(1) Let \( \{ \mathcal{F}(t) : t \geq 0 \} \) be the continuous time branching where each individual has a Poisson rate one offspring distribution. Let 

\[ T_n = \inf \{ t : |\mathcal{F}(t)| = n \} \]

Describe the attachment scheme corresponding to the tree process \( t_n = \mathcal{F}(T_n) \).

(2) **Master equation:** Refer to the random tree model in the previous question. Recall that we think of edges pointed from parents to children in the tree. Fix \( k \geq 0 \) and let \( N_k(n) \) be the number of vertices with out-degree \( k \) in \( t_n \). Write \( p_k(n) = \mathbb{E}(N_k(n)/n) \) denote the expected proportion of vertices with out-degree \( k \). Start with \( k = 0 \) and find a recursion relation between \( p_0(n+1) \) and \( p_0(n) \). What does this suggest about \( \lim_{n \to \infty} p_0(n) \)? Now understand limits \( \lim_{n \to \infty} p_k(n) \) for general \( k \).

(3) Refer to the random tree model in question 1. In the above case, argue intuitively that \( e^{-t|\mathcal{F}(t)|} \) is a martingale (with respect to the natural filtration). Using this derive the limiting degree distribution.

(4) Let \( M_n \) be the maximum degree of the tree \( t_n \) in the above case. What is the magnitude of \( M_n \)?

(5) Fix a positive sequence \( \{ f(k) : k \geq 1 \} \). Suppose one has a branching process with offspring distribution

\[ P = (Y_1, Y_2, \ldots); \quad Y_i = \sum_{k=1}^{i} L_i \]

where \( L_i \) has exponential rate \( f(k) \) distribution. For any \( t \geq 0 \), \( P[0,t] := \sup \{ i : Y_i \leq t \} \) be the number of points in the interval \([0,t]\). Recall that the Malthusian rate of growth \( \lambda \) which controls how quickly this branching process grows is given by the equation

\[ \mathbb{E}(P[0,T_\lambda]) = 1 \]

where \( T_\lambda \) is an \( \exp(\lambda) \) random variable independent of \( P \). Find an equation in terms of \( \{ f(k) : k \geq 1 \} \) that \( \lambda \) solves.
(6) Recall the superstar model $\mathcal{T}_n(p)$ where each new vertex with probability $p$ connects to the vertex $v_0$ with probability $p$ and with probability $(1-p)$ connects to any other vertex with proportional to their current (out)-degree. Show that this model can be embedded in the two-type continuous time branching process $\{\mathcal{F}(t): t \geq 0\}$ in the slides with surgery namely:

(a) Start with one red vertex $v_1$ at time $t = 0$.
(b) Every vertex $v$ reproduces at rate proportional to the number of their current blue children $+1$.
(c) Each new vertex is colored red with probability $p$ and blue with probability $(1-p)$.
(d) From $\mathcal{F}(T_{n-1})$ one can get $\mathcal{T}_n(p)$ as follows: Insert new vertex $v_0$. Delete the edges of each red vertex to their parent in $\mathcal{F}(T_{n-1})$ and re-attach to $v_0$.

(7) In the context of the previous example, let $B(t)$ be the number of blue vertices at time $t$ and $Z(t) := |\mathcal{F}(t)|$ be the size of the tree. Find a constant $\lambda_p$ and constants $a, b$ such that $e^{-\lambda_p t}(aZ(t) + bB(t))$ is a martingale. This tells us about the relative rate of growth of this process.