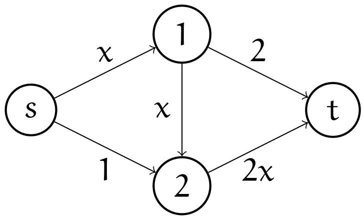
Selfish Routing in Networks

Exercise Sheet

M. Klimm, P. Warode

August 25, 2017

Exercise 1. Consider the given single commodity network



with the demand $d = 1$.
Compute the directed Wardrop Equilibrium flow, the socially optimal flow as well as the social cost of both flows and the Price of Anarchy.

Exercise 2. For the class of quadratic cost functions with offsets and non-negative coefficients

$$\mathcal{C} = \{c(x) = ax^2 + b : a, b \geq 0\}$$

compute the Price of Anarchy by computing the anarchy value

$$\beta = \sup_{c \in C} \sup_{f,g \geq 0} \frac{(c(f) - c(g))g}{c(f)f}.$$

Give an example network that proves that this Price of Anarchy bound is tight.

Exercise 3. For some class of cost functions $\mathcal{C}$, let β be the anarchy value as in the exercise above.
Let f be the Wardrop Equilibrium in some network and for some demands $(d_i)_{i \in I}$ and let furthermore be g a optimal flow in the same network for the demands $((1+\beta)d_i)_{i \in I}$.
Show that

$$C(f) \leq C(g).$$

Exercise 4. Consider a graph $G = (V, E)$ with constant edge cost $k_e > 0$ for every edge $e \in E$ and n players. A strategy for every player is to choose a path between some designated nodes $u_i, v_i \in V$. The edge costs are equally distributed between players that use an edge and the private cost of every player is the sum of all shares of the costs, i.e.

$$\pi_i(s) = \sum_{e \in s_i} \frac{k_e}{x_e(s)}$$

where $x_e(s) := |\{i : e \in s_i\}|$. Let

$$C(s) = \sum_{i \in N} \pi_i(s)$$

denote the social cost of some strategy profile.
Prove that there is a Nash Equilibrium s^* such that $C(s^*) \leq H_n \cdot \min_{s \in S} C(s)$ where

$$H_n := \sum_{k=1}^{n} \frac{1}{k}$$

is the n-th harmonic number.

Exercise 5. Prove that every weighted congestion game with affine linear costs $c_e(x) = a_e x + b_e$ has a pure Nash Equilibrium by defining a suitable potential function.

Exercise 6. A congestion game is called singleton if $|s_i| = 1$ for all $i \in N$. Show that a singleton weighted congestion game has a pure Nash Equilibrium by showing that the vector containing the player's private costs sorted in non-increasing order decreases lexicographically along of any sequence of unilateral improvement.