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Surprises in Mathematics[1]

(1) *Examples* of (possibly) surprising mathematical results: (a) Tarski-Banach paradox; (b) existence of mathematical monsters like nowhere differentiable continuous functions or space-filling curves; (c) any finite-dimensional real division algebra must be of dimension 1, 2, 4 or 8; (d) there are only five regular polyhedra (the Platonic solids); (e) the line has the same cardinality as the plane; (f) Gödel's incompleteness theorems (?); (g) there are coverings of the rational numbers in the unit interval, that do not cover this whole interval, and the 'length' of these coverings can be as small as possible, such that the 'length' of the rest of the interval approaches 1; (f) any set can be well-ordered.

(2) *Toy example*: the harmonic series 1+1/2+1/3+1/4+… *diverges*, in spite of the fact that its partial sums are growing so slowly and $1+(1/2)^{\alpha}+(1/3)^{\alpha}+(1/4)^{\alpha}+\ldots$ converges for any $\alpha > 1$.

(3) Frege in *Grundlagen der Arithmetik*, § 105: "We are concerned in arithmetic not with objects that become known to us through the medium of the senses as something foreign from outside, but with objects that are immediately given to reason, which can fully comprehend them, as its own."

(4) "If you are surprised, then you have not understood it yet. For surprise is not legitimate here, as it is with the issue of an experiment. *There* – I should like to say – it is permissible to yield to its charm; but not when the surprise comes to you at the end of a chain of inference [as in mathematics]. For here it is only a sign that unclarity or some misunderstanding still reigns." (Wittgenstein, § 2)

(5) *Proof* of (2): 1+1/2+(1/3+1/4)+(1/5+1/6+1/7+1/8)+(1/9+…+1/16)+…; and, as one immediately understands, every bracketed sum is at least 1/2. q.e.d.

(6) *R-surprise*: arises merely from a certain *representation* and might be dispelled by a more perspicuous (etc.) representation given by an appropriate proof which leads to a better understanding of the mathematical situation.

F-surprise: arises from the discovery of a new *fact*, expressed by a proposition considered in great independence of a possible proof. ("The mathematician says as it were: »Do you see, this is surely important, this you would never have known without me.« As if, by means of [his] considerations, as by means of a higher experiment, astonishing, nay *the most* astonishing facts were brought to light." (Wittgenstein, §1))

(7) Possible difference between *mathematics* and the *empirical sciences*: in mathematics *only* R-surprises (»If you are surprised, then you have not understood it yet«), in contrast to, say, quantum mechanics (Bohr: »If you are not surprised, then you have not understood it yet«). Compare also Frege: "My very much adored and long ago deceased Karl Schnell of Jena often enunciated the principle: in mathematics everything must be as clear as 2 times 2 equals 4. As soon as there appears something mysterious, this is a sign that not everything is in order" ("Neuer Versuch über die Grundlegung der Arithmetik" of 1924/25); and Gian-Carlo Rota: "The quest for ultimate triviality is characteristic of the mathematical enterprise" (*Indiscrete Thoughts*, p. 93).

(8) *Dispelling* the appearance of an F-surprise: (a) uncovering the *presuppositions* of the result and showing that they do not express 'surprising facts'; (b) showing that the *way* from these presuppositions to the result does not involve 'surprising facts'; (c) devising new *conceptual means* to give a clearer view of results and their proofs (linear algebra with respect to systems of linear equations; differential forms with respect to Stokes' theorem; etc.); (d) *exorcising* the appearance of a mystery ("[I]n a certain sense he had not believed that there was a mystery in the case. But he was under the *impression* of mystery […]. In *one* sense he was indeed acquainted with the situation, but he related to it (in feeling an in action) 'as if there were something else involved' – as we would say" (Wittgenstein, § 8); compare the story of a the magician).

(9) The *prose* of mathematicians (verbal statements of mathematical results considered as similar to the statements of scientists who report facts of nature) can produce surprises, but can also dispel them: see Zermelo's warning that "the uninformed are only too prone to look for some mystical meaning behind Cantor's relation [of well-ordering]" and his therapeutic advice to think of the choices, allowed by the axiom of choice, as "*simultaneous* choices" (Zermelo 1908).

(10) There *may* remain *stubborn* examples, like the result that in the world of Euclidean geometry there are only five regular polyhedra. Examples like this should to be further investigated by mathematicians.

[1] All Wittgenstein quotations are from Appendix II of Part I of the *Remarks on the Foundations of Mathematics*.