Stochastic geometry in telecommunications

Benedikt Jahnel
Challenges

- High complexity in space and time
- Large number of network components
- Random positioning and mobility of components
- Common communication technology
Let us consider networks with the following properties:

- random spatial distribution of network components
- static networks without time
- no additional infrastructure
Idea since the 1960's (Gilbert): Use stochastic geometry to model telecommunication networks.
The Poisson point process

A Poisson point process $X$ is a random cloud of points without cluster points (configuration of network components) with the following properties:

1. Point clouds in disjoint areas are stochastically independent.
2. The number of points in an area $A \subset \mathbb{R}^d$ is Poisson distributed with parameter $\lambda \text{Vol}(A)$:

$$
\mathbb{P}_\lambda (X \text{ has } k \text{ points in } A) = e^{-\lambda \text{Vol}(A)} \frac{(\lambda \text{Vol}(A))^k}{k!}
$$
Gilbert graph

- Gilbert (1961): First network model $g_r(X)$ based on Poisson point process $X$.
- Two network components $x, y$ can communicate if their distance is smaller than a connectivity parameter $r > 0$: $|x - y| < r$
Percolation

- Quality of network connectivity measured via size of connected components: clusters
- Existence of infinite cluster is called percolation

\[ \mathbb{P}_\lambda (g_r (X) \text{ percolates}) > 0 \]
Percolation is a phase-transition phenomena in the intensity parameter $\lambda$.

There exists $0 < \lambda_c < \infty$ with the property that

$$\lambda_c = \lambda_c(r) = \inf\{\lambda : \mathbb{P}(g_r(X) \text{ percolates}) > 0\}.$$

In the sub-critical regime $\lambda < \lambda_c$ we have local communication.

In the super-critical regime $\lambda > \lambda_c$ global communications is possible.

There is no known closed form expression for $\lambda_c$ as a function of $r$.

Numerical approximations suggest that $\lambda_c \approx 1.436$ for $r = 1$. 
Poisson tessellations

Based on the Poisson point process, a large number of tessellations can be defined.

Voronoi tessellation
Delaunay tessellation
Line tessellation
Johnson-Mehl tessellation
Relative neighborhood graph
Minimum spanning forest
Poisson Voronoi tesselation

- Also other names: Voronoi diagram, Voronoi decomposition, Voronoi partition, Dirichlet tessellation or Thiessen Polygon
- Formal definition: The Voronoi cell around the point \( x \in X \) is given by
  \[
  Z(x) = \{ z \in \mathbb{R}^d : |z - x| < |z - y| \text{ für alle } y \in X \setminus \{x\} \}.
  \]
- Used in a great number of scientific fields (Algorithmic geometry, material sciences, ...) and applications (biology, chemistry, meteorology, crystallography, architektur, ...).
Poisson tessellations for urban networks
Poisson tessellations for urban networks
Poisson tessellations for urban networks
Poisson tessellations for urban networks
Poisson tessellations for urban networks
Fundamental network characteristics

1. What is the probability that two users are connected dependent on their distance?

\[ p_s = \mathbb{P}(o \leftrightarrow s e_1) \]

2. What is the proportion of pairs of connected users?

\[ \pi_s = \mathbb{E}(\#(X \leftrightarrow Y) \in B_s(o)) \]

3. What is the probability that two users are connected if the number of hops is constraint?

\[ \hat{p}_s = \mathbb{P}(o \leftrightarrow s e_1 | \# \text{ Hops} < \alpha s) \]
Let's get to it.
Voronoi structure: Avignon (France)

From: Open street maps
Manhattan grid structure: Bouake (Ivory Coast)

From: Open street maps
Manhattan grid structure: Xian (China)

From: Open street maps
Nested Manhattan grid structure with users
Poisson tessellations for urban networks
Essentially asymptotically connected Cox point processes

**Theorem**

If the random intensity measure is essentially asymptotically connected, then $0 < \lambda_c < \infty$.

- Examples are Poisson Voronoi tessellation (PVT) or the Poisson Delaunay tessellation.
- Manhattan and nested Manhattan grids are not stabilizing and proofs for non-triviality should be much harder.
- Continuum percolation for general Cox processes can exhibit pathological effects, for example $\lambda_c = 0$ (see Błaszczyszyn & Yogeshwaran, 2013).
Approximations for $\lambda_c$

- Users form Cox point process $X$ with random intensity $\lambda|du \cap S|$ where
- $S$ realization of a street system, e.g., PVT
- PVT is characterized by length intensity $\gamma = \mathbb{E}[|S \cap [-1/2, 1/2]^2|]$
Approximations for $\lambda_c$

- Users form Cox point process $X$ with random intensity $\lambda \mid du \cap S$ where $S$ realization of a street system, e.g., PVT
- PVT is characterized by length intensity $\gamma = \mathbb{E}[|S \cap [-1/2, 1/2]^2|]$

- Dense streets: approximate $X$ by 2D Poisson point process with spatial intensity $\gamma \lambda$
  - $4.51\pi^{-1}r^{-2}$ is the approximate critical intensity for percolation of the Boolean model
  - Approximation I:
    $\lambda_c \approx 4.51\pi^{-1}\gamma^{-1}r^{-2}$ becomes exact for $\gamma \uparrow \infty$ with $\lambda \gamma$ fixed.
Approximations for $\lambda_c$

- Users form Cox point process $X$ with random intensity $\lambda|du \cap S|$ where
- $S$ realization of a street system, e.g., PVT
- PVT is characterized by length intensity $\gamma = \mathbb{E}[|S \cap [-1/2, 1/2]^2|]$

- Sparse streets: approximate $X$ by inhomogenous Bernoulli bond percolation with parameter $b^l$ where $l$ is edge length
  - $b_{\text{crit}}$ is critical percolation threshold. For PVT with distance parameter 1, by simulations $b_{\text{crit}} \approx 0.725$
  - Approximation II:
    $$\frac{\lambda_c}{\gamma} \exp(-\lambda_c r) \approx -\log(b_{\text{crit}})$$ becomes exact for $\gamma \downarrow 0$ with $\frac{\lambda}{\gamma} \exp(-\lambda r)$ fixed.
Fundamental network characteristics

1. What is the probability that two users are connected dependent on their distance?
   
   \[ p_s = \mathbb{P}^0(o \leftrightarrow se_1)? \]

2. What is the expected number of pairs of connected users?
   
   \[ \pi_s = \mathbb{E}(\#(X \leftrightarrow Y) \in B_s(o))? \]

3. What is the probability that two users are connected if the number of hops is constraint?
   
   \[ \hat{p}_s = \mathbb{P}^0(o \leftrightarrow se_1 | \# \, \text{Hops} < \alpha s)? \]
Connection probability as a function of distance

- Palm calculus to ensure $o$ on streets: Define Palm version of $S$ via

$$\mathbb{E}^0 f(S) = \frac{1}{\gamma} \mathbb{E} \int_{Q_1(o) \cap S} f(S - u)du$$

- Define device connection probability at relative position $B$ via

$$p_B(\lambda, r, \gamma) = \frac{\mathbb{E}^0 \int_{S \cap B} \mathbb{1}\{o \leftrightarrow v \text{ in } g_r(X \cup \{v\})\} dv}{\mathbb{E}^0|S \cap B|}$$

Where $\mathbb{E}^0$ denote the Palm measure for $S$ and $X$.

**Theorem (Scaling invariance)**

Let $\lambda, r, \gamma > 0$ be arbitrary. Then, for every $a > 0$,

$$p_{aB}(a^{-1}\lambda, ar, a^{-1}\gamma) = p_B(\lambda, r, \gamma).$$
Large distance approximation

\( p_s = p_{Q_1(se_1)} \) converges to the square of the percolation probability

\[
\theta(\lambda, r, \gamma) = \mathbb{P}^0(o \leftrightarrow \infty \text{ in } g_r(X))
\]

**Theorem**

Let \( \lambda, r, \gamma > 0 \) be arbitrary. Assume some cluster uniqueness and vacancy condition, then

\[
\lim_{s \uparrow \infty} p_s = \theta^2.
\]

More precisely, there exists \( c > 0 \) such that \( |p_s - \theta^2| \leq \exp(-cs) \) for all sufficiently large \( s \).
Percolation approximation w.r.t. $\lambda$ - universality

- no closed form expression available for $\theta$

$$\lim_{c \to c_\ast} \log \theta(c) / \log(c) = \frac{5}{36}.$$
Percolation approximation w.r.t. $\lambda$ - universality

- No closed form expression available for $\theta$

Conjecture

Let $r, \gamma > 0$. Then, $\theta(\lambda) \approx (\lambda - \lambda_c)^{5/36}$ as $\lambda \to \lambda_c$.

More precisely, \( \lim_{\lambda \to \lambda_c} \frac{\log \theta(\lambda)}{\log(\lambda - \lambda_c)} = 5/36 \).

- Smirnov and Werner 2001 for the triangular lattice
- Assumed to be universal, i.e., depend only on the local structure of the graph and the dimension
Percolation approximation w.r.t. $\lambda$ - large deviations

- finite box crossing: $\theta_K(\lambda) = \mathbb{P}^0(o \leftrightarrow X_i \text{ for some } X_i \in X \setminus Q_K(o))$

**Theorem (large $\lambda_U$)**

Let $r, \gamma > 0$ and $K > 2r_U$ be arbitrary. Then,

$$\theta_K(\lambda) \approx 1 - \exp(-2r\lambda)$$

as $\lambda \to \infty$. More precisely,

$$\lim_{\lambda \to \infty} \lambda^{-1} \log(1 - \theta_K(\lambda)) = -2r.$$
Thank you for your attention.