Dynamic network models
Worksheet for Lecture 1
Shankar Bhamidi

(1) **Proof of deterministic Lemma:** Prove the deterministic lemma from the slides for the exploration walk $Z_n(i) = Z_n(i-1) + c(i) - 1$ with $Z_n(0) = 0$, namely the first time you hit -1 is when you finish the first component, the first time you hit -2 is when you finish the second component and so on.

(2) **Walks and supercritical Erdos-Renyi random graph:** Walks or exploration processes can be used not just at criticality. Consider $\mathcal{G}_n(n, \lambda/n)$ where $\lambda > 1$. Start from a random vertex i_1 and explore the graph sequentially. Let

$$R_{t+1} = R_t \cup \{i_t\}$$
$$A_{t+1} = A_t \cup \{\text{ children of } i_t\} - \{i_t\}$$
$$U_{t+1} = U_t - (\{\text{ children of } i_t\} \cup \{i_t\})$$

Thus when we finish exploring a component, we chose a vertex uniformly from the unexplored set U_t and start exploring that vertex's component. Define for $s \geq 0$, $u_s = U_{ns}/n$, the density of unexplored vertices. Argue intuitively that $u_s \to \exp(-\lambda s)$. From this see what this should imply about the size of the giant component in the graph.

(3) **Size-biased re-ordering:** Suppose $w_1, w_2, \ldots w_n$ are positive random variables, iid with common distribution F. Let $\sigma_3 = \mathbb{E}(w_1^3) < \infty$ and $\nu = \sigma_2/\mu = \mathbb{E}(w_1^2)/\mathbb{E}(w_1) = 1$. Write $\mu = \mathbb{E}(w_1)$. Let $(v(1), v(2), \ldots, v(n))$ be a size-biased re-ordering of $\{1, 2, \ldots, n\}$ using the weight sequence $\{w_i\}_{1 \leq i \leq n}$. Show for any fixed $u > 0$,

$$\frac{1}{n^{2/3}} \sum_{i=1}^{n^{2/3}u} w_{v(i)}^2 \sim \frac{\sigma_3 u}{\mu}$$

Note that without size-biased re-ordering, one would have

$$\frac{1}{n^{2/3}} \sum_{i=1}^{n^{2/3}u} w_i^2 \sim u\sigma_2 = u\mu$$

(4) **Erdos-Renyi process:** Consider the dynamic version of the Erdos-Renyi random graph where you start with the empty graph (no edges) at time t and where each edge has a rate $1/n$ Poisson clock. When this rings that edge is formed. Let $\{\mathcal{F}_t\}_{t \geq 0}$ be

the filtration of the process. Let $\mathcal{C}_i(t)$ denote the size of the i-th largest component at time t. Define the susceptibility

$$s_2^n(t) := \frac{1}{n}\sum_i (\mathcal{C}_i(t))^2, \qquad \Delta s_2^n(t) = s_2^n(t+) - s_2^n(t). \tag{0.1}$$

Calculate the infinitesimal expectation $\mathbb{E}(\Delta s_2^n(t)|\mathcal{F}_t)$. This suggests $s_2^n(t) \to s_2(t)$ for some limiting deterministic function. Find $s_2(t)$. What does this suggest about the critical time t_c for the Erdos-Renyi random graph process?

(5) **Random graph with immigration:** Consider the following dynamic random graph process. At time $t = 0$ you start with an empty system with **no vertices** or edges. New vertices enter the system at rate n. Edges form between pre-existing vertices at rate $1/n$. Derive a limiting differential equation for $s_2^n(t)$ in this case and solve it. What does this suggest about the critical time t_c for this model?

(6) **Rigorous proof for density of singletons:** Recall the Bohman-Frieze process, where at time $t = 0$ we start with n vertices. For all ordered pairs of edges (e_1, e_2) we have a rate $2/n^3$ Poisson process. When one of these ring, if e_1 connects two singletons at that stage use e_1, else use e_2.

Let $X_n(t)$ be the number of singletons at time t and let $x_n(t) := X_n(t)/n$. Give a rigorous proof to show that for any fixed T,

$$\sup_{0\le t\le T} |x_n(t) - x(t)| \xrightarrow{\mathbb{P}} 0$$

where $x(t)$ solves the ODE

$$x'(t) = -x^2(t) - (1 - x^2(t))x(t), \qquad x(0) = 1$$