Computational Harmonic Analysis meets Imaging Sciences Part I

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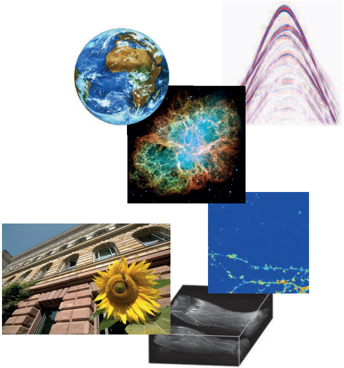
Imaging Science Today

Due to the data deluge, the area of imaging science is of tremendous importance in today's world.

Main Tasks

- Acquisition
- Preprocessing
  - Denoising, Inpainting, ...
- Analysis
  - Feature Detection, ...
- Storing
  - Compression, ...



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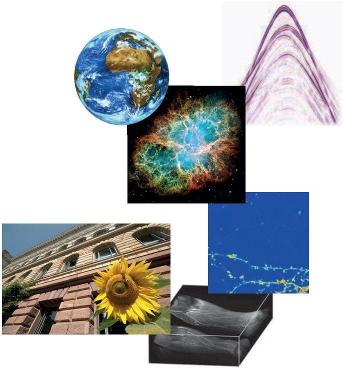


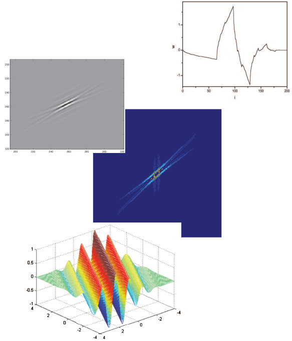
What has Computational Harmonic Analysis to offer?

Computational Harmonic Analysis

Representation systems designed by Computational Harmonic Analysis concepts have established themselves as a standard tool in applied mathematics, computer science, and engineering.

Examples:

- Wavelets.
- Ridgelets.
- Curvelets.
- Shearlets.
- ...



Key Property:

Fast Algorithms combined with Sparse Approximation Properties!

Outline

An Computational Harmonic Analysis Viewpoint

Exploit a carefully designed representation system $(\psi_\lambda)_{\lambda \in \Lambda} \subseteq \mathcal{H}$:

$$\mathcal{H} \ni f \longrightarrow (\langle f, \psi_\lambda \rangle)_{\lambda \in \Lambda} \longrightarrow \sum_{\lambda \in \Lambda} \langle f, \psi_\lambda \rangle \, \psi_\lambda = f.$$

Desiderata:

- Special features encoded in the "large" coefficients $|\langle f, \psi_\lambda \rangle|$.
- Efficient representations:

$$f \approx \sum_{\lambda \in \Lambda_N} \langle f, \psi_\lambda \rangle \, \psi_\lambda, \quad \#(\Lambda_N) \text{ small}$$

Goals:

- Modification of the coefficients according to the task.
- Derive high compression by considering only the "large" coefficients.

Two Main Viewpoints

Decomposition:

$$\mathcal{H} \ni f \longrightarrow (\langle f, \psi_\lambda \rangle)_{\lambda \in \Lambda}.$$

- Preprocessing (e.g. denoising).
- Analysis (e.g. feature detection).
- Clustering/Classification.
- ...

Efficient/Sparse Representations:

$$f = \sum_{\lambda \in \Lambda} c_\lambda \psi_\lambda.$$

- Compression.
- Regularization of inverse problems.
- Ansatz functions for PDE solvers.
- ...

Decomposition

Denoising (Preprocessing):

$$\mathcal{H} \ni f \longrightarrow (\langle f, \psi_\lambda \rangle)_{\lambda \in \Lambda} \longrightarrow (\langle f, \psi_\lambda \rangle)_{\lambda \in \Lambda \setminus \Lambda_0} \rightsquigarrow \tilde{f}.$$

$\longrightarrow$

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Edge Detection (Analysis):

$$\mathcal{H} \ni f \longrightarrow (\langle f, \psi_\lambda \rangle)_{\lambda \in \Lambda} \longrightarrow (\langle f, \psi_\lambda \rangle)_{\lambda \in \Lambda_1} \rightsquigarrow \tilde{f}.$$

$\longrightarrow$

Two Main Viewpoints

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Representation Systems

Functional Analytic Properties:

- $(\psi_\lambda)_\lambda$ can be an orthonormal basis.
  - Unique expansion.
  - Optimal stability.
- $(\psi_\lambda)_\lambda$ can form a frame.
  - Non-unique/redundant expansions.
  - Flexibility in expansions $x = \sum_{\lambda \in \Lambda} c_\lambda \psi_\lambda$.
  - Stability.
  - Robustness against loss of coefficients $\langle x, \psi_\lambda \rangle$.

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Definition: A sequence $(\psi_\lambda)_{\lambda \in \Lambda} \subset \mathcal{H}$ is a frame for $\mathcal{H}$ with frame bounds $0 < A \leq B < \infty$, if

$$A\|x\|^2 \leq \sum_{\lambda \in \Lambda} |\langle x, \psi_\lambda \rangle|^2 \leq B\|x\|^2 \quad \text{for all } x \in \mathcal{H}.$$

We call a frame tight, if $A = B$, and Parseval, if $A = B = 1$.

Frame Theory

Analysis Operator:

$$T : \mathcal{H} \to \ell_2(\Lambda),\; x \mapsto (\langle x, \psi_\lambda \rangle)_{\lambda \in \Lambda}$$

Synthesis Operator:

$$T^* : \ell_2(\Lambda) \to \mathcal{H},\; (c_\lambda)_{\lambda \in \Lambda} \mapsto \sum_{\lambda \in \Lambda} c_\lambda \psi_\lambda$$

Frame Operator:

$$S = T^* T : \mathcal{H} \to \mathcal{H},\; x \mapsto \sum_{\lambda \in \Lambda} \langle x, \psi_\lambda \rangle \, \psi_\lambda$$

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Theorem: The frame operator is a positive, self-adjoint, and invertible operator and satisfies $A \cdot Id \leq S \leq B \cdot Id$. Thus, the following reconstruction/expansion formula holds:

$$x = \sum_{\lambda\in\Lambda} \langle x, \psi_\lambda \rangle S^{-1}\psi_\lambda = \sum_{\lambda\in\Lambda} \left\langle x, S^{-1}\psi_\lambda \right\rangle \psi_\lambda.$$

Sparse Representations

Situation of Orthonormal Bases:

$$f = \sum_{\lambda \in \Lambda} c_\lambda \psi_\lambda = \sum_{\lambda \in \Lambda} \langle f, \psi_\lambda \rangle \psi_\lambda,$$

with rapidly decaying $(\langle f, \psi_\lambda \rangle)_{\lambda \in \Lambda}$.

Situation of Frames:

$$f = \sum_{\lambda \in \Lambda} c_\lambda \psi_\lambda = \sum_{\lambda \in \Lambda} \langle x, \psi_\lambda \rangle S^{-1} \psi_\lambda$$

with rapidly decaying $(c_\lambda)_{\lambda \in \Lambda}$ or $(\langle f, \psi_\lambda \rangle)_{\lambda \in \Lambda}$.

Sparsity

Novel Paradigm:

For each class of data, there exists a sparsifying system!

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Two Viewpoints of 'Sparsifying System':

Let $\mathcal{C} \subseteq \mathcal{H}$ and $(\psi_\lambda)_\lambda \subseteq \mathcal{H}$.

- Decay of Coefficients. Consider the decay for $n \to \infty$ of the sorted sequence of coefficients

$$(|\langle x, \psi_{\lambda_n} \rangle|)_n \quad \text{for all } x \in \mathcal{C}.$$

- Approximation Properties. Consider the decay for $N \to \infty$ of the error of best N-term approximation, i.e.,

$$\inf_{\#\Lambda_N = N, (c_\lambda)_\lambda} \left\| x - \sum_{\lambda \in \Lambda_N} c_\lambda \psi_\lambda \right\| \quad \text{for all } x \in \mathcal{C}.$$

Notion of Optimality

Two Viewpoints of Optimality of $(\psi_\lambda)_\lambda$: Let $\mathcal{C} \subseteq \mathcal{H}$.

- Decay of Coefficients. $\beta > 0$ is largest (for all systems) with

$$|\langle x, \psi_{\lambda_n} \rangle| \lesssim n^{-\beta} \text{ as } n \to \infty, \quad \text{for all } x \in \mathcal{C}.$$

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Situation of an ONB: For the best N-term approximation x_N of x, we have

$$\|x - x_N\|^2 = \sum_{\lambda \notin \Lambda_N} |c_\lambda|^2 = \sum_{n > N} |\langle x, \psi_{\lambda_n} \rangle|^2$$

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Situation of a Frame: For the N-term approximation $x_N = \sum_{\lambda \in \Lambda_N} \langle x, \psi_\lambda \rangle \tilde{\psi}_\lambda$ of x consisting of the N largest coefficients $|\langle x, \psi_\lambda \rangle|$, we *only* have

$$\|x - x_N\|^2 \leq \frac{1}{A} \sum_{n > N} |\langle x, \psi_{\lambda_n} \rangle|^2 .$$

Two Main Viewpoints

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Regularization of Inverse Problems

General Setting:

Given $K : X \to Y$ and $y \in Y$, compute $x \in X$ with $Kx = y$.

Well-Posedness Conditions (Hadamard):

- Existence: For each $y \in Y$, there exists some $x \in X$ with $Kx = y$.
- Uniqueness: Such an $x \in X$ is unique.
- Stability: $\lim_{n\to\infty} Kx_n \to Kx$ implies $\lim_{n\to\infty} x_n \to x$.

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Regularization Strategy:

A family of linear and bounded operators $R_\alpha : Y \to X$, $\alpha > 0$, such that

$$\lim_{\alpha\to 0} R_\alpha Kx (=: x^\alpha) = x \quad \text{for all } x \in X.$$

Tikhonov Regularization

Standard Tikhonov Regularization:

Given an ill-posed inverse problem $Kx = y$, where $K : X \to Y$, an approximate solution $x^\alpha \in X$, $\alpha > 0$, can be determined by minimizing

$$J_\alpha(x) := \|Kx - y\|^2 + \alpha\|x\|^2, \quad x \in X.$$

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Generalization:

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The penalty term $\mathcal{P}$

- ensures continuous dependence on the data,
- incorporates properties of the solution.

Some Examples for $\mathcal{P}$:

$$\|x\|_{TV}, \;\; \|x\|_{H^s}, \;\; \|(\langle x, \psi_\lambda \rangle)_\lambda\|_1, \ldots$$

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Main Goal: Solve an underdetermined linear problem

$$y = Ax, \quad A \text{ an } n \times N\text{-matrix with } n \ll N,$$

for a solution $x \in \mathbb{R}^N$ admitting a sparsifying system $(\psi_\lambda)_\lambda$.

Approach: Recover x by the ℓ_1-analysis minimization problem

$$\min_{\tilde{x}} \|(\langle \tilde{x}, \psi_\lambda \rangle)_\lambda\|_1 \text{ subject to } y = A\tilde{x}$$

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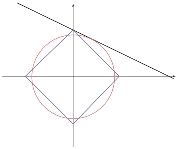
Why ℓ_1?



Compressed Sensing (Candès, Romberg, Tao and Donoho; 2006)

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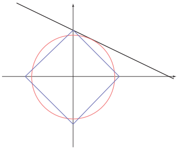

Meta-Result: If $(\langle x, \psi_\lambda \rangle)_\lambda$ is sufficiently sparse, and A is sufficiently incoherent, then x can be recovered from Ax by ℓ_1 minimization.

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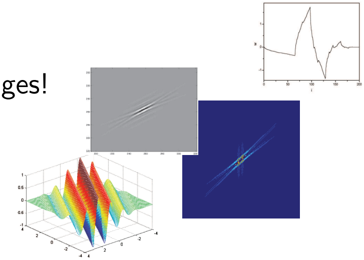
Computational Harmonic Analysis

Desiderata:

- Multiscale representation system.
- Convenient structure: Operators applied to one generating function.
- Partition of Fourier domain.
- Space/frequency localization.
- Fast algorithms: $x \mapsto (\langle x, \psi_\lambda \rangle)_\lambda \rightsquigarrow x$.
- Optimality for the considered class.
  $\rightsquigarrow$ In this Talk: Modeling natural images!



Modelling Anisotropic Structures

What is an Image?

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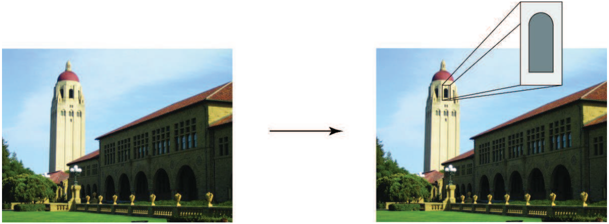

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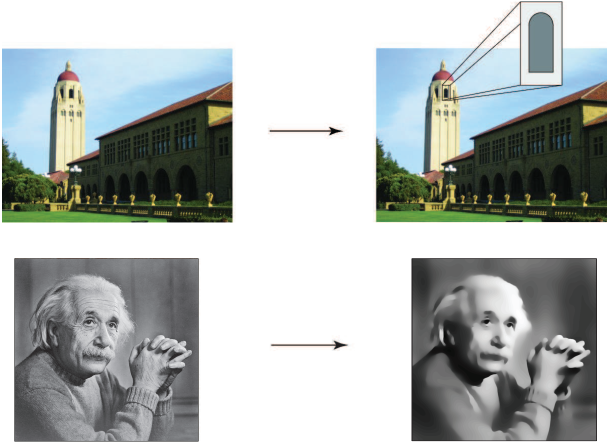

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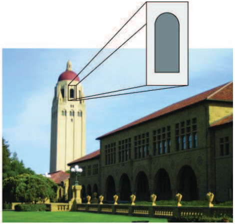
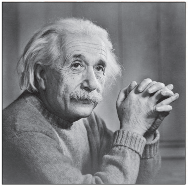
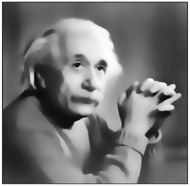
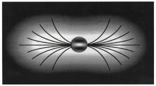






- Intuitively edges are main structure.
- Justified by neurophysiology.



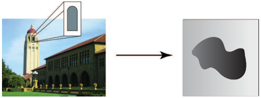
Field et al., 1993

Fitting Model

Definition (Donoho; 2001):
The set of cartoon-like functions $\mathcal{E}^2(\mathbb{R}^2)$ is defined by

$$\mathcal{E}^2(\mathbb{R}^2) = \{f \in L^2(\mathbb{R}^2) : f = f_0 + f_1 \cdot \chi_B\},$$

where $\emptyset \neq B \subset [0,1]^2$ simply connected with C^2-boundary and bounded curvature, and $f_i \in C^2(\mathbb{R}^2)$ with supp $f_i \subseteq [0,1]^2$ and $\|f_i\|_{C^2} \leq 1$, $i = 0, 1$.



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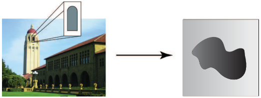

Theorem (Donoho; 2001):
Let $(\psi_\lambda)_\lambda \subseteq L^2(\mathbb{R}^2)$. Allowing only polynomial depth search, we have the following optimal behavior for $f \in \mathcal{E}^2(\mathbb{R}^2)$:

$$\|f - f_N\|_2^2 \asymp N^{-2} \quad \text{and} \quad |\langle f, \psi_{\lambda_n}\rangle| \lesssim n^{-\frac{3}{2}} \quad \text{as } N, n \to \infty.$$

Review of 2-D Wavelets

Definition (1D): Let $\phi \in L^2(\mathbb{R})$ be a scaling function and $\psi \in L^2(\mathbb{R})$ be a wavelet. Then the associated wavelet system is defined by

$$\{\phi(x - m) : m \in \mathbb{Z}\} \cup \{2^{j/2}\,\psi(2^j x - m) : j \geq 0, m \in \mathbb{Z}\}.$$

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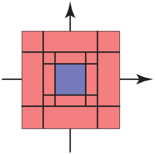
Definition (2D): A wavelet system is defined by

$$\{\phi^{(1)}(x-m) : m \in \mathbb{Z}^2\} \cup \{2^j \psi^{(i)}(2^j x - m) : j \geq 0, m \in \mathbb{Z}^2, i = 1, 2, 3\},$$

where

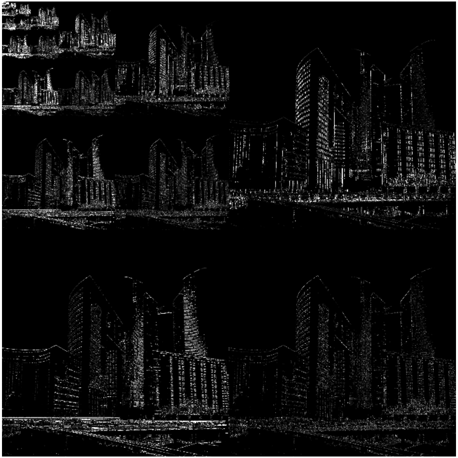
$$\phi^{(1)}(x) = \phi(x_1)\phi(x_2) \quad \text{and} \quad \begin{aligned} \psi^{(1)}(x) &= \phi(x_1)\psi(x_2), \\ \psi^{(2)}(x) &= \psi(x_1)\phi(x_2), \\ \psi^{(3)}(x) &= \psi(x_1)\psi(x_2). \end{aligned}$$



Theorem: Wavelets provide optimally sparse approximations for functions $f \in L^2(\mathbb{R}^2)$, which are C^2 apart from point singularities:

$$\|f - f_N\|_2^2 \asymp N^{-1}, \quad N \to \infty.$$

Wavelet Decomposition: JPEG2000



Wavelet Decomposition: JPEG2000

Original

25% Compression

5% Compression

What can Wavelets do?

Problem:

- For $f \in \mathcal{E}^2(\mathbb{R}^2)$, wavelets only achieve $\|f - f_N\|_2^2 \asymp N^{-1}$, $N \to \infty$.
- Isotropic structure of wavelets:

$$\{2^j \psi(\begin{pmatrix} 2^j & 0 \\ 0 & 2^j \end{pmatrix} x - m) : j \geq 0, m \in \mathbb{Z}^2\}.$$

- Wavelets cannot sparsely represent cartoon-like functions.

Intuitive explanation:

Main Goal

Design a Representation System which...

- ...fits into the framework of affine systems,
- ...provides an optimally sparsifying system for cartoons,
- ...allows for compactly supported analyzing elements,
- ...is associated with fast decomposition algorithms,
- ...treats the continuum and digital 'world' uniformly.

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Non-Exhaustive List of Approaches:

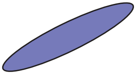
- Ridgelets (Candès and Donoho; 1999)
- Curvelets (Candès and Donoho; 2002)
- Contourlets (Do and Vetterli; 2002)
- Bandlets (LePennec and Mallat; 2003)
- Shearlets (K and Labate; 2006)

What is a Shearlet?

Scaling and Orientation

Parabolic scaling ('width $\approx$ length^2'):

$$A_{2^j} = \begin{pmatrix} 2^j & 0 \\ 0 & 2^{j/2} \end{pmatrix}, \quad j \in \mathbb{Z}.$$



Historical remark:

- 1970's: Fefferman und Seeger/Sogge/Stein.

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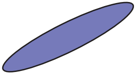


Historical remark:

- 1970's: Fefferman und Seeger/Sogge/Stein.

Orientation via shearing:

$$S_k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}, \quad k \in \mathbb{Z}.$$

Advantage:

- Shearing leaves the digital grid $\mathbb{Z}^2$ invariant.
- Uniform theory for the continuum and digital situation.

Affine systems:

$$\{|\det M|^{1/2}\psi(M\cdot - m) : M \in G \subseteq GL_2,\ m \in \mathbb{Z}^2\}.$$

Definition (K, Labate; 2006):
For $\psi \in L^2(\mathbb{R}^2)$, the associated shearlet system is defined by

$$\{2^{\frac{3j}{4}}\psi(S_k A_{2^j}\cdot - m) : j, k \in \mathbb{Z}, m \in \mathbb{Z}^2\}.$$

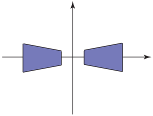
Remarks:

- Advantage: Generated by a unitary representation of the locally compact group $(\mathbb{R}^+ \times \mathbb{R}) \ltimes \mathbb{R}^2$, the so-called shearlet group.
- Disadvantage: Non-uniform treatment of directions.

Example of Classical (Band-Limited) Shearlet

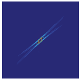
Let $\psi \in L^2(\mathbb{R}^2)$ be defined by

$$\hat{\psi}(\xi) = \hat{\psi}(\xi_1, \xi_2) = \hat{\psi}_1(\xi_1)\,\hat{\psi}_2(\tfrac{\xi_2}{\xi_1}),$$



where

- ψ_1 wavelet, $\mathrm{supp}(\hat{\psi}_1) \subseteq [-2, -\frac{1}{2}] \cup [\frac{1}{2}, 2]$ and $\hat{\psi}_1 \in C^\infty(\mathbb{R})$,
- ψ_2 'bump function', $\mathrm{supp}(\hat{\psi}_2) \subseteq [-1, 1]$ and $\hat{\psi}_2 \in C^\infty(\mathbb{R})$.



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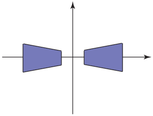

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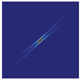
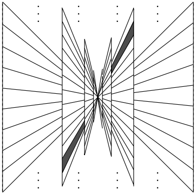


Induced tiling of Fourier domain:



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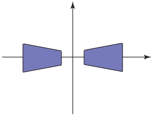

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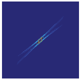

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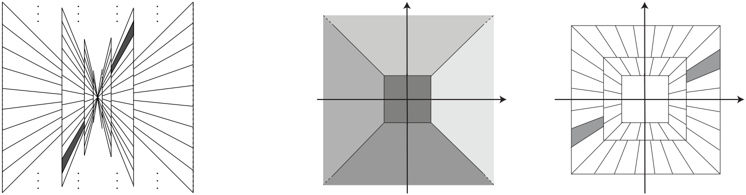


(Cone-adapted) Shearlet Systems

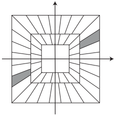
Definition (K, Labate; 2006):

The (cone-adapted) shearlet system $\mathcal{SH}(c;\phi,\psi,\tilde{\psi})$, $c>0$, generated by $\phi \in L^2(\mathbb{R}^2)$ and $\psi, \tilde{\psi} \in L^2(\mathbb{R}^2)$ is the union of

$$\{\phi(\cdot - cm) : m \in \mathbb{Z}^2\},$$

$$\{2^{3j/4}\psi(S_k A_{2^j}\, \cdot\, - cm) : j \geq 0, |k| \leq \lceil 2^{j/2} \rceil, m \in \mathbb{Z}^2\},$$

$$\{2^{3j/4}\tilde{\psi}(\tilde{S}_k \tilde{A}_{2^j}\, \cdot\, - cm) : j \geq 0, |k| \leq \lceil 2^{j/2} \rceil, m \in \mathbb{Z}^2\}.$$



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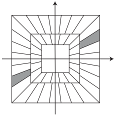


Theorem (K, Labate, Lim, Weiss; 2006):
For $\psi, \tilde{\psi}$ classical shearlets, $\mathcal{SH}(1; \phi, \psi, \tilde{\psi})$ is a Parseval frame for $L^2(\mathbb{R}^2)$:

$$A\|f\|_2^2 \leq \sum_{\sigma \in \mathcal{SH}(\phi, \psi, \tilde{\psi})} |\langle f, \sigma \rangle|^2 \leq B\|f\|_2^2 \quad \text{for all } f \in L^2(\mathbb{R}^2)$$

holds for $A = B = 1$.

Proof of Parseval Frame Property

Specific conditions on a classical shearlet $\hat{\psi}(\xi_1, \xi_2) = \hat{\psi}_1(\xi_1)\,\hat{\psi}_2(\frac{\xi_2}{\xi_1})$:

- Wavelet: $\sum_{j \in \mathbb{Z}} |\hat{\psi}_1(2^{-j}\xi)|^2 = 1$ for a.e. $\xi \in \mathbb{R}$.
- 'Bump Function': $\sum_{k=-1,0,1} |\hat{\psi}_2(\xi + k)|^2 = 1$ for a.e. $\xi \in [-1, 1]$.

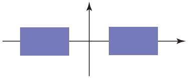
By the above properties of ψ_1 and ψ_2, we have

$$\begin{aligned}
&\sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} |\hat{\psi}(S_{-k}^T A_{-j}\xi)|^2 \\
&= \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} |\hat{\psi}(2^{-j}\xi_1, 2^{-j/2}\xi_2 - 2^{-j}\xi_1 k)|^2 \\
&= \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} |\hat{\psi}_1(2^{-j}\xi_1)|^2 |\hat{\psi}_2(2^{j/2}\tfrac{\xi_2}{\xi_1} - k)|^2 \\
&= \sum_{j \in \mathbb{Z}} |\hat{\psi}_1(2^{-j}\xi_1)|^2 = 1.
\end{aligned}$$

Compactly Supported Shearlets

Theorem (Kittipoom, K, Lim; 2012):

Let $\phi, \psi, \tilde{\psi} \in L^2(\mathbb{R}^2)$ be compactly supported, and let $\hat{\phi}$, $\hat{\psi}$, $\hat{\tilde{\psi}}$ satisfy certain decay conditions. Then there exists c_0 such that $\mathcal{SH}(c; \phi, \psi, \tilde{\psi})$ forms a shearlet frame with controllable frame bounds for all $c \leq c_0$.



Remark: Exemplary class with $B/A \approx 4$.

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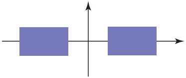

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Theorem (Guo, Labate; 2007)(K, Lim; 2011):

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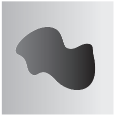
$$\|f - f_N\|_2^2 \lesssim N^{-2}(\log N)^3 \text{ and } |\langle f, \sigma_{\eta_n}\rangle| \lesssim n^{-\frac{3}{2}}(\log n)^{\frac{3}{2}}.$$

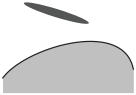


Heuristic Argument

Estimate:

$$\|f - f_N\|_2^2 \lesssim \sum_{n>N} (|\langle f, \sigma_{\eta_n} \rangle|)^2 \lesssim \sum_{n>N} (n^{-\frac{3}{2}})^2 \lesssim N^{-2}.$$

Case 1:



$|\langle f, \sigma_\eta \rangle|$ negligible!

Case 2:

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Case 3:

$|\langle f, \sigma_\eta \rangle| \leq \|f\|_\infty \|\sigma_\eta\|_1 \lesssim 2^{-\frac{3}{4}j}$

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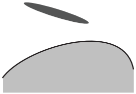

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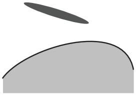

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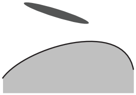

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Recent Approaches to Fast Shearlet Transforms

www.ShearLab.org:

- Separable Shearlet Transform *(Lim; 2009)*
- Digital Shearlet Transform *(K, Shahram, Zhuang; 2011)*
- 2D&3D (parallelized) Shearlet Transform *(K, Lim, Reisenhofer; 2013)*

Additional Code:

- Filter-based implementation *(Easley, Labate, Lim; 2009)*
- Fast Finite Shearlet Transform *(Häuser, Steidl; 2014)*
- Shearlet Toolbox 2D&3D *(Easley, Labate, Lim, Negy; 2014)*

Theoretical Approaches:

- Adaptive Directional Subdivision Schemes *(K, Sauer; 2009)*
- Shearlet Unitary Extension Principle *(Han, K, Shen; 2011)*
- Gabor Shearlets *(Bodmann, K, Zhuang; 2013)*

What about Curvelets...?

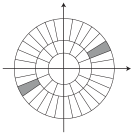
Definition (Candès, Donoho; 2002):

Let

- $W \in C^\infty(\mathbb{R})$ be a wavelet with $\mathrm{supp}(W) \subseteq \left(\frac{1}{2}, 2\right)$,
- $V \in C^\infty(\mathbb{R})$ be a 'bump function' with $\mathrm{supp}(V) \subseteq (-1, 1)$.

Then the curvelet system $(\gamma_{(j,l,k)})_{(j,l,k)}$ is defined by

$$\hat{\gamma}_{(j,0,0)}(r,\omega) := 2^{-3j/4} W\left(2^{-j} r\right) V(2^{\lfloor j/2 \rfloor}\omega)$$



and

$$\gamma_{(j,l,k)}(\cdot) := \gamma_{(j,0,0)}(R_{\theta_{(j,l,k)}}(\cdot - x_{(j,l,k)})).$$

Theorem (Candès, Donoho; 2002):

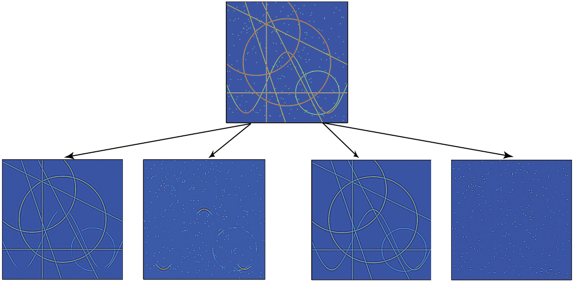
The curvelet system forms a Parseval frame for $L^2(\mathbb{R}^2)$.

Comparison with Shearlets, I

Main Differences to Shearlets:

- Not affine systems.
- Based on rotation in contrast to shearing.
- Only band-limited version available.

Performance on Separation:



Curvelets + Wavelets   Shearlets + Wavelets

Comparison with Shearlets, II

But there are also many Similarities...

Theorem (Candès, Donoho; 2002):
Curvelets provide optimally sparse approximations of $f \in \mathcal{E}^2(\mathbb{R}^2)$, i.e.,

$$\|f - f_N\|_2^2 \leq C \cdot N^{-2} \cdot (\log N)^3, \quad N \to \infty.$$

Towards a General Framework...

General Framework

Introduce a Framework which...

- *...covers all systems known to sparsify cartoons.*
- *...enables easy transfer of (sparsity) results between systems.*
- *...allows categorization of systems with respect to sparsity behaviors.*
- *...is general enough to allow construction of novel systems.*

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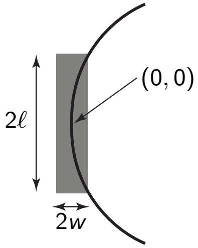
Crucial Ingredient: Parabolic scaling, i.e., a scaling matrix of the type

$$A_{2^j} = \begin{pmatrix} 2^j & 0 \\ 0 & 2^{j/2} \end{pmatrix}, \quad j \in \mathbb{Z},$$



since from

$$E(x_2) \approx \frac{1}{2}\kappa x_2^2 \quad \text{and} \quad E(\ell) = w$$

follows

$$w \approx \frac{\kappa}{2}\ell^2 \qquad (\text{'}\textit{width} \approx \textit{length}^2\text{'}).$$

Parametrization

Parameter space:

$$\mathbb{P} := \mathbb{R}_+ \times \mathbb{T} \times \mathbb{R}^2,$$

where $(s, \theta, x) \in \mathbb{P}$ describes scale 2^s, orientation θ, and location x.

Definition: A parametrization is a pair (Λ, Φ_Λ), where Λ is a discrete index set and Φ_Λ is a mapping

$$\Phi_\Lambda : \begin{cases} \Lambda & \rightarrow \quad \mathbb{P}, \\ \lambda & \mapsto \quad (s_\lambda, \theta_\lambda, x_\lambda) . \end{cases}$$

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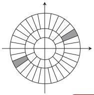
Example: The canonical parametrization $(\Lambda^0, \Phi^0(\lambda))$ is defined by

$$\Lambda^0 := \left\{ (j, \ell, k) \in \mathbb{Z}^4 \; : \; j \geq 0, \; \ell = -2^{\lfloor \frac{j}{2} \rfloor - 1}, \dots, 2^{\lfloor \frac{j}{2} \rfloor - 1} \right\},$$

and

$$\Phi^0(j, \ell, k) = (s_\lambda, \theta_\lambda, x_\lambda) = (j, \ell 2^{-\lfloor j/2 \rfloor} \pi, R_{-\theta_\lambda} A_{2^{-s_\lambda}} k).$$



Definition (Grohs, K; 2014):
Let (Λ, Φ_Λ) be a parametrization. Then $(m_\lambda)_{\lambda\in\Lambda}$ is a system of parabolic molecules of order $(L, M, N_1, N_2) \in (\mathbb{Z}_+ \cup \{\infty\})^2 \times \mathbb{Z}_+^2$, if, for all $\lambda \in \Lambda$,

$$m_\lambda(x) = 2^{3s_\lambda/4} g^{(\lambda)}\left(A_{2^{s_\lambda}} R_{\theta_\lambda}(x - x_\lambda)\right), \quad \Phi_\Lambda(\lambda) = (s_\lambda, \theta_\lambda, x_\lambda),$$

such that, for all $|\beta| \le L$,

$$\left|\partial^\beta \hat{g}^{(\lambda)}(\xi)\right| \lesssim \min\left(1, 2^{-s_\lambda} + |\xi_1| + 2^{-s_\lambda/2}|\xi_2|\right)^M \langle|\xi|\rangle^{-N_1} \langle\xi_2\rangle^{-N_2}.$$

Control Parameters:

- L: Spatial localization.
- M: Number of directional (almost) vanishing moments.
- N_1, N_2: Smoothness of m_λ.

Parabolic Molecules

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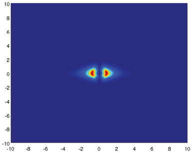
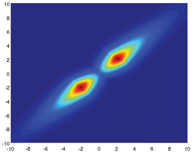
Illustration:





Special Cases

This framework includes...

- Parabolic Frame (Smith; 1998)
- Second Generation Curvelets (Candès and Donoho; 2002)
- Curvelet Molecules (Candès and Demanet; 2002)
- Bandlimited Shearlets (K and Labate; 2006)
- Frame Decompositions (Borup and Nielsen; 2007)
- Shearlet Molecules (Guo and Labate; 2008)
- Compactly Supported Shearlets (Kittipoom, K, and Lim; 2012)
- ...

What about Wavelets, Ridgelets,...?

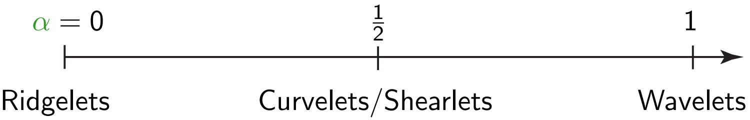
Extension of Framework

Main Idea:

- Introduction of a parameter $\alpha \in [0,1]$ to measure the amount of anisotropy.
- For $a > 0$, define

$$A_{\alpha,a} = \begin{pmatrix} a & 0 \\ 0 & a^{\alpha} \end{pmatrix}.$$

Illustration:



α-Molecules

Definition (Grohs, Keiper, K, Schäfer; 2016):
Let $\alpha \in [0,1]$, and let (Λ, Φ_Λ) be a parametrization. Then $(m_\lambda)_{\lambda\in\Lambda}$ is a system of α-molecules of order $(L, M, N_1, N_2) \in (\mathbb{Z}_+ \cup \{\infty\})^2 \times \mathbb{Z}_+^2$, if, for all $\lambda \in \Lambda$,

$$m_\lambda(x) = s_\lambda^{(1+\alpha)/2} g^{(\lambda)}\left(A_{\alpha,s_\lambda} R_{\theta_\lambda}(x - x_\lambda)\right), \quad \Phi_\Lambda(\lambda) = (s_\lambda, \theta_\lambda, x_\lambda),$$

such that, for all $|\beta| \le L$,

$$\left|\partial^\beta \hat{g}^{(\lambda)}(\xi)\right| \lesssim \min\left(1, s_\lambda^{-1} + |\xi_1| + s_\lambda^{-(1-\alpha)}|\xi_2|\right)^M \langle|\xi|\rangle^{-N_1} \langle\xi_2\rangle^{-N_2}.$$

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Examples:

- Wavelets ($\alpha = 1$)
- Ridgelets ($\alpha = 0$)
- Shearlets, parabolic molecules in general ($\alpha = \frac{1}{2}$)
- α-Curvelets ($\alpha \in [0,1]$)

Metric Properties of Parametrizations

Definition:
Let $\alpha \in [0,1]$, and let (Λ, Φ_Λ) and (Δ, Φ_Δ) be parametrizations. For $\lambda \in \Lambda$ and $\mu \in \Delta$, we define the index distance by

$$\omega_\alpha(\lambda,\mu) := \omega_\alpha(\Phi_\Lambda(\lambda), \Phi_\Delta(\mu)) := \max\left\{\frac{s_\lambda}{s_\mu}, \frac{s_\mu}{s_\lambda}\right\}(1 + d_\alpha(\lambda,\mu)),$$

with $d_\alpha(\lambda,\mu)$ defined by

$$s_0^{2(1-\alpha)}|\theta_\lambda - \theta_\mu|^2 + s_0^{2\alpha}|x_\lambda - x_\mu|^2 + \frac{s_0^2}{1 + s_0^{2(1-\alpha)}|\theta_\lambda - \theta_\mu|^2}|\langle e_\lambda, x_\lambda - x_\mu\rangle|^2.$$

where $s_0 = \min\{s_\lambda, s_\mu\}$ and $e_\lambda = (\cos(\theta_\lambda), -\sin(\theta_\lambda))^T$.

Remark: $d_{\frac{1}{2}}$ is the Hart Smith's phase space metric on $\mathbb{T} \times \mathbb{R}^2$.

Main Result: Decay of Cross-Grammian

Theorem (Grohs, Keiper, K, Schäfer; 2016):
Let $\alpha \in [0,1]$, $N > 0$, and let $(m_\lambda)_{\lambda\in\Lambda}$, $(p_\mu)_{\mu\in\Delta}$ be systems of α-molecules of order (L, M, N_1, N_2) with

$$L \geq 2N, \quad M > 3N - \frac{3-\alpha}{2}, \quad N_1 \geq N + \frac{1+\alpha}{2}, \quad N_2 \geq 2N.$$

Then, for all $\lambda \in \Lambda$ and $\mu \in \Delta$,

$$|\langle m_\lambda, p_\mu \rangle| \lesssim \omega_\alpha(\lambda, \mu)^{-N}.$$

...towards Sparse Approximation Properties!

Sparsity Equivalence

Definition:

Let $(m_\lambda)_{\lambda\in\Lambda}$ and $(p_\mu)_{\mu\in\Delta}$ be systems of α-molecules of order (L, M, N_1, N_2) and $(\tilde{L}, \tilde{M}, \tilde{N}_1, \tilde{N}_2)$, respectively, and let $0 < p \leq 1$. If

$$\left\| \left(\langle m_\lambda, p_\mu \rangle \right)_{\lambda\in\Lambda, \mu\in\Delta} \right\|_{\ell^p \to \ell^p} < \infty,$$

then $(m_\lambda)_{\lambda\in\Lambda}$ and $(p_\mu)_{\mu\in\Delta}$ are sparsity equivalent in ℓ^p.

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then $(m_\lambda)_{\lambda\in\Lambda}$ and $(p_\mu)_{\mu\in\Delta}$ are sparsity equivalent in ℓ^p.

Definition:

Let $\alpha \in [0,1]$ and $k > 0$. Two parametrizations (Λ, Φ_Λ) and (Δ, Φ_Δ) are (α, k)-consistent, if

$$\sup_{\lambda\in\Lambda} \sum_{\mu\in\Delta} \omega_\alpha(\lambda,\mu)^{-k} < \infty \quad \text{and} \quad \sup_{\mu\in\Delta} \sum_{\lambda\in\Lambda} \omega_\alpha(\lambda,\mu)^{-k} < \infty.$$

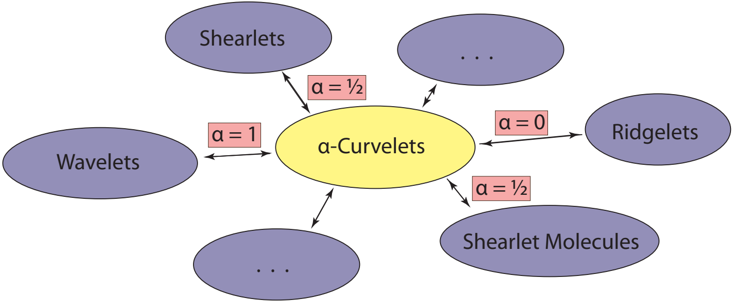
Sufficient Condition for Sparsity Equivalence

Theorem (Grohs, Keiper, K, Schäfer; 2016):
Let $0 < p \leq 1$, and let $(m_\lambda)_{\lambda \in \Lambda}$ and $(p_\mu)_{\mu \in \Delta}$ be frames of α-molecules of order (L, M, N_1, N_2) with (α, k)-consistent parametrizations (Λ, Φ_Λ) and (Δ, Φ_Δ) for some $k > 0$. If

$$L \geq 2\frac{k}{p}, \quad M > 3\frac{k}{p} - \frac{3-\alpha}{2}, \quad N_1 \geq \frac{k}{p} + \frac{1+\alpha}{2}, \quad N_2 \geq 2\frac{k}{p},$$

then $(m_\lambda)_{\lambda \in \Lambda}$ and $(p_\mu)_{\mu \in \Delta}$ are sparsity equivalent in ℓ^p.

Strategy


Shearlets
. . .
α = ½
α = 0
Ridgelets
α = 1
α-Curvelets
Wavelets
α = ½
Shearlet Molecules
. . .

Generalized Image Model

Definition (K, Lemvig, Lim; 2012), (Keiper; 2012):
The set of cartoon-like functions $\mathcal{E}^\beta(\mathbb{R}^2)$, $\beta \in (1,2]$ is defined by

$$\mathcal{E}^\beta(\mathbb{R}^2) = \{f \in L^2(\mathbb{R}^2) : f = f_0 + f_1 \cdot \chi_B\},$$

where $B \subset [0,1]^2$ with ∂B a closed C^β-curve, $f_0, f_1 \in C_0^\beta([0,1]^2)$.

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Theorem (Grohs, Keiper, K, Schäfer; 2016):
Let $\alpha \in [\frac{1}{2}, 1)$, $\beta = \alpha^{-1}$. The Parseval frame of α-curvelets provides an optimally sparse approximation of $f \in \mathcal{E}^{\beta}(\mathbb{R}^2)$, i.e.,

$$\|f - f_N\|_2^2 \leq C \cdot N^{-\beta} \cdot (\log N)^{\beta+1}, \quad N \to \infty.$$

Sparse Approximation with α-Molecules

Theorem (Grohs, Keiper, K, Schäfer; 2016):
Let $\alpha \in [\frac{1}{2}, 1)$, $\beta = \alpha^{-1}$, and let $(m_\lambda)_{\lambda \in \Lambda}$ be a system of α-molecules of order (L, M, N_1, N_2) such that

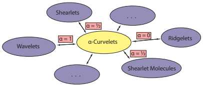
(i) $(m_\lambda)_{\lambda \in \Lambda}$ constitutes a frame for $L^2(\mathbb{R}^2)$,

(ii) (Λ, Φ_Λ) is (α, k)-consistent with the parametrization of α-curvelets for all $k > 0$,

(iii) it holds that

$$L \geq k(1+\beta), M \geq \frac{3k}{2}(1+\beta) + \frac{\alpha - 3}{2}, N_1 \geq \frac{k}{2}(1+\beta) + \frac{1+\alpha}{2}, N_2 \geq k(1+\beta).$$

Then, for any $\varepsilon > 0$ and for any $f \in \mathcal{E}^\beta(\mathbb{R}^2)$, $(m_\lambda)_{\lambda \in \Lambda}$ satisfies



$$\|f - f_N\|_2^2 \leq C \cdot N^{-\beta + \varepsilon}, \quad N \to \infty,$$

Let's conclude...

What to take Home...?

- Computational Harmonic Analysis provides various representation systems such as wavelets, ridgelets, curvelets, and shearlets.
- They provide sparse approximation for certain classes of images, leading to
  - Efficient decompositions for, e.g., the analysis/processing of images.
  - Sparse representations for, e.g., regularization of inverse problems.
- Shearlets provide an optimally sparsifying system for a model class of functions being governed by anisotropic features.
- α-Molecules provide a general framework for various systems from computational harmonic analysis.
- Sparse approximation results can be derived in a unified manner.

THANK YOU!

References available at:

www.math.tu-berlin.de/∼kutyniok

Code available at:

www.ShearLab.org

Related Books:

- Y. Eldar and G. Kutyniok
  Compressed Sensing: Theory and Applications
  Cambridge University Press, 2012.
- G. Kutyniok and D. Labate
  Shearlets: Multiscale Analysis for Multivariate Data
  Birkhäuser-Springer, 2012.