BOUNDS ON STABLE SETS
IN GRAPHS AND CODES

Alexander Schrijver

CWI and University of Amsterdam

Joint work with
Dion Gijswijt (Amsterdam & Leiden)
and
Hans Mittelmann (Arizona)
STABLE SETS IN GRAPHS

Let $G = (V, E)$ be a graph.

$S \subseteq V$ is a stable set if no two vertices in $V$ are adjacent.

$$\alpha(G) := \max\{|S| \mid S \text{ stable}\}$$

is hard to compute.

CODES

A code (of length $n$) is a subset $C$ of $\{0, 1\}^n$.

The (Hamming) distance $d_H(u, v)$ of $b, c \in \{0, 1\}^n$ is the number of $i$ with $b_i \neq c_i$.

The minimum distance of a code $C$ is the minimum distance of any two distinct elements of $C$.

$A(n, d) :=$ the maximum cardinality of any code of length $n$ and minimum distance at least $d$.

CODES AS STABLE SETS

Let $G_{n,d}$ be the graph with vertex set $\{0, 1\}^n$, two of them being adjacent if their distance is less than $d$.

Then: $A(n, d) = \alpha(G_{n,d})$

$\alpha(G)$ is hard to compute and $G_{n,d}$ itself is exponentially large ......

but $G_{n,d}$ has a large symmetry.
EIGENVALUE METHODS

Let \( G = (V, E) \) be a graph.

**Lovász bound:**

\[
\vartheta(G) := \max \{ \mathbf{1}^T X \mathbf{1} \mid X \text{ nonnegative and positive semidefinite } V \times V \text{ matrix with trace } 1 \text{ and with } X_{u,v} = 0 \text{ if } u \text{ and } v \text{ are adjacent vertices} \}.
\]

\[\alpha(G) \leq \vartheta(G)\]

**Proof:** Let \( S \) be a maximum-size stable set.

Take \( X := |S|^{-1} \mathbf{1}_S \mathbf{1}_S^T. \)

\( \vartheta(G) \) can be computed in polynomial time (with semidefinite programming).

\[A(n, d) \leq \vartheta(G_{n,d}) \quad \text{(Delsarte bound)}\]

but \( G_{n,d} \) is exponentially large,

but we can exploit the symmetry of \( G_{n,d} \):

SYMMETRY

If $X$ is an optimum solution for $\vartheta(G)$ and $P \in \text{Aut}(G)$ then $PXPT$ is again an optimum solution.

(We identify an automorphism of $G$ with its permutation matrix in $\mathbb{R}^{V \times V}$.)

Averaging over all $P \in \text{Aut}(G)$ gives that we may assume that $X$ is $\text{Aut}(G)$-invariant.

For $G = G_{n,d}$, the algebra of $\text{Aut}(G)$-invariant matrices in $\mathbb{R}^{V \times V}$ is $\leq n + 1$-dimensional and can be simultaneously diagonalized (since it is commutative).

Hence the semidefinite programming problem for $\vartheta(G_{n,d})$ can be reduced to a linear programming problem, with $\leq n + 1$ variables and $\leq n + 1$ constraints.

**Corollary:** the Delsarte bound can be computed fast.
GENERALIZATION

Let $G = (V, E)$ be a graph and let $k \in \mathbb{N}$.

Define $\mathcal{S}_k := \{ U \subseteq V \mid U \text{ stable}, |U| \leq k \}$.

$$\vartheta_k(G) := \max \left\{ \sum_{v \in V} X_{\{v\}, \{v\}} \mid X \text{ nonnegative and positive semidefinite } S_k \times S_k \text{ matrix with } X_{\emptyset, \emptyset} = 1, \right.$$  \[ X_{U,W} = 0 \quad \text{if } U \cup W \text{ is not stable}, \]
\[ X_{U,W} = X_{U',W'} \quad \text{if } U \cup W = U' \cup W' \] \}

It can be proved that $\vartheta_1(G) = \vartheta(G)$.

Also: $\alpha(G) \leq \vartheta_k(G)$

Proof: Let $S$ be a maximum-size stable set.

Define $X_{U,W} := \begin{cases} 1 & \text{if } U \cup W \subseteq S, \\ 0 & \text{else.} \end{cases}$

\square
EXPLOITING SYMMETRY

$\text{Aut}(G)$ acts on $S_k$, hence on $\mathbb{R}^{S_k \times S_k}$.

Again we can assume $X$ to be $\text{Aut}(G)$-invariant.

Let $\mathcal{A}$ be the algebra of $\text{Aut}(G)$-invariant matrices in $\mathbb{C}^{S_k \times S_k}$.

Then there exists a unitary matrix $M \in \mathbb{C}^{S_k \times S_k}$ such that

$$MAM^* = \bigoplus_{i=1}^{t} \mathbb{C}^{a_i \times a_i} \otimes I_{b_i}$$

for some $a_1, b_1, \ldots, a_t, b_t \in \mathbb{N}$.

Note that $\mathbb{C}^{a \times a} \otimes I_b = \{ \begin{pmatrix} Y & 0 & \cdots & 0 \\ 0 & Y & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Y \end{pmatrix} \mid Y \in \mathbb{C}^{a \times a} \}$.

So $\dim \mathcal{A} = a_1^2 + \cdots + a_t^2$. 
APPLICATION TO CODES GIVES:

(taking $k := 2$)

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HOW TO FIND THE BLOCK DIAGONALIZATION?

\[ \mathcal{A} = \text{End}_{\text{Aut}(G)}(\mathbb{C}^{S_2}) \]

= the set of all linear functions \( f : \mathbb{C}^{S_2} \to \mathbb{C}^{S_2} \)
such that \( f \circ g = g \circ f \) for each \( g \in \text{Aut}(G) \).

Find the canonical decomposition into isotypical components
\( C_1, \ldots, C_t \) of the action of \( \text{Aut}(G) \) on \( \mathbb{C}^{S_2} \).

(An isotypical component is the sum
of an equivalence class of irreducible subrepresentations.)

Then

\[ \text{End}_{\text{Aut}(G)}(\mathbb{C}^{S_2}) \cong \bigoplus_{i=1}^{t} \text{End}_{\text{Aut}(G)}(C_i) \cong \bigoplus_{i=1}^{t} \mathbb{C}^{a_i \times a_i} \]

for some \( a_1, \ldots, a_t \in \mathbb{N} \).
THE CODE BOUNDS USE AN EXTRA CONDITION

For $k = 2$, you can add the condition:

for all $s, t \in V$, the matrix

$$(X_{\{s,u\},\{t,v\}})_{u,v \in V}$$

is positive semidefinite.