Lecture 1: Dynamic network models
Probabilistic and statistical methods for networks
Berlin Bath summer school for young researchers

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University of North Carolina

August, 2017

Motivation

Dynamic network models

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- Show up in an enormous number of areas: real world networks (Twitter/Facebook); statistical physics; combinatorial optimization (e.g. Minimal spanning tree algorithms) etc.

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Brain Plasticity

News

Brains re-wire after injury

GARVAN INSTITUTE
WEDNESDAY, 15 MAY 2013

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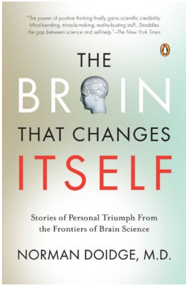
Scientists from the United States and Australia have advanced our understanding of brain plasticity by showing that the brain forms complex new circuits after damage, often far from the damaged site, to compensate for lost function.

A new study by Drs Moriel Zelikowsky and Michael Fanselow from the University of California Los Angeles (UCLA), in collaboration with Dr Bryce Vissel, from Sydney's Garvan Institute of Medical Research, identified the exact regions of the brain that take over when a learning and memory centre, known as the hippocampus, is damaged.

Researchers have identified the regions of the brain that take over when the hippocampus, a learning and memory centre, is damaged. This finding will help scientists develop better treatments for stroke victims and patients with Alzheimer's disease.

Image: Jezper/Shutterstock

Their breakthrough, the first demonstration of such circuit plasticity, is published in the early online edition of the **Proceedings of the Academy of Science (PNAS)**, the journal of the United States National Academy of Sciences.



Twitter event networks I

Twitter event networks II

Course contents

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Brief discussion of the order of topics.

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This talk

- What if you had n vertices, new edges entering the system at random (say 2 at a time).
- You could decide which edges to use based on the current configuration.
- Phase transition? Emergence of the giant?

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- Preferential attachment type models.
- Shall see the power of continuous time branching processes and local weak convergence.

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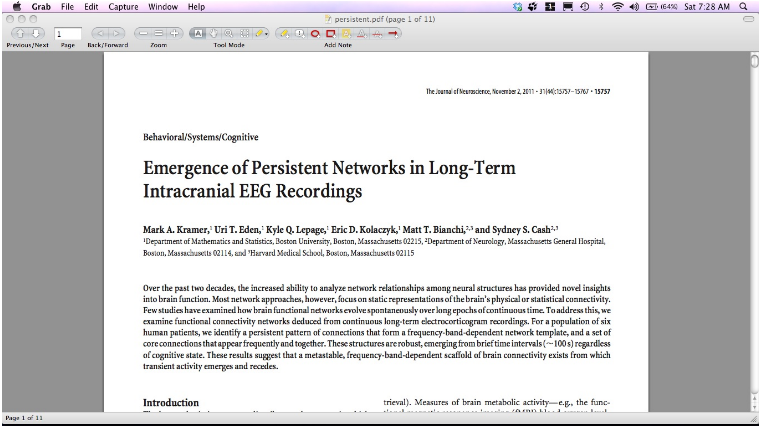
Applied context



The Journal of Neuroscience, November 2, 2011 • 31(44):15757–15767 • 15757

Behavioral/Systems/Cognitive

Emergence of Persistent Networks in Long-Term Intracranial EEG Recordings

Mark A. Kramer,[1] Uri T. Eden,[1] Kyle Q. Lepage,[1] Eric D. Kolaczyk,[1] Matt T. Bianchi,[2,3] and Sydney S. Cash[2,3]
[1]Department of Mathematics and Statistics, Boston University, Boston, Massachusetts 02215, [2]Department of Neurology, Massachusetts General Hospital, Boston, Massachusetts 02114, and [3]Harvard Medical School, Boston, Massachusetts 02115

Over the past two decades, the increased ability to analyze network relationships among neural structures has provided novel insights into brain function. Most network approaches, however, focus on static representations of the brain's physical or statistical connectivity. Few studies have examined how brain functional networks evolve spontaneously over long epochs of continuous time. To address this, we examine functional connectivity networks deduced from continuous long-term electrocorticogram recordings. For a population of six human patients, we identify a persistent pattern of connections that form a frequency-band-dependent network template, and a set of core connections that appear frequently and together. These structures are robust, emerging from brief time intervals (~100 s) regardless of cognitive state. These results suggest that a metastable, frequency-band-dependent scaffold of brain connectivity exists from which transient activity emerges and recedes.

Introduction

trieval). Measures of brain metabolic activity—e.g., the func-

Mathematical questions

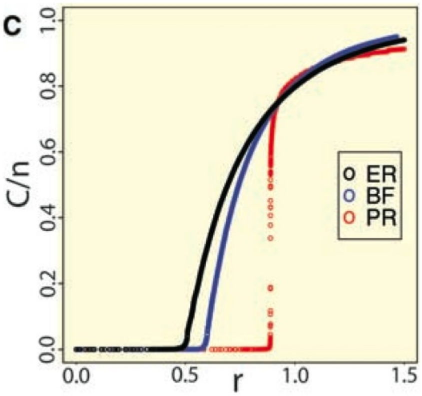


Figure: New phenomena in phase transition?

Outline

1. Problem formulation

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2. Erdos-Renyi random graph at criticality: Static regime.

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4. Erdos-Renyi random graph: Dynamic regime.
5. Bounded size-rules: Emergence of the giant component.

Teaching goals/outcomes of this Lecture

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3. Introduce fundamental probabilistic proof techniques in the area including random walks and exploration processes, differential equations technique etc.

Some notation

- $\mathscr{G}_n = (\mathscr{V}_n, \mathscr{E}_n)$ graph on n vertices with vertex set $\mathscr{V}_n$ and edge set $\mathscr{E}_n$. Typically $\mathscr{V}_n = [n] := \{1, 2, \ldots, n\}$.

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- $\mathcal{C} \subseteq \mathcal{G} :=$ connected component. $|\mathcal{C}| :=$ size (number of vertices) in a component.
- Main functionals of interest for this talk: maximal components. $\mathcal{C}_n^{(k)} :=$ is the k-th largest component.
- $\mathcal{C}_n^{(1)}$: maximal component.

Erdos-Renyi random graph

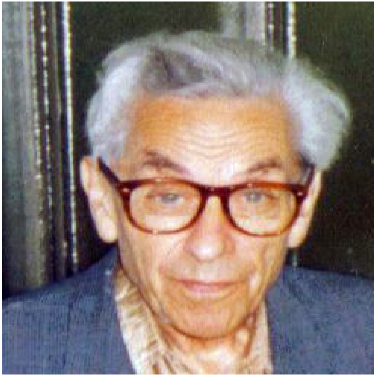

Figure: Paul Erdos. **By Topsy Kretts - Own work, CC BY 3.0, https://commons.wikimedia.org/w/index.php?curid=2874719**

Figure: Alfred Renyi. **Taken from** https://alchetron.com/Alfred-Renyi-738936-W.

Phase transition

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Beautiful math theory. Limits depend on λ.

Bounded size rules

Dynamic version of Erdős-Rényi

- $\mathscr{G}_n(0) = \mathbf{0}_n$ the graph with n vertices but no edges
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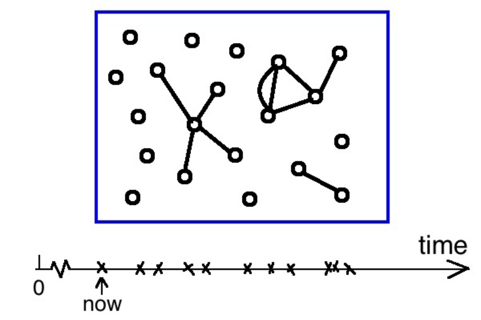

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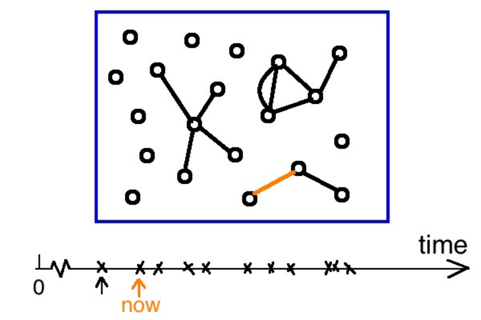

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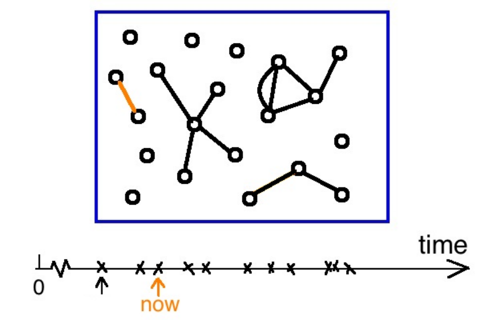

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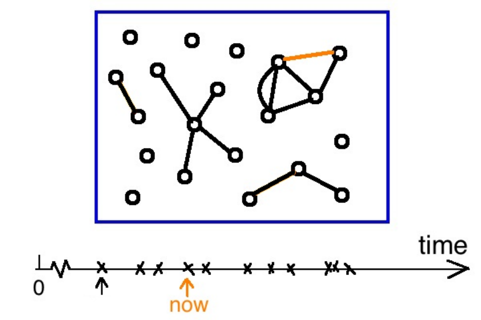

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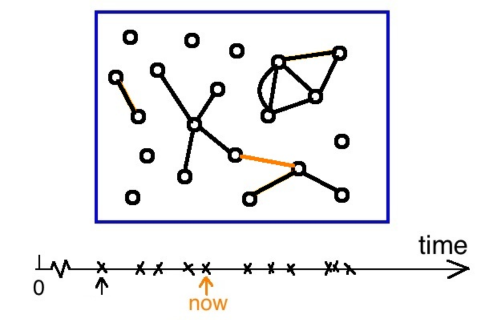

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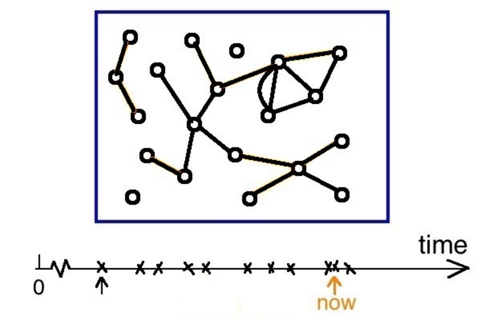

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- $t_c = t_c^{ER} = 1$ is the critical time.
- (super-critical) when $t > 1$, $\mathscr{C}_n^{(1)}(t) = \Theta(n)$, $\mathscr{C}_n^{(2)}(t) = O(\log n)$.
- (sub-critical) when $t < 1$, $\mathscr{C}_n^{(1)}(t) = O(\log n)$, $\mathscr{C}_n^{(2)}(t) = O(\log n)$.
- (critical) when $t = 1$, $\mathscr{C}_n^{(1)}(t) \sim n^{2/3}$, $\mathscr{C}_n^{(2)}(t) \sim n^{2/3}$.

Bounded size rules: Effect of limited choice

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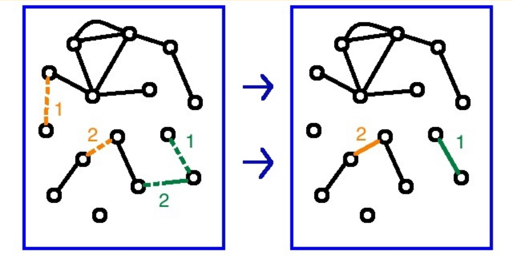

Erdos-Renyi random graph at criticality

History

- after initial work by Erdos-Renyi[60], Bollobas[84], Luczak[90],Janson-Luczak-Knuth-Pittel [94], finally proved by Aldous[97].
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Problem statement

- Connection probability $p_n := \frac{1}{n} + \frac{\lambda}{n^{4/3}}$.
- $\mathscr{C}_n^{(i)}(\lambda)$ size of the i-th largest component.
- Surplus (Complexity) of a component

$$\xi_n^{(i)}(\lambda) = E(\mathscr{C}_n^{(i)}(\lambda)) - (\mathscr{C}_n^{(i)}(\lambda) - 1)$$

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- $l_\downarrow^2 = \left\{(x_i)_{i\geq 1} : x_1 \geq x_2 \geq \cdots \geq 0, \sum_i x_i^2 < \infty\right\}$

- $$\mathbf{C}_n^*(\lambda) := n^{-2/3}(\mathscr{C}_n^{(1)}(\lambda), \mathscr{C}_n^{(2)}(\lambda), \ldots)$$
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Aldous (97)

As $n \to \infty$, in $l^2_\downarrow$ one has

$$\mathbf{C}_n^*(\lambda) \xrightarrow{d} X(\lambda)$$

In pictures

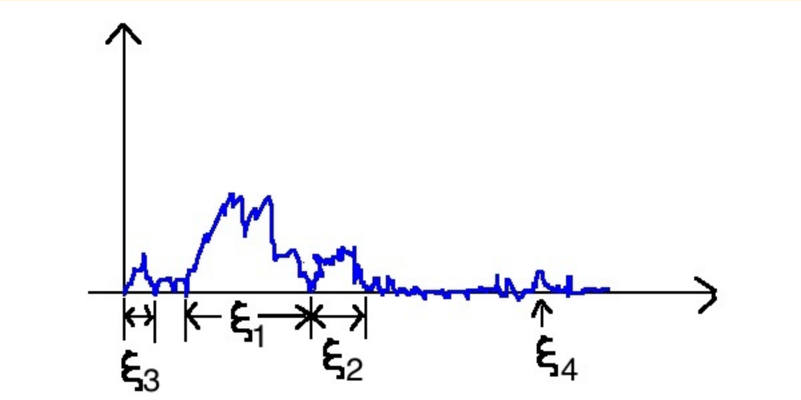

Figure: Reflected process

Proof techniques

Outline

- **Branching process methods:** Great tool above and below criticality.

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Exploration process

- *Start* with a vertex.
- Explore component of vertex keeping track of *various functionals* whilst exploring.
- Move to *next* component.

Typical method of proof: Exploration

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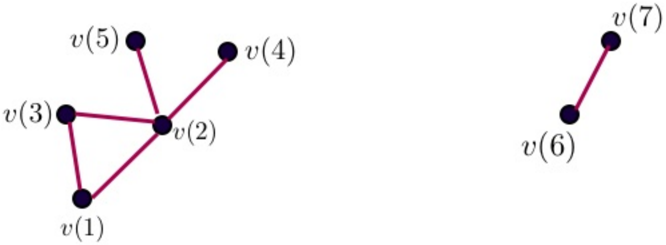
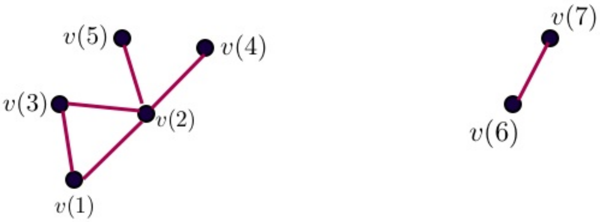


Typical method of proof: Exploration

$c(1) = 2$
$c(2) = 2$
$c(3) = 0$
$\cdots$



Proof: Deterministic Lemma

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Functional

- *Define* $Z_n(0) = 0, \qquad Z_n(i) = Z_n(i-1) + c(i) - 1.$
- **Amazing fact:** $Z(\cdot) = -1$ *for the first time when we finish exploring component* 1. *Walk then hits* -2 *for first time when exploring component* 2 *and so on.*

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- Same as lengths of excursions from zero of the reflected process. This is Aldous's result.

Showing process convergence

Infinitesimal mean

- $\mathscr{F}_i$ what we know till time i in the exploration process. So $\{\mathscr{F}_i : i \geq 0\}$ natural filtration.
- Exploring number of children of $v(i)$ at time $i-1$.
- Number of free vertices $= n - (i + Z_n(i))$.
- Conditional on $\mathscr{F}_{i-1}$,

$$c(i) \sim \mathrm{Bin}\left(n - (i + Z_n(i)), \frac{1}{n} + \frac{\lambda}{n^{4/3}}\right)$$

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- Thus if $\bar{Z}_n(s) = n^{-1/3} Z_n(sn^{2/3})$ then

$$\mathbb{E}(\bar{Z}_n(s) | \mathscr{F}_{sn^{2/3}}) = n^{2/3} n^{-1/3} \mathbb{E}(\Delta Z_n(sn^{2/3}) | \mathscr{F}_{sn^{2/3}}) \approx \lambda - s$$

Martingale CLT

- Thus $\bar{Z}_n(t) - \int_0^t (\lambda - s)\,ds = \bar{Z}_n(t) - (\lambda t - t^2/2)$ is basically a martingale.

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- Proof completed!

Rank-1 inhomogeneous random graphs/Norros-Reittu/Chung-Lu

- Vertex set $[n]$ with each vertex having weight w_i ("popularity or affinity"). Let $l_n = \sum_i w_i$. Simplest example: $w_i \sim_{iid} F$ with finite third moments.
- Connect vertex i, j with probability

$$p_{ij} := 1 - \exp(-w_i w_j / l_n) \sim \frac{w_i w_j}{l_n}$$

- **Known:** Phase transition at $\nu = \mathbb{E}(W^2)/\mathbb{E}(W) = 1$.

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- Turns out one can use previous exploration process.
- Now have to be careful of the order in which vertices are seen $\{v(1), v(2), \ldots, v(n)\}$.
- In calculating infinitesimal means, variances, have terms like $\sum_{j=1}^{i} w_{v(j)}/l_n$ and $\sum_{j=1}^{i} w_{v(j)}^2/l_n$.

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Size biased reordering

- $(v(1), v(2), \ldots v(n))$ is a *size biased random re-ordering of* $[n]$..
- $\mathbb{P}(v(1) = i) \propto w_i$, $\mathbb{P}(v(2) = i | v(1)) \propto w_i, i \neq v(1)$ etc.
- Allows us to understand asymptotics of $\sum_{j=1}^{i} w_{v(j)}/l_n$ etc.

Erdos-Renyi: Dynamic regime

The Erdős-Rényi random graph of $\mathscr{G}_n^{ER}$

- $\mathscr{G}_n(0) = \mathbf{0}_n$ the graph with n vertices but no edges
- Each step, choose one edge e uniformly among all $\binom{n}{2}$ possible edges, and add it to the graph.

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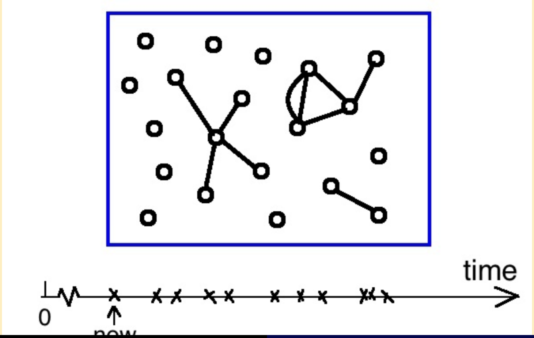
- $\mathscr{G}_n(t)$: add edges at rate $n/2$.



Back to Erdos-Renyi processes

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- Not hard to believe that for fixed λ

$$\mathbf{C}_n^*(\lambda) := n^{-2/3}(\mathscr{C}_n^{(1)}(1+\lambda/n^{1/3}), \mathscr{C}_n^{(2)}(1+\lambda/n^{1/3}), \ldots) \xrightarrow{d} X(\lambda) := \text{ Excursion lengths }.$$

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Important question

What happens to $\{\mathbf{C}_n^*(\lambda) : -\infty < \lambda < \infty\}$ as a a process in λ?

Rate of mergers

- Recall we are looking at the new time scale $t = 1 + \lambda/n^{1/3}$

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- Recall we are looking at the new time scale $t = 1 + \lambda/n^{1/3}$
- In this time scale, in time interval $[\lambda, \lambda + d\lambda)$, components a and b merge at rate

$$\frac{1}{n^{1/3}} \times \frac{\mathscr{C}_a(1+\lambda/n^{1/3})\mathscr{C}_b(1+\lambda/n^{1/3})}{n} = \bar{\mathscr{C}}_a(\lambda)\bar{\mathscr{C}}_a(\lambda)$$

- Aldous showed there exists an $l^2_\downarrow$ valued Markov process $\{X(\lambda) : -\infty < \lambda < \infty\}$ called the **Standard multiplicative coalescent** such that

$$\{\mathbf{C}^*_n(\lambda) : -\infty < \lambda < \infty\} \stackrel{d}{\Longrightarrow} \{X(\lambda) : -\infty < \lambda < \infty\}$$

Standard Multiplicative coalescent

Dynamics

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- For each fixed λ, $X(\lambda)$ has distribution given by excursion lengths.
- Suppose $\mathbf{X}(\lambda) = (x_1, x_2, x_3, \ldots)$, each x_l is viewed as the size of a cluster.
- Each pair of clusters of sizes (x_i, x_j) merges at rate $x_i \cdot x_j$ into a cluster of size $x_i + x_j$.
- If x_i, x_j merge, then $(x_1, x_2, x_3, \ldots) \rightsquigarrow (x_1', x_2', x_3', \ldots)$ where the latter is the re-ordering of $\{x_i + x_j, x_l : l \neq i, j\}$.

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- *If initial configuration at time "$\lambda = -\infty$" has good properties and follows the merging dynamics of the multiplicative coalescent then,*

$$\{\mathbf{C}_n^*(\lambda) : -\infty < \lambda < \infty\} \stackrel{d}{\Longrightarrow} \{X(\lambda) : -\infty < \lambda < \infty\}$$

Bounded size rules: Effect of limited choice

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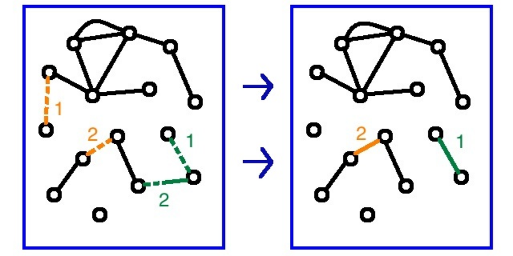

The Bohman-Frieze process

[Bohman, Frieze 2001] Delay in phase transition

Consider the continuous time version $\mathscr{G}_n^{BF}(t)$, then there exists $\epsilon > 0$ such that at time $t_c^{ER} + \epsilon$,

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[Spencer, Wormald 2004] Critical time

- $t_c^{BF} \approx 1.1763 > t_c^{ER} = 1$.
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Near Criticality: Susceptibility functions

- Janson and Spencer (2011) analyzed how $s_2(\cdot), s_3(\cdot) \to \infty$ (*defined below*) as $t \uparrow t_c$.
- Kang, Perkins and Spencer (2011) analyze the near subcritical $(t_c - \epsilon)$ regime.

General bounded size rules

- Fix $K \geq 1$
- Let $\Omega_K = \{1, 2, \ldots, K, \omega\}$
- General bounded size rule: subset $F \subset \Omega_K^4$.
- Pick 4 vertices uniformly at random. If $(c(v_1), c(v_2), c(v_3), c(v_4)) \in F$ then choose edge e_1 else e_2

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BF model

$K = 1, \; F = \left\{(1, 1, \alpha, \beta)\right\}$.

Main questions for bounded size rules

- Question: when $t = t_c$, do we have $\mathscr{C}_n^{(1)}(t_c) \sim n^{2/3}$? How do components merge?

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- *Who cares?!?!*

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Who cares?!

Turns out: even if you don't care about the model, the proof technique has far reaching consequences. *Stay tuned for Lecture 3.*

Bounded size rules

Theorem (Bhamidi, Budhiraja, Wang, 2012)

Let $(\mathscr{C}_n^{(1)}(t), \mathscr{C}_n^{(2)}(t), \ldots)$ be the component sizes of $\mathscr{G}_n^{BSR}(t)$ in decreasing order. Define the rescaled size vector $\mathbf{C}_n(\lambda)$, $-\infty < \lambda < +\infty$ as the vector

$$\mathbf{C}_n(\lambda) := (\bar{\mathscr{C}}_i(\lambda) : i \geq 1) = \left(\frac{\beta^{1/3}}{n^{2/3}} \mathscr{C}_n^{(i)}(t_c + \frac{\beta^{2/3} \alpha \lambda}{n^{1/3}}) : i \geq 1 \right)$$

where α, β are constants determined by the BSR process. Then

$$\{\mathbf{C}_n(\lambda) : -\infty < \lambda < \infty\} \xrightarrow{d} \{X(\lambda) : -\infty < \lambda < \infty\}$$

where $(\mathbf{X}(\lambda), -\infty < \lambda < +\infty)$ is the standard multiplicative coalescent and convergence happens in $l^2_{\downarrow}$ with metric d.

Proof Technique: Differential equations

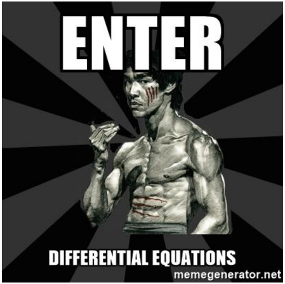

Figure: From Memegenerator.net

- Hard to think about exploration process especially at criticality.

Proof Technique: Differential equations

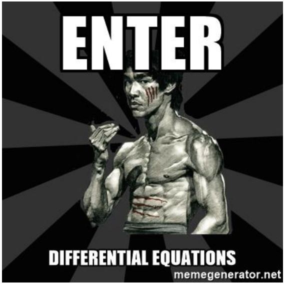

Figure: From Memegenerator.net

- Hard to think about exploration process especially at criticality.
- Turns out: Easier to analyze the entire process!

Proof idea: The Bohman-Frieze process

Question: Where does t_c come from ?

- Define

$$X_n(t) = \#\text{ of singletons}, \qquad S_2(t) = \sum_i (\mathscr{C}_n^{(i)}(t))^2, \qquad S_3(t) = \sum_i (\mathscr{C}_n^{(i)})^3.$$

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- Let $\bar{x}_n(t) = X_n(t)/n$, $\bar{s}_2(t) = S_2/n$, $\bar{s}_3(t) = S_3/n$.

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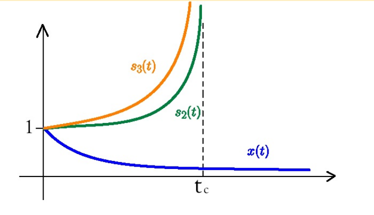
- Let $\bar{x}_n(t) = X_n(t)/n$, $\bar{s}_2(t) = S_2/n$, $\bar{s}_3(t) = S_3/n$.
- Spencer, Wormald [04]: For any fixed $t > 0$,
$$\bar{x}_n(t) \xrightarrow{\mathbb{P}} x(t), \qquad \bar{s}_2(t) \xrightarrow{\mathbb{P}} s_2(t), \qquad \bar{s}_3(t) \xrightarrow{\mathbb{P}} s_3(t)$$



Why?

Behavior of $x_n(t)$

- In small time interval $[t, t+\Delta(t))$, $x_n(t) \to x_n(t) - 1/n$ at rate

$$\frac{2}{n^3}\left(\binom{n}{2} - \binom{X_n(t)}{2}\right) X_n(t)(n - X_n(t)) \sim n(1 - x_n^2(t)) x_n(t)(1 - x_n(t))$$

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- $[t, t+\Delta(t))$, $x_n(t) \to x_n(t) - 2/n$ at rate

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- Suggests that $x_n(t) \to x(t)$ where

$$x'(t) = -x^2(t) - (1 - x^2(t))x(t) \qquad \text{for } t \in [0, \infty,) \qquad x(0) = 1.$$

Susceptibility functions

Limit functions

- Similar analysis suggests that for the susceptibility functions, the limits should satisfy, $\bar{s}_2(t), \bar{s}_3(t)$

$$s_2'(t) = x^2(t) + (1 - x^2(t))s_2^2(t) \qquad \text{for } t \in [0, t_c), \qquad s_2(0) = 1$$
$$s_3'(t) = 3x^2(t) + 3(1 - x^2(t))s_2(t)s_3(t) \qquad \text{for } t \in [0, t_c), \qquad s_3(0) = 1.$$

- Analysis of the ODEs gives the picture on the previous page.

The Bohman-Frieze process

Scaling exponents of s_2 and s_3 (Janson, Spencer 11)

- Functions $x(t), s_2(t), s_3(t)$ are determined by some differential equations
- Differential equations imply $\exists$ constants α, β such that $t \uparrow t_c$

$$s_2(t) \sim \frac{\alpha}{t_c - t}$$

$$s_3(t) \sim \beta (s_2(t))^3 \sim \beta \frac{\alpha^3}{(t_c - t)^3}$$

Now we enter the analysis carried out in Bhamidi, Budhiraja and Wang (2012).

I: Regularity conditions of the component sizes at "$-\infty$"

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- Need to verify the three conditions

$$\frac{\sum_i \left(\bar{\mathscr{C}}_i(\lambda_n)\right)^3}{\left[\sum_i \left(\bar{\mathscr{C}}_i(\lambda_n)\right)^2\right]^3} \xrightarrow{\mathbb{P}} 1 \quad \Leftrightarrow \quad \frac{n^2 S_3(t_n)}{S_2^3(t_n)} \xrightarrow{\mathbb{P}} \beta$$

$$\frac{1}{\sum_i \left(\bar{\mathscr{C}}_i(\lambda_n)\right)^2} + \lambda_n \xrightarrow{\mathbb{P}} 0 \quad \Leftrightarrow \quad \frac{n^{4/3}}{S_2(t_n)} - \frac{n^{-\delta+1/3}}{\alpha} \xrightarrow{\mathbb{P}} 0$$

$$\frac{\bar{\mathscr{C}}_1(\lambda_n)}{\sum_i \left(\bar{\mathscr{C}}_i(\lambda_n)\right)^2} \xrightarrow{\mathbb{P}} 0 \quad \Leftrightarrow \quad \frac{n^{2/3}\mathscr{C}_n^{(1)}(t_n)}{S_2(t_n)} \xrightarrow{\mathbb{P}} 0$$

II: Dynamics of merging in the critical window

The dynamic of merging

- In any small time interval $[t, t+dt)$, two components i and j merge at rate

$$\frac{2}{n^3}\left[\binom{n}{2} - \binom{X_n(t)}{2}\right]\mathscr{C}_i(t)\mathscr{C}_j(t)$$
$$\sim \frac{1}{n}(1-\bar{x}^2(t))\mathscr{C}_i(t)\mathscr{C}_j(t)$$

Let $\lambda = (t-t_c)n^{1/3}/\alpha\beta^{2/3}$ be rescaled time paramter, rate at which two components merge

$$\gamma_{ij}(\lambda) \sim \frac{(1-x^2(t_c+\beta^{2/3}\alpha\frac{\lambda}{n^{1/3}}))}{n}\frac{\beta^{2/3}\alpha}{n^{1/3}}\mathscr{C}_i\left(t_c+\frac{\beta^{2/3}\alpha\lambda}{n^{1/3}}\right)\mathscr{C}_j\left(t_c+\frac{\beta^{2/3}\alpha\lambda}{n^{1/3}}\right)$$

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$$= \bar{\mathscr{C}}_i(\lambda)\bar{\mathscr{C}}_j(\lambda) \qquad \text{since } \alpha(1-x^2(t_c)) = 1$$

How to check regularity conditions

Analysis of $\mathscr{C}_n^{(1)}(t)$

- Key point: need to get refined bounds on maximal component in barely subcritical regime.
- known result: for fixed $t < t_c$, $\mathscr{C}_n^{(1)}(t) = O(\log n)$.
- not enough, since we want a sharp upper bound when $t \uparrow t_c$.

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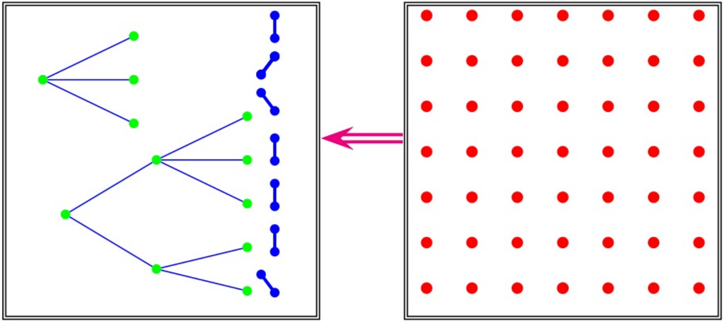
Lemma (Bounds on the largest component)

Let $\delta \in (0, 1/5)$, t_c be the critical time for the BF process, $\mathscr{C}_n^{(1)}(t)$ be the size of the largest component. Then there exists a constant $B = B(\delta)$ such that as $n \to +\infty$,

$$\mathbb{P}\{\mathscr{C}_n^{(1)}(t) \leq \frac{B \log^4 n}{(t_c - t)^2} \text{ for all } t < t_c - n^{-\delta}\} \to 1$$

Proof strategy: Coupling with a near critical multi-type branching process on an infinite dimensional type space. delicate analysis of the maximal eigenvalue.

Random graph with Immigrating doubletons



Sketch of the proof

Regularity condition at time $\lambda = -\infty$

Check the following properties for the un-scaled component sizes. For $\delta \in (1/6, 1/5)$, and $t_n = t_c - n^{-\delta}$,

$$\frac{n^2 S_3(t_n)}{S_2^3(t_n)} \xrightarrow{\mathbb{P}} \beta \tag{6.1}$$

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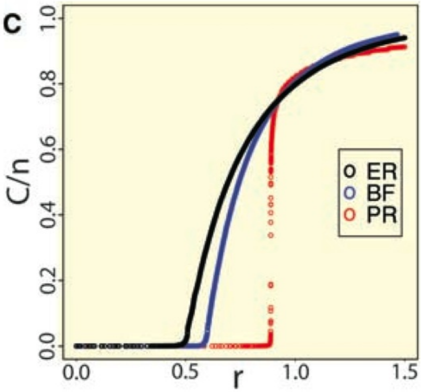
Analysis of $S_2(t)$, $S_3(t)$

- above relations hold for limiting functions s_2, s_3 for t_n.
- Delicate stochastic analytic argument combined with result on $\mathscr{C}_n^{(1)}(t)$ to show this holds for S_2, S_3 near criticality.

Where do go from here? Work in progress

Explosive percolation

In 2009, Achlioptas,D'Souza and Spencer considered "product rule". Conjectured that this process exhibits Explosive percolation



Truncated product rule

Fix K

- Choose 2 edges $e_1 = (v_1, v_2)$ and $e_2 = (v_3, v_4)$ at random
- If $\max\{C(v_1), C(v_2)C(v_3), C(v_4)\} \leq K$, then use the edge which minimizes of $\min\{C(v_1)C(v_2), C(v_3)C(v_4)\}$.
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Work in progress

- Consider the rescaled and re-centered component sizes

$$\mathbf{C}_K(\lambda) = \left(\frac{1}{n^{2/3}}\mathscr{C}_K^{(i)}\left(t_c(K) + \gamma(K)\frac{\lambda}{n^{1/3}}\right) : i \geq 1\right) \qquad \lambda \in \mathbb{R}$$

  Then we have $(\mathbf{C}_K(\lambda) : \lambda \in \mathbb{R}) \xrightarrow{d} (X(\lambda) : \lambda \in \mathbb{R})$ as $n \to \infty$.
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Other questions

"Natural questions"

- What happens if we start with a configuration other than the empty graph?
- Related to the entrance boundary of the multiplicative coalescent.

Unnatural next questions

- Scaling limits?

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- What happens if we start with a configuration other than the empty graph?
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Unnatural next questions

- Scaling limits?
- Conjecture: Rescale each edge by $n^{-1/3}$
- Largest components converge to random fractals (Gromov-Hausdorff sense), the same limits as for Erdos-Renyii

Minimal spanning tree

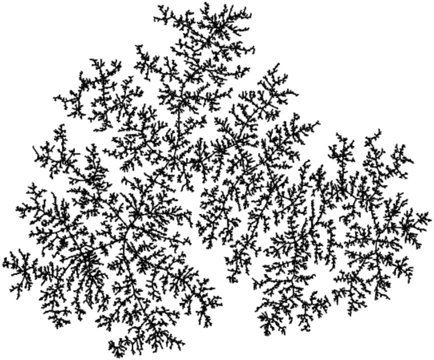

Figure: Minimal spanning tree on the complete graph on $n = 100{,}000$ vertices. Generated by Nicolas Broutin.

Minimal spanning tree

- Connected Graph, put distinct positive edge lengths (road network)
- Want to get a spanning graph,

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- Enormous literature in applied sciences
- Deep connections to statistical physics models of disorder

Statistical physics models of disorder

Weak disorder (First passage percolation)

- Weight of a path = sum of weight on edges
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Statistical physics models of disorder

Weak disorder (First passage percolation)

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Strong disorder (Minimal spanning tree)

- Weight of path = max edge on the path
- Choose optimal path

Adario-Berry, Broutin, Goldschmidt + Miermont

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- Consider complete graph, each edge given iid continuous edge length
- $\mathscr{M}_n$ minimal spanning tree
- Asymptotics?

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- For the Erdos-Renyi, the largest components at criticality rescaled by $n^{-1/3}$ converge to limiting random fractals [BBG]
- This implies that $n^{-1/3}\mathscr{M}_n$ converges to a limiting random fractal [BBGM]
- **Open Problem:** Show that for these models, have same limiting structure

More in Lecture 3.