f_v(k + 1)$.
   - (b) **Additive fitness:** $f(k) = k + 1 + f_v$. 
## Examples of attractiveness functions

### Attachment trees

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4. **Sublinear Pref Attachment**: \[ f(k) = (k + 1)^\alpha, \ 0 < \alpha < 1 \]
Main math idea

Simple idea [Karlin-Athreya]

- Suppose we have vertex set \( \{1, 2, \ldots, m\} \) with associated (strictly positive) weights \( \{d_1, d_2, \ldots, d_m\} \).
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- Let \( J \) be index
  \[
  X_J = \min_{1 \leq i \leq m} X_i.
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- Then \( \mathbb{P}(J = i) \propto d_i \).
Outline

1. **Preferential attachment: Base model**
   - Continuous time construction
   - Local weak convergence

2. **Twitter event networks and the superstar model**
   - Retweet Graph and Superstar Model
   - Main Results
   - Comparison with Preferential Attachment Model
   - Superstar Model: Tools for Analysis

3. **Power of choice and random trees**

4. **Conclusion**
Point process corresponding to attractiveness function $f$

- $\mathcal{P}$ is Markov pure birth process with rate description

$$\mathbb{P}(\mathcal{P}(t, t + dt] = 1|\mathcal{P}(t) = k) = f(k) dt.$$
Point process corresponding to attractiveness function \( f \)

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- For example, for \( f(k) = k + 1 \) (usual preferential attachment model) we get the Yule process.
Continuous time construction

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Continuous time branching process $\mathcal{F}(t)$

1. Start with a single node at time 0 giving birth to children at times of $\mathcal{P}$.
2. Each node born behaves in the same manner (has it’s own independent point process of births).
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Key connection

\[ \tau_n = \inf\{t : \mathcal{F}(t) = n\} \text{ then } \mathcal{F}(\tau_n) \overset{d}{=} \mathcal{T}_n^f. \]
Continuous time and discrete time in pictures

```
\begin{align*}
\rho & \quad 2 \quad 3 \quad 5 \\
& \quad 4 \\
& \quad 6 \\
\end{align*}
```

```
\begin{align*}
\rho (1) & \quad 2 \quad 3 \quad 5 \\
& \quad 4 \\
& \quad 6 \\
\end{align*}
```

```
\text{time} \quad \tau_1 \quad \tau_2 \quad \tau_3 \quad \tau_4 \quad \rightarrow \quad \tau_{15}
```
Asymptotics

Conjectured by Euler. Developed by Jagers and Nerman.

- Processes grow exponentially: $|\mathcal{F}(t)| \sim e^{\lambda t}$
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- Here $\lambda$ is a very important characteristic: called the *Malthusian rate of growth*
Branching process theory

Asymptotics

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- Processes grow exponentially: $|\mathcal{F}(t)| \sim e^{\lambda t}$
- Here $\lambda$ is a very important characteristic: called the Malthusian rate of growth
- Given by the formula:
  $$\mathbb{E}(\mathcal{P}(T_\lambda)) = 1,$$
  where $T_\lambda \sim \exp(\lambda)$ independent of $\mathcal{P}$.

Exact result

Under technical conditions ($\mathbb{E}(\mathcal{P}(T_\lambda), \log^+ \mathcal{P}(T_\lambda)) < \infty$):

- $|\mathcal{F}(t)| e^{-\lambda t} \xrightarrow{a.s.} W$.
- In our settings: $W > 0$ a.s.
- Bottom line:
  $$\tau_n \sim \frac{1}{\lambda} \log n \pm O_P(1)$$
Case Study: Usual preferential attachment

\[ f(k) = k + 1 \]

- Offspring distribution: \( \mathcal{P}(\cdot) = \text{Yule process} \)
## Case Study: Usual preferential attachment

$$f(k) = k + 1$$

- Offspring distribution: $\mathcal{P}(\cdot) = \text{Yule process}$
- Malthusian rate of growth: $\lambda = 2$. 
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Root degree asymptotics

- Degree of the root = \( \mathcal{P}_\rho(\tau_n) \)
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- \( \tau_n \sim \frac{1}{2} \log n + O_P(1) \). Yule process also grows exponentially: \( \mathcal{P}(t) \sim e^t \)
- \( \deg_n(\rho) = \mathcal{P}(\frac{1}{2} \log n + O_P(1)) \sim O_P(e^{\frac{1}{2} \log n}) = O_P(\sqrt{n}) \)
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- Degree of the root = \( \mathcal{P}_\rho(\tau_n) \)
- \( \tau_n \sim \frac{1}{2} \log n \pm OP(1) \). Yule process also grows exponentially: \( \mathcal{P}(t) \sim e^t \)
- \( \deg_n(\rho) = \mathcal{P}(\frac{1}{2} \log n + OP(1)) \sim OP(e^{\frac{1}{2} \log n}) = OP(\sqrt{n}) \)
- More refined analysis (Mori/Pekoz+Rollin+Ross) gives

\[
\frac{\deg_n(\rho)}{\sqrt{n}} \xrightarrow{a.s.} Z \quad Z \text{ has explicit recursive construction.}
\]
Maximal degree

Basic heuristics for models with “heavy tails”

- Due to exponential growth of the models in the natural “time scale”, maximal degree occurs in a finite neighborhood of the root.
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- Due to exponential growth of the models in the natural “time scale”, maximal degree occurs in a finite neighborhood of the root.
- For usual preferential attachment model, using explicit distributional properties of Yule process easy to conclude, for any given $\epsilon > 0 \exists K_\epsilon$,

$$\limsup_{n \to \infty} \mathbb{P} \left( \frac{\max\deg n}{\sqrt{n}} > K_\epsilon \right) < \epsilon$$
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- For usual preferential attachment model, using explicit distributional properties of Yule process easy to conclude, for any given $\epsilon > 0 \exists K_\epsilon$,
  \[
  \limsup_{n \to \infty} P \left( \frac{\text{maxdeg}_n}{\sqrt{n}} > K_\epsilon \right) < \epsilon
  \]
- With a bit more work, possible to deduce distributional convergence for the maximal degree.
- Example of interesting results: Sublinear pref attachment $f(k) = (k + 1)^\alpha$
  \[
  \frac{\text{deg}_n(\rho)}{(\log n)^{1-\alpha}} \xrightarrow{P} \left( \frac{1}{\theta(\alpha)} \right)^{\frac{1}{1-\alpha}}
  \]
Height asymptotics

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Kingman’s result and Pittel’s “proof from the book”

Let \( B_k := \text{first time that an individual in the } k^{th} \text{ generation (namely an individual at graph distance } k \text{ from the root) is born.} \)
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[Kingman]: There exists a (model dependent) limit constant \( \gamma \) such that:

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\frac{B_k}{k} \xrightarrow{a.s.} \gamma_{\text{model}}
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For us we have

$B_{h_n} \leq T_n \leq B_{h_n+1}$
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Thus

$$\frac{B_{h_n}}{h_n} \leq \frac{\tau_n}{h_n} \leq \frac{B_{h_n+1}}{h_n}$$

Now use the fact [Pittel's argument] that

$$\frac{\tau_n}{\frac{1}{\lambda} \log n} \xrightarrow{a.s.} 1 \Rightarrow \frac{h_n}{\log n} \xrightarrow{p} C_{\text{model}}$$
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Local asymptotics (Jagers+Norman/Aldous)

Conceptual point

- Construction of infinite (locally finite) rooted trees with a single infinite path. (Trees with one end).
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Starting point: Age of an individual

\[ \mathbb{P}(\text{Age}(V_t) > 10|\mathcal{F}(t)) = \frac{|\mathcal{F}(t - 10)|}{|\mathcal{F}(t)|} \]
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\[ \mathbb{P}(\text{Age}(V_t) > 10 \mid \mathcal{F}(t)) = \frac{|\mathcal{F}(t-10)|}{|\mathcal{F}(t)|} \sim \frac{W e^{\lambda(t-10)}}{W e^{\lambda t}} \]
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**Conceptual point**

- Construction of infinite (locally finite) rooted trees with a single infinite path. (Trees with one end).
- Local neighborhood of a random node asymptotically looks like neighborhood of the *root* of appropriately constructed sin-tree.
- *Infinite path* represents the “*path to the root*”.
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**Starting point: Age of an individual**

\[
P(\text{Age}(V_t) > 10 | \mathcal{F}(t)) = \frac{|\mathcal{F}(t-10)|}{|\mathcal{F}(t)|} \sim \frac{We^\lambda (t-10)}{We^\lambda t} = e^{-10\lambda} = P(T_\lambda > 10)
\]

*Suggests tree “below” random node looks like \( \mathcal{F}(T_\lambda) \) i.e. branching process run for random exponential amount of time.*
Sin-tree

$$f_2(0, \text{sin-tree})$$
Can think of random sin-trees
$T$ is a tree with root $r$. Given a vertex $v$, there exists a unique path $v_0 = v, v_1, ..., v_h = r$ from $v$ to the root.
$t_0$

$\cdots$

$t_1$

$\cdots$

$t_2$
Decompose tree into a sequence of finite rooted subtrees or fringes $(T_0(v), T_1(v), T_2(v), \ldots)$. For each $k \geq 1$,

$$\frac{1}{n} \sum_{v \in T} 1(f_k(v, T) = (t_0, t_1, \ldots, t_k)) \xrightarrow{P} \mathbb{P}_\mu(f_k(0, T) = (t_0, t_1, \ldots, t_k)).$$

**Diagram**

- $v = v_0$
- $v_1$
- $v_2$
- $r = V_h$

Tree $T$

Function $f_2(v, T)$
Convergence in probability fringe sense

Decompose tree into a sequence of finite rooted subtrees or fringes \((T_0(v), T_1(v), T_2(v), \ldots)\). For each \(k \geq 1\),

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Construction

\( T_{\alpha, \mu, \rho}^{\sin} \): Random tree with single infinite path.
Construction

\( \mathcal{T}^{\text{sin}}_{\alpha, \mu, \mathcal{P}} \): Random tree with single infinite path.

- \( X_0 \sim \exp(\alpha) \) and for \( i \geq 1 \), \( X_i \sim \mu \). \( S_n = \sum_{0}^{n} X_i \).
- Conditional on the sequence \( (S_n)_{n \geq 0} \),
  1. \( \mathcal{T}_{X_0} \): continuous time branching process driven by \( \mathcal{P} \) observed up to time \( X_0 \).
  2. For \( n \geq 1 \) let \( \mathcal{T}_{S_n, S_{n-1}} \): continuous time branching process observed up to time \( S_n \).
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- \text{sin-tree construction}: Infinite path is \( \mathbb{Z}^+ = 0,1,2,\ldots \).
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- \( \text{sin-tree construction: Infinite path is } \mathbb{Z}^+ = 0,1,2,\ldots \)
  0 designated as the root. \( \mathcal{F}_X^0 \) to be rooted at 0 and for \( n \geq 1 \) consider \( \mathcal{F}_{S_n,S_{n-1}} \) to be rooted at \( n \).

- \( f_k(\mathcal{T}^{\text{sin}}_{\alpha,\mu,P}) = (\mathcal{F}_X^0,\mathcal{F}_{S_1,S_0},\mathcal{F}_{S_2,S_3},\ldots,\mathcal{F}_{S_k,S_{k-1}}) \).
Let's recall that the offspring distribution is given by a Yule process $\mathcal{P}(\cdot)$. This distribution is crucial in understanding the degree distribution in preferential attachment models.
Recall $\lambda = 2$. Offspring distribution: Yule process $\mathcal{P}(\cdot)$.

From above discussion: expect degree of a randomly selected vertex to converge to:

$$1 + \mathcal{P}(T_2) := 1 + C \text{ say.}$$
Case study: degree distribution in preferential attachment

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$$\tilde{p}_k := \mathbb{P}(C \geq k) = \mathbb{P}(E_0 + E_1 + \cdots E_{k-1} \leq T_2) = \mathbb{E}(\exp(-2(E_0 + E_1 + \cdots + E_{k-1}))).$$
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$$p_k := \mathbb{P}(C = k) = \frac{2}{k+3} \prod_{i=0}^{k-1} \left( \frac{i+1}{i+1+2} \right) = \frac{2}{(k+3)(k+2)(k+1)} \sim \frac{C}{k^3}$$
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**Power law degree distribution!**
Random matrices

What more can one do with this machinery?

Notation

- $A_n$ adjacency matrix of tree $T_n$. 
Random matrices

What more can one do with this machinery?

Notation

- $A_n$ adjacency matrix of tree $T_n$.
- $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the $n$ eigen values.
What more can one do with this machinery?

**Notation**

- $A_n$ adjacency matrix of tree $\mathcal{T}_n$.
- $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the $n$ eigen values.
- $F_n = \frac{1}{n} \sum_1^n \delta_{\lambda_i}$ spectral distribution.

**Setting**

- For the convergence of spectral distribution can take general families of trees satisfying sin-tree convergence.
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Setting

- For the convergence of spectral distribution can take general families of trees satisfying sin-tree convergence.
- For maximal eigen value convergence talking about preferential attachment with $f(v, n) = \text{Deg}(v, n) + a$. 
Main result

Theorem (SB, Evans, Sen 08)

(a) Consider a sequence of trees converging in fringe since to a random infinite \( \sin \)-tree. Then there exists a model dependent probability distribution function \( F \) such that

\[
\text{as } n \to \infty, \quad d(F_n, F) \xrightarrow{P} 0.
\]

(b) Let \( \gamma \alpha = \alpha + 2 \). Then for the linear preferential attachment model

\[
\left( \frac{\lambda_1}{n^{1/2\gamma \alpha}}, \frac{\lambda_2}{n^{1/2\gamma \alpha}}, \ldots, \frac{\lambda_k}{n^{1/2\gamma \alpha}} \right) \xrightarrow{d} \nu_k
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Main result

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(a) Consider a sequence of trees converging in fringe since to a random infinite sin-tree. Then there exists a model dependent probability distribution function $F$ such that

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$$\left( \frac{\lambda_1}{n^{1/2}\gamma_\alpha}, \frac{\lambda_2}{n^{1/2}\gamma_\alpha}, \ldots, \frac{\lambda_k}{n^{1/2}\gamma_\alpha} \right) \xrightarrow{d} \nu_k$$

Spectral distribution turns out to be a local property of random node, maximal eigen values, local property about the root.
Spectral distribution: Method of proof

Stieltjes transform

\[ s(z) = \int_{\mathbb{R}} \frac{1}{x-z} \, dF_n(x) \]
Spectral distribution: Method of proof

Stieltjes transform

\[ s(z) = \int_{\mathbb{R}} \frac{1}{x - z} dF_n(x) \]

For eigen value distribution

\[ s(z) = \frac{1}{n} \text{Tr} (A - zI)^{-1} \]
\[ = \frac{1}{n} \sum_{v=1}^{n} R_{vv}(z) \]
Spectral distribution: Method of proof

**Stieltjes transform**

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\[
= \frac{1}{n} \sum_{v=1}^{n} R_{vv}(z)
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\[
R_{uv}(z) = \frac{1}{-z + \sum_{1}^{N(v)} R_{vu}v_i(z) + R_{A(v)}^{\text{big}}(z)}
\]
Fix $\text{Im}(z) > 1$. Iterate the above expansion $d$ times. Get a continued fraction upto $d$ terms and some error term.
Spectral distribution contd

- Fix $\text{Im}(z) > 1$. Iterate the above expansion $d$ times. Get a continued fraction upto $d$ terms and some error term.
- Not hard to see that for $\text{Im}(z) > 1$, this implies that $s_n(z)$ “depends” on the first $K$ terms.
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- Not hard to see that for $\text{Im}(z) > 1$, this implies that $s_n(z)$ “depends” on the first $K$ terms.
- Fringe convergence of the random trees tells you what happens upto distance $K$ for any fixed $K$.
- So not hard to show that there exists a fixed Stieltjes transform $s(z)$ such that,

$$ s_n(z) \xrightarrow{P} s(z). $$
Properties and questions

Open question

- We have established sufficient conditions for a point $a \in \mathbb{R}$ to be an atom of limiting $F$.
- Implies that for most standard models, limiting $F$ has dense set of atoms.
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- We have established sufficient conditions for a point $a \in \mathbb{R}$ to be an atom of limiting $F$.
- Implies that for most standard models, limiting $F$ has dense set of atoms.
- **Open Question:** Does limiting $F$ have absolutely continuous part?
- Connections to areas such as Random Schrodinger operators?

At this point

- If one can embed things in a continuous time all good things happen.
- How far can one push such embeddings? Can these continuous time branching processes arise in the limit even when no such embeddings exist?
Outline

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   - Continuous time construction
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4 Conclusion
From the Retweet Graph to the Superstar Model

- Joint work with J Michael Steele (Wharton) and Tauhid Zaman (MIT).
From the Retweet Graph to the Superstar Model

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- **Retweet graph**: Given a topic and a time frame — form all the (undirected) *retweet arcs* and look at the graph you get.
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Some Empirical Retweet Graphs

- Retweet graphs were constructed for 13 different public events\(^1\)

\(^1\)Data courtesy of Microsoft Research, Cambridge, MA
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- We treat the graph as undirected

---

1Data courtesy of Microsoft Research, Cambridge, MA
The superstar model

BET Awards

Superstar Model: Tools for Analysis
The superstar model
The superstar model

- Max degree in retweet graph is on the order of graph size (i.e. $M_G \sim pn$)
- Preferential attachment predicts sub-linear max degree
The Superstar Model

\[ G_2 \]

\[ v_0 \quad \text{(superstar)} \]

\[ v_1 \]
The Superstar Model

$G_3$

$v_0$ (superstar)

$v_1$

$v_2$
The Superstar Model

- Attach to superstar with probability $p$
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- Attach to superstar with probability \( p \)
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- Attach to superstar with probability $p$
- Else with probability $1 - p$ attach to one of the non-superstar vertices.

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The only model parameter is $p$: The superstar parameter

This is a very simple model: But (1) it has empirical benefits and (2) it is tractable — though not particularly easy.
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Superstar Degree

**Theorem**

Let $\deg(v_0, G_n)$ be the superstar degree. Then we have that

$$\frac{\deg(v_0, G_n)}{n} \rightarrow p \quad \text{with probability 1 as } n \rightarrow \infty$$
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Let $\deg(v_0, G_n)$ be the superstar degree. Then we have that

$$\frac{\deg(v_0, G_n)}{n} \to p \quad \text{with probability 1 as } n \to \infty$$

- Empirically the Superstar degree is $\Theta(n)$ and the Superstar Model “Bakes this into the Cake"
Superstar Degree

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- Empirically the Superstar degree is \( \Theta(n) \) and the Superstar Model “Bakes this into the Cake”
- But that is ALL that is baked in...
- The value of \( p \) determines other features of the graph — the Superstar Model is *testable*. 
Let $\text{deg}_{\text{max}}(G_n)$ be the maximal non-superstar degree:

$$\text{deg}_{\text{max}}(G_n) = \max_{1 \leq i \leq n} \deg(v_i, G_n)$$

and let

$$\gamma = \frac{1 - p}{2 - p}.$$

Then there exists a non-degenerate, strictly positive random variable $\Delta^*$ such that

$$n^{-\gamma} \text{deg}_{\text{max}}(G_n) \rightarrow \Delta^* \quad \text{with probability 1 as } n \rightarrow \infty.$$
Non-Superstar Degree

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\[
n^{-\gamma} \deg_{\text{max}}(G_n) \to \Delta^* \quad \text{with probability 1 as } n \to \infty
\]

- Maximal non-superstar degree = \( \Theta(n^\gamma) \)
Realized Degree Distribution in the Superstar Model

Theorem

Let $f(k, G_n)$ be the realized degree distribution of $G_n$ under the Superstar model,

$$f(k, G_n) = n^{-1} \sum_{1 \leq j \leq n : \deg(v_j, G_n) = k}$$

and introduce the superstar model scaling constant

$$f_{SM}(k, p) = \frac{2 - p}{1 - p} (k - 1)! \prod_{i=1}^{k} \left( i + \frac{2 - p}{1 - p} \right)^{-1}.$$

We then have

$$f(k, G_n) \rightarrow f_{SM}(k, p) \quad \text{with probability 1 as } n \rightarrow \infty$$
Realized Degree Distribution in the Superstar Model

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- The degree distribution scales like $k^{-\beta}$, where $\beta = 3 + p/(1 - p)$

- This contrasts with the preferential attachment model which scales like $k^{-3}$
Height result

**Theorem**

Let $W(\cdot)$ be the Lambert special function with $W(1/e) \approx 0.2784$. Then with probability one we have

$$\lim_{n \to \infty} \frac{1}{\log n} \mathcal{H}(G_n) = \frac{1 - p}{W(1/e)(2 - p)}.$$
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- **Superstar Degree**: $\Theta(n)$ versus $NA$
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- Comparison: Preferential Attachment always predicts...
  \[ f_{PA}(k) = \frac{4}{k(k + 1)(k + 2)} \]
Degree distribution

Lebron, $p' = 0.09$

Brazil Portugal, $p' = 0.28$

BET Awards, $p' = 0.58$

Federer, $p' = 0.37$
The Superstar Model and the Realized Degree Distribution: Bottom Line

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- Next: How Can one Analyze the Superstar Model?
Outline

1. Preferential attachment: Base model
   - Continuous time construction
   - Local weak convergence

2. Twitter event networks and the superstar model
   - Retweet Graph and Superstar Model
   - Main Results
   - Comparison with Preferential Attachment Model
   - **Superstar Model: Tools for Analysis**

3. Power of choice and random trees

4. Conclusion
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- **News You Can Use?** One can see the benefits of using multi-type branching processes. One can see that the connection between the Yule process and preferential attachment is natural.
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Surgery: From BP Model to Superstar Model

\[ \mathcal{F}(\tau_6) \]

- Add an exogenous superstar vertex \( v_0 \) to the vertex set.
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Relating the BP Construction with the Superstar Model

Claim: \( S(\emptyset n) \) is "probabilistically the same" as \( G_{n+1} \)

Base case: 

\[
S(\emptyset 1) = G_2
\]

Need to show that \( S(\emptyset n) \) and \( G_{n+1} \) have the same probabilistic evolution.

Superstar: probability of joining superstar = probability of red vertex being born = \( p \)

Same probability for \( S \) and \( G \)

Non-superstars: degree of vertex = number of blue children + 1

\[
\deg(v_k, G_{n+1}) = c_B(v_k, \emptyset n) + 1
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\[
F(\emptyset 6) = c_B(v_1, \emptyset 6) + 1 = 2
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---

**Diagram 1:**

- **Vertex $v_1$:** Red square, $c_B(v_1, \tau_6 - \tau_1) + 1 = 2$
- **Degree relation:** $\deg(v_1, G_7) = 2$
Further Linking of the BP Model and the Superstar Model
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Power of choice in random trees [D’Souza, Mitzenmacher]

Model: Motivation and construction

- Usual pref. attachment: Basic assumption: every new vertex has knowledge of entire network
- Each stage new vertex chooses 2 vertices uniformly at random
- Connect to vertex with maximal degree **amongst** the ones chosen (breaking ties with probability $1/2$)
- Model which incorporates randomness as well as limited choice
- Let $T_n$ denote the tree on $n$ vertices
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Theorem (Angel, Pemantle, SB)

There exists a rooted limiting random tree $T_\infty$, described by Jagers-Nerman stable age distribution theory such that such that $T_n$ converges locally $T_\infty$. 
Main idea

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- So have interval $[0, T]$ with $T \sim \exp(1/2)$ where queries come at uniform times
- If still a leaf, for each query, no connection made which happens with probability 
  $$1 - p_0 + p_0/2 = 1 - p_0/2.$$
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  \[ 1 - p_0 + \frac{p_0}{2} = 1 - \frac{p_0}{2}. \]
- Rate one poisson process, marking each with probability $p_0/2$, time of first point:
  \[ X_0 \sim \exp(p_0/2) \]
- So probability not a leaf: \[ 1 - p_0 = \mathbb{P}(T > X_0) \]
Description of the limit tree

Recursive construction of the degree

- Let \( p_0 \) limiting fraction of leaves
- Define \( q_0 = p_0/2 \)
- Then \( p_0 \) obtained by doing the following: Let \( T \sim \exp(1/2) \) and \( X_0 \sim \exp(q_0) \). Then
  \[
  1 - p_0 = \Pr(T > X_0)
  \]
- \[
  p_0 = \frac{\sqrt{5} - 1}{2}
  \]
- General, having obtained \( p_k \), get \( p_{k+1} \) by solving
  \[
  1 - (p_0 + \cdots + p_{k+1}) = \Pr(X_0 + \cdots X_{k+1} > T)
  \]
  where
  \[
  X_{k+1} \sim \exp(p_0 + \cdots + p_k + \frac{p_{k+1}}{2})
  \]
After having obtained $p_i$, let $L_i = \sum_{j=0}^{i} X_j$

Consider the point process $\mathcal{P}_{\text{max}} = (L_0, L_1, \ldots)$

Define

\[
\mu_{\text{max}}(0, t) = \mathbb{E}(\#i : L_i < t)
\]

\[
\nu_{\text{max}}(dx) = \exp(-\frac{x}{2})\mu(dx)
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Description of $\mathcal{F}_\infty$

- After having obtained $p_i$, let $L_i = \sum_{j=0}^{i} X_j$
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$$

Theorem

- Then $\mathcal{F}_\infty$ is the Jagers-Nerman stable age distribution tree with offspring distribution $\mathcal{P}_{\text{max}}$, age distribution $\exp(1/2)$ and time to nearest ancestor $\nu_{\text{max}}$
- Implies convergence of global functionals as well such as the spectral distribution of adjacency matrix
Lots of interesting questions

Understanding what happens for general **unbounded size** rules such as product rule (**explosive percolation**).

Small variants of standard models turn out to be technically much more challenging, requiring the development of new machinery.

For the superstar model, a simple tweak gave much better fit to the data (one parameter $p$).
Dynamic random graphs

- Lots of interesting questions
- Understanding what happens for general **unbounded size** rules such as product rule (*explosive percolation)*.
- Small variants of standard models turn out to be technically much more challenging, requiring the development of new machinery.
- For the superstar model, a simple tweak gave much better fit to the data (one parameter $p$).

Next lecture

Back to critical random graphs: suppose we were interested in the metric structure of maximal components. What can we say? Why should one care.