Counting curves: the hunting of generating functions

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"A generating function is a clothesline on which we hang up a sequence of numbers for display."

H.S.Wilf
generatingfunctionology
Academic Press Inc., 1990

Partitions

Let $p(n)$ denote the number of partitions of the integer n, i.e., $p(n)$ is the number of ways we can write $n = n_1 + n_2 + \ldots$, where $n_1 \geq n_2 \geq \cdots > 0$.

Then $p(n)$ is clearly the coefficient of q^n in the power series expansion of the product

$$(1 + q + q^2 + \cdots)(1 + q^2 + q^4 + \cdots) \cdots (1 + q^m + q^{2m} + \cdots) \cdots$$

$$= 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + 11q^6 + 15q^7 + 22q^8 + \cdots$$

The *generating function* of integer partitions is

$$f(q) = \sum_{n \geq 0} p(n)\, q^n = \prod_{m \geq 1} (1 - q^m)^{-1}$$

(all this goes back to Euler).

If we instead look at

$$(1+q+q^2+\cdots)(1+q^2+q^4+\cdots)^2\cdots(1+q^m+q^{2m}+\cdots)^m\cdots$$

it turns out that the coefficient of q^n in the expansion is the number of plane partitions (or 3D Young diagrams) of n.

So the generating function for this problem is the MacMahon function

$$M(q)=\prod_{m\geq 1}(1-q^m)^{-m},$$

$$M(q)=1+q+3q^2+6q^3+13q^4+24q^5+\cdots$$

Recursions

If a sequence of numbers is defined recursively, we might be able to find the generating functions, and maybe even a closed formula for the numbers.

Let $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$. Then

$$\sum F_n\, q^n = q + \sum F_{n-1}\, q^n + \sum F_{n-2}\, q^n$$

Hence the generating function for the Fibonacci numbers is

$$f(q) = \sum F_n q^n = \frac{q}{1 - q - q^2}$$

and $F_n = \frac{1}{\sqrt{5}}\big(\varphi^n - (1-\varphi)^n\big)$, where $\varphi = \frac{1}{2}(1+\sqrt{5})$ is the golden ratio.

Enumerative geometry

Unlike the case of enumerative combinatorics, generating functions did not use to play a prominent role in enumerative geometry. One reason is that there often are not natural sequences that need to be displayed, rather one looks for specific numbers, or general formulas.

Schubert calculus: Enumerate linear spaces in a given projective space satisfying a certain number of incidence conditions with other linear spaces.

Example

How many lines intersect four given lines in $\mathbb{P}^3$?

Plane rational curves

Let N_d denote the number of plane rational curves of degree d passing through $3d-1$ given points.

Kontsevich's recursion formula:

$$N_d = \sum_{d_1+d_2=d} N_{d_1} N_{d_2} \left(d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \right)$$

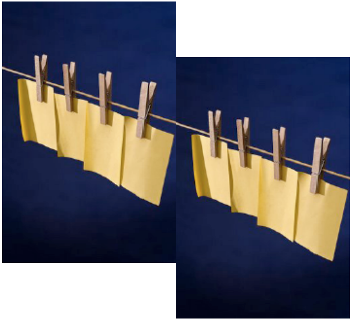
What can we say about the generating function

$$f(q) = \sum_d N_d \, q^d \ ?$$

or the *exponential* generating function

$$f(q) = \sum_d N_d \, q^d / d! \ ?$$

To display on the clothesline?



The sequence of numbers

$$N_1 = 1$$
$$N_2 = 1$$
$$N_3 = 12$$
$$N_4 = 620$$
$$N_5 = 87304$$
$$N_6 = 26312976$$
$$N_7 = 14616808192$$
$$N_8 = 13525751027392$$
$$N_9 = 19385778269260800$$
$$N_{10} = 40739017561997799680$$
$$N_{11} = 120278021410937387514880$$
$$N_{12} = 482113680618029292368686080$$

Curves on Calabi–Yau threefolds

Clemens' conjecture: There are only finitely many rational curves of degree d on a general quintic hypersurface in $\mathbb{P}^4$.

(Proved for $d \leq 10$ by Clemens, Katz, Kleiman–Johnsen, Cotterill.)

The physicists enter the scene:
In string theory, rational curves on Calabi–Yau threefolds are instantons. Using the principle of *mirror symmetry*, curves on a Calabi–Yau threefold (such as the quintic) can be counted by computing integrals on the mirror manifold. This way Candelas, de la Ossa, Green, and Parkes predicted the generating function of the Clemens problem!

Gromov–Witten invariants "replaced" the numbers of curves, and this way the theory was made sound.

Curves on a K3 surface

The 2-dimensional analogue of a CY threefold is a K3 surface, for example a quartic surface $S \subset \mathbb{P}^3$.

Let $C \subset S$ be a curve, $n = C^2/2 + 1$, and set

$$N_n := \#\{n\text{-nodal rational curves in } |C|\}.$$

The generating function is (Yau–Zaslow, Bryan–Leung)

$$f(q) = \sum_n N_n\, q^n = \prod_m (1 - q^m)^{-24} = \Big(\prod_m (1 - q^m)^{-1}\Big)^{\chi(S)}$$

where $\chi(S)$ is the (topological) Euler characteristique of S.

WHY??

Reduce to combinatorics!

Degenerate the pair (C, S) to (C', S'), where $S' \to \mathbb{P}^1$ is an elliptic fibration with 24 nodal (hence rational) fibers F_j, and $|C'| = |s(\mathbb{P}^1) + nF|$, where F is a fiber.

Hence the solutions in the degenerate case is equal to the number of curves of the form $s(\mathbb{P}^1) + \sum_{j=1}^{24} a_j F_j$, with $n = a_1 + \cdots + a_{24}$.

Each degenerate solution comes with a weight: there are $p(a_j)$ ways to cover F_j a_j-fold by $\leq a_j$ $\mathbb{P}^1$'s, hence each degenerate solution has weight $p(a_1) \cdots p(a_{24})$, and

$$N_n = \sum_{a_1 + \cdots + a_{24} = n} p(a_1) \cdots p(a_{24})$$

And what about MacMahon?

The number of partitions $p(n)$ is the same as the number of Ferrers–Young diagrams of size n and their generating function appears in the count of curves on K3 surfaces. So the generating function of *plane* partitions, or 3D Young diagrams, should appear when counting curves on three-dimensional Calabi–Yau varieties?? Behrend–Bryan–Szendroi: "…the MacMahon function …whose appearance permeats Donaldson–Thomas theory."

For example,

$$\sum_{n\geq 0} DT_n(\mathbb{C}^3)\, q^n = \prod_{m\geq 1} (1-(-q)^m)^{-m} = M(-q)$$

It is conjectured that for a Calabi–Yau threefold X,

$$Z_{DT,0}(q) = M(-q)^{\chi(X)}$$

and there are further refinements, conjectures and theorems.

Curves on surfaces (joint with Steven Kleiman)

Let S be a smooth projective surface, $Y = |C|$ a linear system. Let n_r denote the number of r-nodal curves in the system passing through $\dim Y - r$ given points. The *Chern numbers* are $m = C^2$, $k = K_S \cdot C$, $s = K_S^2$, and $x = c_2(S)$.

We (and everybody) conjectured (based on Vainsencher's work) that there should exist universal polynomials ϕ_r – the node polynomials – of degree r such that $n_r = \phi_r(m, k, s, x)$.

Göttsche's conjectured that the generating function can be expressed in terms of three quasimodular forms and two universal (but unknown) power series. This has been proved, as we have seen, for K3 surfaces (the only quasimodular function appearing in genus 0 for a K3 is the discriminant), and also for abelian surfaces.

Plane curves revisited

In the case of plane curves of degree d, the Chern numbers are

$$m = d^2, k = -3d, s = 9, x = 3.$$

The n_r seem to be polynomials in d of degree $2r$:

$$\begin{aligned}
n_1 &= 3d^2 - 6d + 3 \\
n_2 &= \tfrac{9}{2}d^4 - 18d^3 + 6d^2 + \tfrac{81}{2}d - 33 \\
n_3 &= \tfrac{9}{2}d^6 - 27d^5 + \tfrac{9}{2}d^4 + \tfrac{423}{2}d^3 - 229d^2 - \tfrac{829}{2}d + 525 \\
n_4 &= \tfrac{27}{8}d^8 - 27d^7 + \tfrac{1809}{4}d^5 - 642d^4 - 2529d^3 + \tfrac{37881}{8}d^2 \\
&\quad + \tfrac{18057}{4}d - 8865 \\
n_5 &= \tfrac{81}{40}d^{10} - \tfrac{81}{4}d^9 - \tfrac{27}{8}d^8 + \tfrac{2349}{4}d^7 - 1044d^6 - \tfrac{127071}{20}d^5 \\
&\quad + \tfrac{128859}{8}d^4 + \tfrac{59097}{2}d^3 - \tfrac{3528381}{40}d^2 - \tfrac{946929}{20}d + 153513
\end{aligned}$$

(Further conjectures by Di Francesco–Itzykson, Göttsche, and Qviller.)

Combinatorial proof?

Yes, by translation to *tropical curves*, Fomin–Mikhalkin have proved that the n_r are indeed polynomials in d of degree $2r$. They reduce the count to that of labeled floor diagrams.

Theorem (F–M). *For any fixed r, there exists a polynomial $Q_r(d) \in \mathbb{Q}[d]$ of degree $2r$ and a threshold value $d_0(r)$ such that for $d \geq d_0(r)$, we have $N_r = Q_r(d)$.*

They give the threshold $d_0(r) = 2r$ (but say they can do better) – the conjectured threshold is $d_0(r) = r/2 + 1$ (we prove it for $r \leq 8$).

Back to the general case

Consider a family of curves $D \subset F = S \times Y$ on S, and the classes d, k, s, x on Y analogous to the Chern numbers. We are looking for the class u_r on Y that enumerates r-nodal curves in the family.

By blowing up the family F along a section, we obtain, by restriction, a family of curves on the family of blown up surfaces, with one node less. To get the expression for u_r, we can push down the expression for u_{r-1} of the new family. Since there are r nodes in each r-nodal curve, there are r ways to blow up:

$$r\, u_r = u_{r-1}u_1 + Du_{r-1}$$

where Du_{r-1} is a "correction term."

$$
\begin{aligned}
u_1 &= u_1 \\
2\,u_2 &= u_1^2 + Du_1 \\
3!u_3 &= (u_1^2 + Du_1)u_1 + 2Du_2 \\
&= u_1^3 + u_1 Du_1 + D(u_1^2 + Du_1) \\
&= u_1^3 + 3u_1 Du_1 + D^2 u_1
\end{aligned}
$$

Is Faà di Bruno hiding here somewhere?

Set $a_1 = u_1$, $a_2 = Du_1$, $a_3 = D^2 u_1$:

$$
\begin{aligned}
u_1 &= a_1 \\
2\,u_2 &= a_1^2 + a_2 \\
3!u_3 &= a_1^3 + 3a_1 a_2 + a_3
\end{aligned}
$$

Yes, but it is really Bell whom we found!

The (complete exponential) Bell polynomials can be recursively defined as

$$P_{r+1}(a_1,\ldots,a_{r+1}) = \sum_{j=0}^{r} \binom{r}{j} P_{r-j}(a_1,\ldots,a_{r-j})a_{j+1}$$

or by the formal identity

$$\sum_{r\geq 0} P_r(a_1,\ldots,a_r)\, q^r/r! = \exp \sum_{j\geq 1} a_i\, q^i/i!$$

or

$$P_r(a_1,\ldots,a_r) = \sum_{k_1+2k_2\cdots+rk_r=r} \frac{r!}{k_1!\cdots k_r!} \Big(\frac{a_1}{1!}\Big)^{k_1} \cdots \Big(\frac{a_r}{r!}\Big)^{k_r}$$

A conjecture and a theorem

Conjecture. The generating function for nodal curves in a given family of curves on a surface S can be written as

$$\sum n_r\, q^r = \sum \frac{1}{r!} P_r(a_1, \ldots, a_r)\, q^r,$$

where the a_i are linear combinations (computable from an algorithm) of the Chern numbers of S and the curves C in the family, and the P_i are the Bell polynomials.

Theorem. *The conjecture is true for $r \leq 8$.*

An advantage of having a generating function of this form is that we only need to compute one new term a_r to pass from n_{r-1} to n_r.

The first eight a_i

$$
\begin{aligned}
a_1 &= 3d + 2k + x \\
a_2 &= -42d - 39k - 6s - 7x \\
a_3 &= 1380d + 1576k + 376s + 138x \\
a_4 &= -72360d - 95670k - 28842s - 3888x \\
a_5 &= 5225472d + 7725168k + 2723400s + 84384x \\
a_6 &= -481239360d - 778065120k - 308078520s + 7918560x \\
a_7 &= 53917151040d + 93895251840k + 40747613760s \\
&\quad -2465471520x \\
a_8 &= -7118400139200d - 13206119880240k - 6179605765200s \\
&\quad +516524964480x
\end{aligned}
$$

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Ingredients of proof

- The functor of infinitely near points on a family of surfaces
- Enriques diagrams of singularities of curves on a surface
- Zariski clusters and their representation in the Hilbert scheme

We would of course like to enumerate curves in a family with *any* given set of singularities.

Example

The number of curves with one triple point and one node, passing through $\dim Y - 5$ points is

$$n(3,2) = 45m^2 + (15s + 90k + 30x - 420)m + 40k^2$$

$$+(10s + 30x - 624)k + (5x - 196)s + 5x^2 - 100x$$

Idea of the proof

A node (an ordinary double point) of a curve C on a nonsingular surface S is resolved by blowing up the singular point. The strict transform $C - 2E$ of the given curve C will thus have one fewer node than C.

Set $Y = |C|$, and consider the family of surfaces

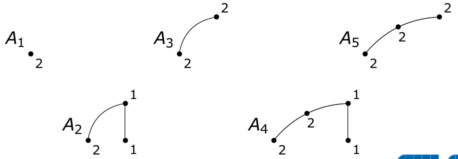
$$\pi' \colon F' \to F = S \times Y,$$

obtained by blowing up the diagonal in $F \times_Y F$. The new family is a family of surfaces S_x, where S_x is the blow up of S in the point x.
Let $X \subset F$ be the set of singular points of the curves C; then the r-nodal curves in $|C|$ correspond to $(r-1)$-nodal curves in $|\pi'^{*}C - 2E|_X$. We get the r-nodal formula by pushing down the $(r-1)$-nodal formula, and hence can use induction on the number r of nodes.

Curve singularities and Enriques diagrams

In the proof we also need to consider worse singularities.
Let $x \in C \subset S$ be a singular point on a curve on a nonsingular surface. The singularity can be resolved by a series of blowups, and we get an associated weighted resolution graph (Enriques diagram) $\mathbf{D}$, which determines the topological type of the singularity.

Examples of diagrams



The functor of infinitely near points

Let S be a nonsingular surface and

$$S^{(n+1)} \to S^{(n)} \to \cdots \to S^{(0)} = S$$

a sequence of blowups at centers $t_i \in S^{(i)}$. Then $(t_0, \ldots, t_n)$ is called a *sequence of infinitely near points* of S.

Theorem. *Let $F \to Y$ be a family of surfaces. The sequences of infinitely near T-points of F/Y form a functor, which is represented by $F^{(n)}/Y$, where $F^{(n)}$ is defined recursively by letting $F^{(i)} \to F^{(i-1)}$ be the blow up of the diagonal in $F^{(i-1)} \times_Y F^{(i-1)}$, composed with projection on the second factor. For each ordered diagram $(\mathbf{D}, \theta)$, the subfunctor of sequences with ordered diagram $(\mathbf{D}, \theta)$ is represented by a subscheme $F(\mathbf{D}, \theta) \subset F^{(n)}$, which is smooth over Y.*

Zariski clusters and the Hilbert scheme

To each point x on the surface S and each diagram $\mathbf{D}$, one can associate an ideal in the local ring $\mathcal{O}_{S,x}$ of finite colength equal to the *degree* d of the diagram $\mathbf{D}$. The corresponding "fat point" (Zariski cluster) has the property that a curve containing this fat point has a singularity of type $\mathbf{D}$.

This gives a map $\Psi_\theta \colon F(\mathbf{D},\theta) \to \mathrm{Hilb}^d_{F/Y}$, which factors through the quotient of the action of $\mathrm{Aut}(\mathbf{D})$.

Theorem. *The map*

$$\Psi \colon F(\mathbf{D},\theta)/\mathrm{Aut}(\mathbf{D}) \to \mathrm{Hilb}^d_{F/Y}$$

is universally injective and an embedding in characteristic 0 (but can be purely inseparable in positive characteristic).

The Hilbert scheme of a diagram

Let $H(\mathbf{D}) \subset \mathrm{Hilb}^d_{F/Y}$ denote the set of fat points of colength d with diagram $\mathbf{D}$.

Corollary. *The subset $H(\mathbf{D}) \subset \mathrm{Hilb}^d_{F/Y}$ is a locally closed subscheme, smooth over Y, with geometrically irreducible fibers of dimension* $\dim(\mathbf{D})$.

APPLICATION: Let $C \subset F$ be a family of curves over Y, and consider the natural embedding $\mathrm{Hilb}^d_{C/Y} \subset \mathrm{Hilb}^d_{F/Y}$. The solution to the problem: *enumerate curves in the family C/Y with singularities of type* $\mathbf{D}$ is the computation of the pushdown to Y of the class

$$[\overline{H(\mathbf{D})}] \cap [\mathrm{Hilb}^d_{C/Y}].$$

PROBLEM: Compute $[\overline{H(\mathbf{D})}]$.

Configuration spaces

Instead of the recursive procedure, using $F^{(r)}$, one could try directly to use intersection theory on $F^{(r)}$ or on Ulyanov's compactification $F\langle r\rangle$ of the configuration space $F^r\setminus$ diagonals.

We want to compute the class

$$m_r := [\overline{X^r \setminus \{\text{diagonals}\}}],$$

where $X \subset F$ is the set of singular points of the curves in the family $C \subset F$; then $n_r = r!\,\pi_* m_r$.

Example

In the case of plane curves

$$F = \mathbb{P}^2 \times \mathbb{P}^{(d+3)d/2}$$

then X is the complete intersection given by the three partial derivatives of the universal polynomial of degree d.

Excess intersection theory

Consider the natural map $F\langle r\rangle \to F^r$ and let $p_i \colon F\langle r\rangle \to F$ be the composition with the i-th projection. Then

$$p_1^*[X]\cdots p_r^*[X] = m_r + \sum_Z (p_1^*[X]\cdots p_r^*[X])^Z,$$

where the sum is taken over all connected components Z of the intersection

$$p_1^*[X] \cap \cdots \cap p_r^*[X],$$

and where $(p_1^*[X]\cdots p_r^*[X])^Z$ denotes the "equivalence" of Z.

For each polydiagonal Δ_I in F^r, there is a divisor D_I in $F\langle r\rangle$. For each D_I, collect the Z's contained in it. This gives a a contribution b_I, a class supported on D_I.

Polydiagonals

How many polydiagonals of each type exist in an r-fold product? By type $\mathbf{k} = (k_1, \ldots k_r)$ we mean that k_2 pairs of points of $(x_1, \ldots, x_r)$ are equal, k_3 triples of points of $(x_1, \ldots, x_r)$ are equal, and so on, with $k_1 + 2k_2 + \cdots + rk_r = r$. There are precisely

$$\frac{r!}{k_1! \cdots k_r!} \Big(\frac{1}{1!}\Big)^{k_1} \cdots \Big(\frac{1}{r!}\Big)^{k_r}$$

polydiagonals of type $(k_1, \ldots k_r)$.

By symmetry, the class b_I only depends on the type of the diagonal. The excess intersection formula therefore gives

$$m_r = p_1^*[X] \cdots p_r^*[X] - \sum_{\mathbf{k}} \frac{r!}{k_1! \cdots k_r!} \Big(\frac{1}{1!}\Big)^{k_1} \cdots \Big(\frac{1}{r!}\Big)^{k_r} b_{\mathbf{k}}.$$

The conjecture

The conjectured formula was

$$n_r = \frac{1}{r!} P_r(a_1, \ldots, a_r) = \sum_{\mathbf{k}} \frac{1}{k_1! \cdots k_r!} \Big(\frac{a_1}{1!}\Big)^{k_1} \cdots \Big(\frac{a_r}{r!}\Big)^{k_r},$$

where the sum is taken over $\mathbf{k} = (k_1, \ldots, k_r) \in \mathbb{Z}^r_{\geq 0}$ with $\sum ik_i = r$.

We know that $\pi_* p_i^*[X] = a_1$, and it remains to show that, more generally, $\pi_* b_{\mathbf{k}} = a_1^{k_1} \cdots a_r^{k_r}$.

This is a current project of my student Nikolay Qviller. The ambition for the time being is to prove the *shape* of the polynomials n_r, not to compute them.

Note that Kazarian has a topological argument, involving Thom polynomials, for showing that such a formula holds.

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Thank you for your attention!

The Hunting of the Snark (Lewis Carrol)

The Beaver had counted with scrupulous care,
Attending to every word:
But it fairly lost heart, and outgrabe in despair,
When the third repetition occurred.

It felt that, in spite of all possible pains,
It had somehow contrived to lose count,
And the only thing now was to rack its poor brains
By reckoning up the amount.

"Two added to one — if that could but be done,"
It said, "with one's fingers and thumbs!"
Recollecting with tears how, in earlier years,
It had taken no pains with its sums.

"The thing can be done," said the Butcher, "I think.
The thing must be done, I am sure.
The thing shall be done! Bring me paper and ink,
The best there is time to procure.