BOUNDS ON STABLE SETS IN GRAPHS AND CODES

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STABLE SETS IN GRAPHS

Let $G = (V, E)$ be a graph.

$S \subseteq V$ is **stable** if no two vertices in V are adjacent.

$\boldsymbol{\alpha(G)} := \max\{|S| \mid S \text{ stable}\}$

is hard to compute.

CODES

A **code** (of length n) is a subset C of $\{0,1\}^n$.

The **(Hamming) distance** $d_H(u, v)$ of $b, c \in \{0,1\}^n$

is the number of i with $b_i \neq c_i$.

The **minimum distance** of a code C is the minimum

distance of any two distinct elements of C.

$\boldsymbol{A(n,d)} :=$ the maximum cardinality of any code of length n

and minimum distance at least d.

CODES AS STABLE SETS

Let $\boldsymbol{G_{n,d}}$ be the graph with vertex set $\{0,1\}^n$,

two of them being adjacent if their distance is less than d.

Then: $\qquad A(n,d) = \alpha(G_{n,d})$

$\alpha(G)$ is hard to compute and $G_{n,d}$ itself is exponentially large .....,

but $G_{n,d}$ has a large symmetry.

EIGENVALUE METHODS

Let $G = (V, E)$ be a graph.

Lovász bound:

$\boldsymbol{\vartheta(G)}$:= $\max\{ \mathbf{1}^T X \mathbf{1} \;\; |$

X nonnegative and positive semidefinite $V \times V$ matrix

with trace 1 and with $X_{u,v} = 0$ if u and v are adjacent vertices $\}$.

$$\alpha(G) \leq \vartheta(G)$$

Proof: Let S be a maximum-size stable set.

Take $X := |S|^{-1}\mathbf{1}_S\mathbf{1}_S^T$. $\qquad \Box$

$\vartheta(G)$ can be computed in polynomial time

(with semidefinite programming).

$A(n, d) \leq \vartheta(G_{n,d})$ **(Delsarte bound)**

but $G_{n,d}$ is exponentially large,

but we can exploit the symmetry of $G_{n,d}$:

SYMMETRY

If X is an optimum solution for $\vartheta(G)$ and $P \in \mathrm{Aut}(G)$
then PXP^T is again an optimum solution.

(We identify an automorphism of G with
its permutation matrix in $\mathbb{R}^{V \times V}$.)

Averaging over all $P \in \mathrm{Aut}(G)$ gives that
we may assume that X is $\mathrm{Aut}(G)$-invariant.

For $G = G_{n,d}$, the algebra of $\mathrm{Aut}(G)$-invariant matrices in $\mathbb{R}^{V \times V}$
is $\leq n+1$-dimensional
and can be simultaneously diagonalized
(since it is commutative).

Hence the semidefinite programming problem for $\vartheta(G_{n,d})$
can be reduced to a linear programming problem,
with $\leq n+1$ variables and $\leq n+1$ constraints.

Corollary: the Delsarte bound can be computed fast.

GENERALIZATION

Let $G = (V, E)$ be a graph and let $k \in \mathbb{N}$.

Define $\boldsymbol{\mathcal{S}_k} := \{U \subseteq V \mid U \text{ stable}, |U| \leq k\}$.

$$\boldsymbol{\vartheta_k(G)} := \max\Big\{ \sum_{v \in V} X_{\{v\},\{v\}} \;\Big|$$

X nonnegative and positive semidefinite $\mathcal{S}_k \times \mathcal{S}_k$ matrix

with $X_{\emptyset,\emptyset} = 1$,

$X_{U,W} = 0$ if $U \cup W$ is not stable,

$X_{U,W} = X_{U',W'}$ if $U \cup W = U' \cup W'$ $\Big\}$

It can be proved that $\vartheta_1(G) = \vartheta(G)$.

Also: $\alpha(G) \leq \vartheta_k(G)$

Proof: Let S be a maximum-size stable set.

Define $X_{U,W} := \begin{cases} 1 & \text{if } U \cup W \subseteq S, \\ 0 & \text{else.} \end{cases}$ $\square$

EXPLOITING SYMMETRY

$\mathrm{Aut}(G)$ acts on $\mathcal{S}_k$, hence on $\mathbb{R}^{\mathcal{S}_k \times \mathcal{S}_k}$.

Again we can assume X to be $\mathrm{Aut}(G)$-invariant.

Let $\mathcal{A}$ be the algebra of $\mathrm{Aut}(G)$-invariant matrices in $\mathbb{C}^{\mathcal{S}_k \times \mathcal{S}_k}$.

Then there exists a unitary matrix $M \in \mathbb{C}^{\mathcal{S}_k \times \mathcal{S}_k}$ such that

$$M \mathcal{A} M^* = \bigoplus_{i=1}^{t} \mathbb{C}^{a_i \times a_i} \otimes I_{b_i}$$

for some $a_1, b_1, \ldots, a_t, b_t \in \mathbb{N}$.

Note that $$\mathbb{C}^{a \times a} \otimes I_b = \{ \begin{pmatrix} Y & 0 & \cdots & 0 \\ 0 & Y & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Y \end{pmatrix} \mid Y \in \mathbb{C}^{a \times a} \}.$$

So $\dim \mathcal{A} = a_1^2 + \cdots + a_t^2$.

APPLICATION TO CODES GIVES:

(taking $k := 2$)

n	d	known lower bound	known upper bound	new upper bound
17	5	512	680	673
18	5	1024	1280	1237
19	5	2048	2372	2279
22	5	8192	13766	13674
18	7	128	142	135
19	7	256	274	256
24	7	4096	5477	5421
20	9	42	48	47
21	9	64	87	84
23	9	128	280	268
24	9	192	503	466
25	9	384	886	836
26	9	512	1764	1585
24	11	52	56	55
25	11	64	98	96

HOW TO FIND THE BLOCK DIAGONALIZATION ?

$$\mathcal{A} = \mathrm{End}_{\mathrm{Aut}(G)}(\mathbb{C}^{\mathcal{S}_2})$$

$=$ the set of all linear functions $f : \mathbb{C}^{\mathcal{S}_2} \to \mathbb{C}^{\mathcal{S}_2}$

such that $f \circ g = g \circ f$ for each $g \in \mathrm{Aut}(G)$.

Find the canonical decomposition into isotypical components $C_1, \ldots, C_t$ of the action of $\mathrm{Aut}(G)$ on $\mathbb{C}^{\mathcal{S}_2}$.

(An **isotypical component** is the sum of an equivalence class of irreducible subrepresentations.)

Then

$$\mathrm{End}_{\mathrm{Aut}(G)}(\mathbb{C}^{\mathcal{S}_2}) \cong \bigoplus_{i=1}^{t} \mathrm{End}_{\mathrm{Aut}(G)}(C_i) \cong \bigoplus_{i=1}^{t} \mathbb{C}^{a_\times a_i}$$

for some $a_1, \ldots, a_t \in \mathbb{N}$.

THE CODE BOUNDS USE AN EXTRA CONDITION

For $k = 2$, you can add the condition:

for all $s, t \in V$, the matrix

$$\left(X_{\{s,u\},\{t,v\}}\right)_{u,v \in V}$$

is positive semidefinite.