**Exercise 1.** Consider the given single commodity network

```
    s ---- x ---- 1 ---- 2 ---- t
      |            |            |
      1           x           2x
```

with the demand \( d = 1 \).
Compute the directed Wardrop Equilibrium flow, the socially optimal flow as well as the social cost of both flows and the Price of Anarchy.

**Exercise 2.** For the class of quadratic cost functions with offsets and non-negative coefficients

\[
\mathcal{C} = \{ c(x) = ax^2 + b : a, b \geq 0 \}
\]
compute the Price of Anarchy by computing the anarchy value

\[
\beta = \sup_{c \in \mathcal{C}} \sup_{f, g \geq 0} \frac{(c(f) - c(g))g}{c(f)}.
\]

Give an example network that proves that this Price of Anarchy bound is tight.

**Exercise 3.** For some class of cost functions \( \mathcal{C} \), let \( \beta \) be the anarchy value as in the exercise above.
Let \( f \) be the Wardrop Equilibrium in some network and for some demands \( (d_i)_{i \in I} \) and let furthermore be \( g \) a optimal flow in the same network for the demands \( ((1+\beta)d_i)_{i \in I} \). Show that

\[
C(f) \leq C(g).
\]
Exercise 4. Consider a graph $G = (V, E)$ with constant edge cost $k_e > 0$ for every edge $e \in E$ and $n$ players. A strategy for every player is to choose a path between some designated nodes $u_i, v_i \in V$. The edge costs are equally distributed between players that use an edge and the private cost of every player is the sum of all shares of the costs, i.e.

$$\pi_i(s) = \sum_{e \in s_i} k_e x_e(s)$$

where $x_e(s) := |\{i : e \in s_i\}|$. Let

$$C(s) = \sum_{i \in N} \pi_i(s)$$

denote the social cost of some strategy profile.

Prove that there is a Nash Equilibrium $s^*$ such that $C(s^*) \leq H_n \cdot \min_{s \in S} C(s)$ where

$$H_n := \sum_{k=1}^{n} \frac{1}{k}$$

is the $n$-th harmonic number.

Exercise 5. Prove that every weighted congestion game with affine linear costs $c_e(x) = a_e x + b_e$ has a pure Nash Equilibrium by defining a suitable potential function.

Exercise 6. A congestion game is called singleton if $|s_i| = 1$ for all $i \in N$. Show that a singleton weighted congestion game has a pure Nash Equilibrium by showing that the vector containing the player’s private costs sorted in non-increasing order decreases lexicographically along of any sequence of unilateral improvement.