Dynamic network models
Worksheet for Lecture 2

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(1) Let $\{\mathcal{F}(t) : t \geq 0\}$ be the continuous time branching where each individual has a Poisson rate one offspring distribution. Let

$$T_n = \inf \{t : |\mathcal{F}(t)| = n\}$$

Describe the attachment scheme corresponding to the tree process $\mathbf{t}_n = \mathcal{F}(T_n)$.

(2) **Master equation:** Refer to the random tree model in the previous question. Recall that we think of edges pointed from parents to children in the tree. Fix $k \geq 0$ and let $N_k(n)$ be the number of vertices with out-degree k in $\mathbf{t}_n$. Write $p_k(n) = \mathbb{E}(N_k(n)/n)$ denote the expected proportion of vertices with out-degree k. Start with $k = 0$ and find a recursion relation between $p_0(n+1)$ and $p_0(n)$. What does this suggest about $\lim_{n\to\infty} p_0(n)$? Now understand limits $\lim_{n\to\infty} p_k(n)$ for general k.

(3) Refer to the random tree model in question 1. In the above case, argue intuitively that $e^{-t}|\mathcal{F}(t)|$ is a martingale (with respect to the natural filtration). Using this derive the limiting degree distribution.

(4) Let M_n be the maximum degree of the tree $\mathbf{t}_n$ in the above case. What is the magnitude of M_n?

(5) Fix a positive sequence $\{f(k) : k \geq 1\}$. Suppose one has a branching process with offspring distribution

$$\mathcal{P} = (Y_1, Y_2, \ldots); \qquad Y_i = \sum_{k=1}^{i} L_i$$

where L_i has exponential rate $f(k)$ distribution. For any $t \geq 0$, $\mathcal{P}[0,t] := \sup\{i : Y_i \leq t\}$ be the number of points in the interval $[0,t]$. Recall that the Malthusian rate of growth λ which controls how quickly this branching process grows is given by the equation

$$\mathbb{E}(\mathcal{P}[0, T_\lambda]) = 1$$

where T_λ is an $\exp(\lambda)$ random variable independent of $\mathcal{P}$. Find an equation in terms of $\{f(k) : k \geq 1\}$ that λ solves.

(6) Recall the superstar model $\mathcal{T}_n(p)$ where each new vertex with probability p connects to the vertex v_0 with probability p and with probability $(1-p)$ connects to any other vertex with proportional to their current (out)-degree. Show that this model can be embedded in the two-type continuous time branching process $\{\mathcal{F}(t) : t \geq 0\}$ in the slides with surgery namely:
  (a) Start with one red vertex v_1 at time $t = 0$.
  (b) Every vertex v reproduces at rate proportional to the number of their current blue children $+1$.
  (c) Each new vertex is colored red with probability p and blue with probability $(1-p)$.
  (d) From $\mathcal{F}(T_{n-1})$ one can get $\mathcal{T}_n(p)$ as follows: Insert new vertex v_0. Delete the edges of each red vertex to their parent in $\mathcal{F}(T_{n-1})$ and re-attach to v_0.

(7) In the context of the previous example, let $B(t)$ be the number of blue vertices at time t and $Z(t) := |\mathcal{F}(t)|$ be the size of the tree. Find a constant λ_p and constants a, b such that $e^{-\lambda_p t}(aZ(t) + bB(t))$ is a martingale. This tells us about the relative rate of growth of this process.