Selfish Routing in Networks

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Exercise Sheet

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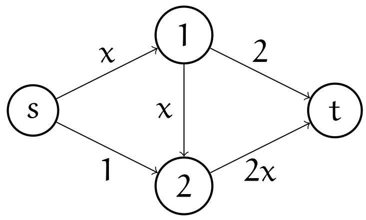
Exercise 1. Consider the given single commodity network

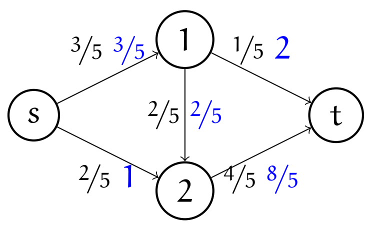


with the demand $d = 1$.
Compute the directed Wardrop Equilibrium flow, the socially optimal flow as well as the social cost of both flows and the Price of Anarchy.

Solution. Let $P_1 = (s, 1, t)$ be the upper path, $P_2 = (s, 1, 2, t)$ the middle path, and $P_3 = (s, 2, t)$ the lower path. Then the Wardrop Equilibrium is

$$f_{P_1} = 1/5, f_{P_2} = 2/5, f_{P_3} = 2/5.$$

The edge flows and latencies (in blue) are:



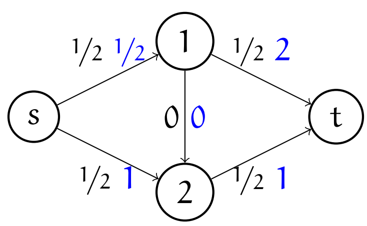
The social cost of the Wardrop Equilibrium f is

$$C(f) = \sum_{e \in E} f_e c_e(f_e) = 3/5 \cdot 3/5 + 1/5 \cdot 2 + 2/5 \cdot 2/5 + 2/5 \cdot 1 + 4/5 \cdot 8/5 = 1 \cdot 13/5 = 13/5$$

The optimal flow g is

$$g_{P_1} = 1/2, g_{P_2} = 0, g_{P_3} = 1/2.$$

The edge flows and latencies (in blue) are:



The social cost of the optimal flow g is

$$C(g) = \sum_{e \in E} g_e c_e(g_e) = 1/2 \cdot 1/2 + 1/2 \cdot 2 + 0 + 1/2 \cdot 1 + 1/2 \cdot 1 = 9/4.$$

So the price of anarchy is

$$\mathrm{PoA} = \frac{C(f)}{C(g)} = \frac{52}{45} \approx 1.156.$$

$\square$

Exercise 2. For the class of quadratic cost functions with offsets and non-negative coefficients

$$\mathcal{C} = \{c(x) = ax^2 + b : a, b \geq 0\}$$

compute the Price of Anarchy by computing the anarchy value

$$\beta = \sup_{c \in C} \sup_{f,g \geq 0} \frac{(c(f) - c(g))g}{c(f)f}.$$

Give an example network that proves that this Price of Anarchy bound is tight.

Solution. We know that the price of anarchy can be computed as

$$\mathrm{PoA} = \frac{1}{1-\beta}.$$

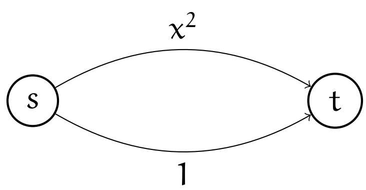
We compute

$$\begin{aligned}
\beta &= \sup_{c \in C} \sup_{f,g} \frac{(c(f) - c(g))g}{c(f)f} \\
&= \sup_{a,b \geq 0} \sup_{f,g \geq 0} \frac{(af^2 + b - (ag^2 + b)) \cdot g}{(af^2 + b) \cdot f} \\
&= \sup_{a,b \geq 0} \sup_{f,g \geq 0} \frac{a \cdot g \cdot (f^2 - g^2)}{(af^2 + b) \cdot f} \\
&= \sup_{a \geq 0} \sup_{f,g \geq 0} \frac{a \cdot g \cdot (f^2 - g^2)}{af^2 \cdot f} \\
&= \sup_{f \geq g \geq 0} \frac{\alpha f \cdot (f^2 - g^2)}{f^3} \\
&= \sup_{f \geq 0, \alpha \in [0,1]} \frac{\alpha f \cdot (f^2 - (\alpha f)^2)}{f^3} \\
&= \sup_{\alpha \in [0,1]} (1 - \alpha^2)\alpha = \frac{2}{3\sqrt{3}}
\end{aligned}$$

Thus the Price of Anarchy is

$$\mathrm{PoA} = \frac{1}{1 - \frac{2}{3\sqrt{3}}} = \frac{9}{9 - 2\sqrt{3}} \approx 1.625.$$

The worst-case network with demand $d = 1$ is:



The Wardrop Equilibrium is obviously $f = (f_1, f_2) = (1, 0)$ with cost $C(f) = 1$. Let $g = (g_1, g_2)$ be the optimal flow. Then we know that the flow over the upper edge of the optimal flow g_1 must solve

$$\min_{x \geq 0} x^2 \cdot x + 1 \cdot (1 - x) = \min_{x \geq 0} x^3 - x + 1$$

and thus obtain $g = (1/\sqrt{3}, 1 - 1/\sqrt{3})$ with cost $C(g) = 1 - \frac{2}{3\sqrt{3}}$. $\square$

Exercise 3. For some class of cost functions $\mathcal{C}$, let β be the anarchy value as in the exercise above.
Let f be the Wardrop Equilibrium in some network and for some demands $(d_i)_{i\in I}$ and let furthermore be g a optimal flow in the same network for the demands $((1+\beta)d_i)_{i\in I}$. Show that

$$C(f) \leq C(g).$$

Solution. Define $x := {}^1\!/_{1+\beta} \cdot g$. Then g is also a flow for demand d. Then we have by the variational inequality (first inequality) and the definition of β (second inequality) that

$$\begin{aligned}
C(f) &= \sum_{e\in E} f_e \cdot c_e(f_e) \\
&\leq \sum_{e\in E} x_e \cdot c_e(f_e) \\
&= \frac{1}{1+\beta} \sum_{e\in E} g_e \cdot c_e(f_e) \\
&= \sum_{e\in E} g_e \cdot c_e(f_e) - g_e \cdot c_e(g_e) + g_e \cdot c_e(g_e) \\
&\leq \frac{1}{1+\beta} \sum_{e\in E} (\beta f_e c_e(f_e) + g_e c_e(f_e)) \\
&= \frac{\beta}{1+\beta} C(f) + \frac{1}{1+\beta} C(g).
\end{aligned}$$

This proves the claim. $\square$

Exercise 4. Consider a graph $G = (V, E)$ with constant edge cost $k_e > 0$ for every edge $e \in E$ and n players. A strategy for every player is to choose a path between some designated nodes $u_i, v_i \in V$. The edge costs are equally distributed between players that use an edge and the private cost of every player is the sum of all shares of the costs, i.e.

$$\pi_i(s) = \sum_{e \in s_i} \frac{k_e}{x_e(s)}$$

where $x_e(s) := |\{i : e \in s_i\}|$. Let

$$C(s) = \sum_{i \in N} \pi_i(s)$$

denote the social cost of some strategy profile.
Prove that there is a Nash Equilibrium s^* such that $C(s^*) \leq H_n \cdot \min_{s \in S} C(s)$ where

$$H_n := \sum_{k=1}^{n} \frac{1}{k}$$

is the n-th harmonic number.

Solution. At first, we observe

$$C(s) = \sum_{i \in N} \pi_i(s) = \sum_{e \in E} k_e.$$

We then define the potential function

$$P(s) := \sum_{e \in E} \sum_{k=1}^{x_e(s)} \frac{k_e}{k}$$

and obtain

$$P(s) = \sum_{e \in E} k_e H_{x_e(s)} \leq H_n \sum_{\substack{e \in E: \\ x_e(s) > 0}} k_e = H_n \cdot C(s)$$

and observe that this is actually a potential function since

$$\begin{aligned} \pi_i(t_i, s_{-i}) - \pi_i(s) &= \sum_{e \in t_i \setminus s_i} \frac{k_e}{x_e(t_i, s_{-i})} - \sum_{e \in s_i \setminus t_i} \frac{k_e}{x_e(s)} \\ &= P(t_i, s_{-i}) - P(s). \end{aligned}$$

Then the strategy profile s^* that minimizes $P(s)$ must be a Nash Equilibrium since there can not be any profitable deviation of any player.

We then compute

$$\begin{aligned}
C(s^*) &= \sum_{i \in N} \pi_i = \sum_{i \in N} \sum_{e \in s_i^*} \frac{k_e}{x_e(s^*)} \\
&= \sum_{\substack{e \in E: \\ x_e(s^*) > 0}} k_e \leq \sum_{\substack{e \in E: \\ x_e(s^*) > 0}} k_e \cdot H_{x_e(s^*)} \\
&= P(s^*) \leq P(s) \\
&\leq H_n \sum_{\substack{e \in E: \\ x_e(s) > 0}} k_e = H_n C(s).
\end{aligned}$$

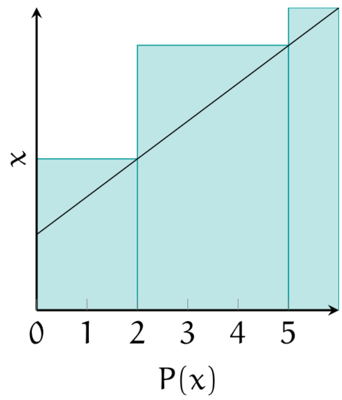
Since this holds true for every strategy profile s, this in particular holds for the social optimal strategy profile s. □

Exercise 5. Prove that every weighted congestion game with affine linear costs $c_e(x) = a_e x + b_e$ has a pure Nash Equilibrium by defining a suitable potential function.

Solution. We define the potential function $P : \mathbf{S} \to \mathbb{R}$ with

$$P(s) = \sum_{e \in E} \sum_{i \in N : e \in s_i} d_i c_e \left(\sum_{j \in \{1,\dots,i\} : e \in s_j} d_j \right).$$

Then P is *independent* of the ordering of the players, see the graph below:



We thus can assume without loss of generality that $i = n$.

$$\begin{aligned}
\pi_i(t_i, s_{-i}) - \pi_i(s) &= \pi_n(t_n, s_{-n}) - \pi_n(s) \\
&= d_n \sum_{t_n \setminus s_n} c_e(x_e(t_n, s_{-n})) - d_n \sum_{s_n \setminus t_n} c_e(x_e(s)) \\
&= P(t_n, s_{-n}) - P(s).
\end{aligned}$$

So if we observe a sequence of strategy profiles where there are unilateral improvements, i.e.

$$\pi_i(t_i, s_{-i}) - \pi_i(s) < 0$$

the potential function decreases along this sequence. Since there are only finitely many possible strategy profiles, we have to reach a minimum. This minimum must be a Nash equilibrium since there are no more profitable deviations for any player. □

Exercise 6. A congestion game is called singleton if $|s_i| = 1$ for all $i \in N$. Show that a singleton weighted congestion game has a pure Nash Equilibrium by showing that the vector containing the player's private costs sorted in non-increasing order decreases lexicographically along of any sequence of unilateral improvement.

Solution. We say some vector $x \in \mathbb{R}^n$ is *lexicographically smaller* than $y \in \mathbb{R}^n$, denoted by $x \prec_L y$, if

$$x_j < y_j \text{ for } j = \min\{i : x_i \neq y_i\}.$$

Let

$$x = (\pi_1(s), \pi_2(s), \dots, \pi_n(s))$$

be the sorted vector of private cost, i.e. the players are sorted such that $\pi_1(s) \leq \pi_2(s) \leq \dots \leq \pi_n(s)$. So assume, some player i changes to another resource and this deviation is profitable, i.e $\pi_i(t_i, s_{-i}) < \pi_i(s)$ and call the new vector of sorted private cost y. Then he must have chosen a resource that is used by either only players $j > i$ or no player. All players $j < i$ have higher cost, so this would not be a profitable deviation.
This in particular means, that the private cost of every player $j < i$ can not increase, meaning that $y_j \leq x_j$ for all $j < i$. In particular, no player $j > i$ will get higher cost then player i because the only way that j's cost change is that i now uses his resource. So in particular no player $j > i$ will be before i in the ordering and thus we know $y_i = \pi_i(t_i, s_{-i}) < \pi_i(s) = x_i$ and thus $y \prec_L x$.
Since there are only finitely many strategy profiles s, there has to be a lexicographically minimal vector $x(s)$. The associated strategy profile has to be a nash equilibrium. □