Exercise 1. Consider the given single commodity network

with the demand $d = 1$.

Compute the directed Wardrop Equilibrium flow, the socially optimal flow as well as the social cost of both flows and the Price of Anarchy.

Solution. Let $P_1 = (s, 1, t)$ be the upper path, $P_2 = (s, 1, 2, t)$ the middle path, and $P_3 = (s, 2, t)$ the lower path. Then the Wardrop Equilibrium is

$$f_{P_1} = \frac{1}{5}, f_{P_2} = \frac{2}{5}, f_{P_3} = \frac{2}{5}.$$

The edge flows and latencies (in blue) are:

The social cost of the Wardrop Equilibrium $f$ is

$$C(f) = \sum_{e \in E} f_e c_e(f_e) = \frac{3}{5} \cdot \frac{3}{5} + \frac{1}{5} \cdot 2 + \frac{2}{5} \cdot \frac{2}{5} + \frac{2}{5} \cdot 1 + \frac{4}{5} \cdot \frac{8}{5} = 1 \cdot \frac{13}{5} = \frac{13}{5}$$

The optimal flow $g$ is

$$g_{P_1} = \frac{1}{2}, g_{P_2} = 0, g_{P_3} = \frac{1}{2}.$$

The edge flows and latencies (in blue) are:

The social cost of the optimal flow $g$ is

$$C(g) = \sum_{e \in E} g_e c_e(g_e) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot 2 + 0 + \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 1 = 9/4.$$
So the price of anarchy is

$$\text{PoA} = \frac{C(f)}{C(g)} = \frac{52}{45} \approx 1.156.$$
Exercise 2. For the class of quadratic cost functions with offsets and non-negative coefficients
\[ C = \{ c(x) = ax^2 + b : a, b \geq 0 \} \]
compute the Price of Anarchy by computing the anarchy value
\[ \beta = \sup_{c \in C} \sup_{f, g \geq 0} \frac{(c(f) - c(g))g}{c(f)f}. \]

Give an example network that proves that this Price of Anarchy bound is tight.

Solution. We know that the price of anarchy can be computed as
\[ \text{PoA} = \frac{1}{1 - \beta}. \]
We compute
\[ \beta = \sup_{c \in C} \sup_{f, g \geq 0} \frac{(c(f) - c(g))g}{c(f)f} \]
\[ = \sup_{a, b \geq 0} \sup_{f, g \geq 0} \frac{(af^2 + b - (ag^2 + b)) \cdot g}{(af^2 + b) \cdot f} \]
\[ = \sup_{a, b \geq 0} \sup_{f, g \geq 0} \frac{a \cdot g \cdot (f^2 - g^2)}{(af^2 + b) \cdot f} \]
\[ = \sup_{a \geq 0} \sup_{f, g \geq 0} \frac{a \cdot g \cdot (f^2 - g^2)}{af^2 \cdot f} \]
\[ = \sup_{f \geq g \geq 0} \frac{\alpha f \cdot (f^2 - g^2)}{f^3} \]
\[ = \sup_{f \geq 0, \alpha \in [0, 1]} \frac{\alpha f \cdot (f^2 - (\alpha f)^2)}{f^3} \]
\[ = \sup_{\alpha \in [0, 1]} (1 - \alpha^2)\alpha = \frac{2}{3\sqrt{3}} \]

Thus the Price of Anarchy is
\[ \text{PoA} = \frac{1}{1 - \frac{2}{3\sqrt{3}}} = \frac{9}{9 - 2\sqrt{3}} \approx 1.625. \]

The worst-case network with demand \( d = 1 \) is:

```
      x^2
     / \  \\
    S --- t
     \  /  \\
      1
```

The Wardrop Equilibrium is obviously \( f = (f_1, f_2) = (1, 0) \) with cost \( C(f) = 1 \). Let \( g = (g_1, g_2) \) be the optimal flow. Then we know that the flow over the upper edge of the optimal flow \( g_i \) must solve
\[ \min_{x \geq 0} x^2 \cdot x + 1 \cdot (1 - x) = \min_{x \geq 0} x^3 - x + 1 \]
and thus obtain $g = (\frac{1}{\sqrt{3}}, 1 - \frac{1}{\sqrt{3}})$ with cost $C(g) = 1 - \frac{2}{3\sqrt{3}}$. □
Exercise 3. For some class of cost functions $C$, let $\beta$ be the anarchy value as in the exercise above. Let $f$ be the Wardrop Equilibrium in some network and for some demands $(d_i)_{i \in I}$ and let furthermore be $g$ a optimal flow in the same network for the demands $((1+\beta)d_i)_{i \in I}$. Show that
\[ C(f) \leq C(g). \]

Solution. Define $x := \frac{1}{1+\beta} \cdot g$. Then $g$ is also a flow for demand $d$. Then we have by the variational inequality (first inequality) and the definition of $\beta$ (second inequality) that
\[
C(f) = \sum_{e \in E} f_e \cdot c_e(f_e) \\
\leq \sum_{e \in E} x_e \cdot c_e(f_e) \\
= \frac{1}{1+\beta} \sum_{e \in E} g_e \cdot c_e(f_e) \\
= \sum_{e \in E} g_e \cdot c_e(f_e) - g_e \cdot c_e(g_e) + g_e \cdot c_e(g_e) \\
\leq \frac{1}{1+\beta} \sum_{e \in E} (\beta f_e c_e(f_e) + g_e c_e(f_e)) \\
= \frac{\beta}{1+\beta} C(f) + \frac{1}{1+\beta} C(g).
\]
This proves the claim. □
Exercise 4. Consider a graph $G = (V, E)$ with constant edge cost $k_e > 0$ for every edge $e \in E$ and $n$ players. A strategy for every player is to choose a path between some designated nodes $u_i, v_i \in V$. The edge costs are equally distributed between players that use an edge and the private cost of every player is the sum of all shares of the costs, i.e.

$$\pi_i(s) = \sum_{e \in s_i} \frac{k_e}{x_e(s)}$$

where $x_e(s) := |\{i : e \in s_i\}|$. Let

$$C(s) = \sum_{i \in N} \pi_i(s)$$
denote the social cost of some strategy profile.

Prove that there is a Nash Equilibrium $s^*$ such that

$$C(s^*) \leq H_n \cdot \min_{s \in S} C(s)$$

where

$$H_n := \sum_{k=1}^{n} \frac{1}{k}$$

is the $n$-th harmonic number.

Solution. At first, we observe

$$C(s) = \sum_{i \in N} \pi_i(s) = \sum_{e \in E} k_e.$$ 

We then define the potential function

$$P(s) := \sum_{e \in E} \sum_{k=1}^{x_e(s)} \frac{k_e}{k}$$

and obtain

$$P(s) = \sum_{e \in E} k_e H_{x_e(s)} \leq H_n \sum_{e \in E, x_e(s) > 0} k_e = H_n \cdot C(s)$$

and observe that this is actually a potential function since

$$\pi_i(t_i, s_{-i}) - \pi_i(s) = \sum_{e \in t_i \setminus s_i} \frac{k_e}{x_e(t_i, s_{-i})} - \sum_{e \in s_i \setminus t_i} \frac{k_e}{x_e(s)}$$

$$= P(t_i, s_{-i}) - P(s).$$

Then the strategy profile $s^*$ that minimizes $P(s)$ must be a Nash Equilibrium since there can not be any profitable deviation of any player.
We then compute

\[ C(s^*) = \sum_{i \in N} \pi_i = \sum_{i \in N} \sum_{e \in s^*_i} \frac{k_e}{x_e(s^*)} \]

\[ = \sum_{e \in E : x_e(s^*) > 0} k_e \leq \sum_{e \in E : x_e(s^*) > 0} k_e \cdot H_{x_e(s^*)} \]

\[ = P(s^*) \leq P(s) \leq H_n \sum_{e \in E : x_e(s) > 0} k_e = H_n C(s). \]

Since this holds true for every strategy profile \( s \), this in particular holds for the social optimal strategy profile \( s \). \qed
**Exercise 5.** Prove that every weighted congestion game with affine linear costs \( c_e(x) = a_e x + b_e \) has a pure Nash Equilibrium by defining a suitable potential function.

**Solution.** We define the potential function \( P : S \to \mathbb{R} \) with

\[
P(s) = \sum_{e \in E} \sum_{i \in N \subseteq S_i} d_i c_e \left( \sum_{j \in \{1, \ldots, i\} \cap S_j} d_j \right).
\]

Then \( P \) is *independent* of the ordering of the players, see the graph below:

![Graph](image)

We thus can assume without loss of generality that \( i = n \).

\[
\pi_i(t_i, s_{-i}) - \pi_i(s) = \pi_n(t_n, s_{-n}) - \pi_n(s)
\]

\[
= d_n \sum_{t_n \setminus s_n} c_e(x_e(t_n, s_{-n})) - d_n \sum_{s_n \setminus t_n} c_e(x_e(s))
\]

\[
= P(t_n, s_{-n}) - P(s).
\]

So if we observe a sequence of strategy profiles where there are unilateral improvements, i.e.

\[
\pi_i(t_i, s_{-i}) - \pi_i(s) < 0
\]

the potential function decreases along this sequence. Since there are only finitely many possible strategy profiles, we have to reach a minimum. This minimum must be a Nash equilibrium since there are no more profitable deviations for any player. \(\square\)
Exercise 6. A congestion game is called singleton if $|s_i| = 1$ for all $i \in N$. Show that a singleton weighted congestion game has a pure Nash Equilibrium by showing that the vector containing the player’s private costs sorted in non-increasing order decreases lexicographically along of any sequence of unilateral improvement.

Solution. We say some vector $x \in \mathbb{R}^n$ is lexicographically smaller than $y \in \mathbb{R}^n$, denoted by $x \prec_L y$, if

$$x_j < y_j \text{ for } j = \min\{i : x_i \neq y_i\}.$$ 

Let

$$x = (\pi_1(s), \pi_2(s), \ldots, \pi_n(s))$$

be the sorted vector of private cost, i.e. the players are sorted such that $\pi_1(s) \leq \pi_2(s) \leq \cdots \leq \pi_n(s)$. So assume, some player $i$ changes to another resource and this deviation is profitable, i.e $\pi_i(t_i, s_{-i}) < \pi_i(s)$ and call the new vector of sorted private cost $y$. Then he must have chosen a resource that is used by either only players $j > i$ or no player. All players $j < i$ have higher cost, so this would not be a profitable deviation.

This in particular means, that the private cost of every player $j < i$ can not increase, meaning that $y_j \leq x_j$ for all $j < i$. In particular, no player $j > i$ will get higher cost then player $i$ because the only way that $j$’s cost change is that $i$ now uses his resource. So in particular no player $j > i$ will be before $i$ in the ordering and thus we know $y_i = \pi_i(t_i, s_{-i}) < \pi_i(s) = x_i$ and thus $y \prec_L x$.

Since there are only finitely many strategy profiles $s$, there has to be a lexicographically minimal vector $x(s)$. The associated strategy profile has to be a nash equilibrium. □