Rough paths and the Gap Between Deterministic and Stochastic Differential Equations

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Donsker’s invariance principle and Brownian motion

Itô integration and SDEs

Doss-Sussman’s ODE approach to SDEs

More on ODEs: Euler estimates

Smart ODE limits: rough differential equations

Brownian motion as a rough path

SDEs driven by non-Brownian noise

Rough path spaces

Donsker’s principle revisited
Donsker's invariance principle

- Let \((\zeta_i)\) be an IID sequence of zero-mean, unit-variance random variables. [Donsker '52] shows that the rescaled, piecewise-linearly-connected, random-walk

\[
W_t^{(n)} = \frac{1}{n^{1/2}} \left( \zeta_1 + \cdots + \zeta_{[tn]} + (nt - [nt]) \zeta_{[nt]+1} \right)
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converges weakly in the space of continuous functions on \([0, 1]\).
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- Straight-forward extension to \(\mathbb{R}^d\)-valued case
- In particular, a \(d\)-dimensional Brownian motion is just an ensemble of \(d\) independent Brownian motions, say

\[
B_t = \left( B_t^1, \ldots, B_t^d \right).
\]
Brownian motion: alternative characterizations

(i): continuous martingale such that \( (B_t^2 - t) \) is also a martingale
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(iii): Markov process with generator \(L = \frac{1}{2} \frac{\partial^2}{\partial x^2}\) in the sense that

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How to define integration against Brownian motion? Itô’s brilliant idea: with some help and intuition from martingale theory,

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\int_0^1 f(t, \omega) \, dB_t(\omega)
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can be defined for reasonable non-anticipating \( f \): start with simple integrands and complete with isometry

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E \left[ \left( \int_0^1 f(t, \omega) \, dB_t(\omega) \right)^2 \right] = E \left[ \int_0^1 f^2(t, \omega) \, dt \right].
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Example: $\int_0^t B_s \, dB_s = \frac{1}{2} (B_t^2 - t)$ ... 2nd order calculus!
Itô integration

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Fact: Itô-integrals have left-point Riemann-sum approximations.

Define Stratonovich-integration via mid-point Riemann-sum approximations \( \implies \int_0^t B_s \, dB_s = \frac{1}{2} B_t^2 \) (1st order calculus!)
Let $B$ be a $d$-dimensional Brownian motion.
Stochastic differential equations

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is, by definition, a solution to corresponding integral equation.
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Existence, uniqueness by fixpoint arguments.
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  dy &= V_0(y) \, dt + \sum_{i=1}^d V_i(y) \, \partial B^i \\
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- Existence, uniqueness by fixpoint arguments.
- For simplicity only: from here on $V_0 = 0$. 
The Doss-Sussman approach

- Let $B$ be a $d$-dimensional Brownian motion, $d = 1$. 

Let $V$ be a nice vector field on $\mathbb{R}$. 

Aim: find solution to SDE

$$
\begin{align*}
  \frac{dy}{dt} &= V(y) \partial dB
\end{align*}
$$

Let $e^{tV}$ be the solution flow to the ODE

$$
\begin{align*}
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\end{align*}
$$

Then

$$
\begin{align*}
  y(t, \omega) &= e^{Bt}(\omega) V y_0
\end{align*}
$$

is the SDE solution. Proof: First order calculus. 

This is an ODE solution method for SDEs. 

Benefit: solution depends in a robust way on $B$ and $y_0$. 

A drift $V_0(y) dt$ can be incorporated (flow decomposition) 

but this method fails when $d > 1$. 

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- Let $(e^{tV})$ be the solution flow to the ODE $\dot{z} = V(z)$. Then

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- ... but this method fails when $d > 1$. 

So far, we have been interested in stochastic differential equations of the type

\[ dy_t = \sum_{i=1}^{d} V_i(y_t) \, dB_t^i \]

Let us now look at such differential equations when \( B \) is replaced by some path \( x \in C^1([0,1], \mathbb{R}) \); that is

\[ \dot{y}_t = \sum_{i=1}^{d} V_i(y_t) \, \dot{x}_t \]

This is a classical setup in system control theory... and in our case the system response is modelled by ODE... How would one simulate... on a computer?
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- How would one simulate \((\ast)\) on a computer?
More precisely: \( x \in C^1 ([0, 1], \mathbb{R}^d) \), \( V_1, \ldots, V_d \in C^{2,b} (\mathbb{R}^e, \mathbb{R}^e) \)

\[
dy = V(y) \, dx \iff \dot{y} = V_i(y) \, \dot{x}^i
\]

(Summation over repeated indices!) Usual Euler-scheme:

\[
y_t - y_s \approx V_i(y_s) \int_s^t dx^i
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Step-2 Euler scheme:

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y_t - y_s \approx V_i(y_s) \int_s^t dx^i + V_i V_j(y_s) \int_s^t \int_s^r dx^i dx^j
\]

\[
= \mathcal{E}(y_s, x_{s,t})
\]

with

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x_{s,t} = \left( \int_s^t dx, \int_s^t \int_s^r dx \otimes dx \right) \in \mathbb{R}^d \oplus \mathbb{R}^{d \times d}.
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Natural scaling assumption. For some \( \alpha \in (0, 1] \),

\[
\left| \int_s^t \, dx^i \right| \vee \left| \int_s^t \int_s^r \, dx^i \, dx^j \right|^{1/2} \leq c_1 |t - s|^\alpha.
\]

[Okay for BM with \( \alpha < 1/2 \) but keep \( x \in C^1 \) for now ...]
**Davie’s Lemma**: Error estimate on Step-2 Euler scheme

\[ |y_t - y_s - \mathcal{E}(y_s, \mathbf{x}_s, t)| \leq c_2 |t - s|^{\theta} \]

with \( \theta = 3\alpha > 1 \) \( \implies \) need \( \alpha > 1/3 \) [Okay for BM ...]. The catch is **uniformity**

\[ c_2 = c_2(c_1) \quad \text{not } c_2(\|\dot{x}\|_{\infty}) \text{ or } c_2\left(\|x\|_{\text{Lip}}\right) \]
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Easy to see that

\[ \mathcal{E}(y_s, x_{s,t}) \leq c_3 |t - s|^{\alpha}, \quad c_3 = c_3(c_1) \]
\[ |y_t - y_s| \leq c_4 |t - s|^{\alpha}, \quad c_4 = c_4(c_1). \]
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Easy to see that

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\[ |y_t - y_s| \leq c_4 |t - s|^{\alpha}, \quad c_4 = c_4(c_1). \]

Take \( x_n \in C^1([0,1], \mathbb{R}^d) \) with uniform bounds

\[ \sup_n \left| \int_s^t dx_n^i \right| \vee \left| \int_s^t \int_s^r dx_n^i dx_n^j \right|^{1/2} \leq c_1 |t - s|^{\alpha} \]

s.t. \( x_n + \text{iterated integrals converge (pointwise)} \) to

\[ x_t = \left( x_t^{(1)}, x_t^{(2)} \right) \in \mathbb{R}^d \oplus \mathbb{R}^{d \times d}; \]

then call \( t \mapsto x_t \) a (geometric) rough path.
Apply Davie’s lemma: get \( \{y_n\} \) with uniform Hölder bound \( c_4 \).
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• More regularity + a bit work \( \implies \exists! \) RDE solution \( y \equiv \Phi(x; y_0) \)
  and write
  
  \[ dy = V(y) \, dx \]

... and this "\textit{Itô-Lyons}" map \( \Phi \) is continuous in the above sense
[Lyons 98].
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- Error estimates for step-\( N \) Euler schemes [F-Victoir, JDE 07]
- RDE smoothness and Malliavin calculus [Cass-F, Annals of Math 09]
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... and this "Itô-Lyons" map \( \Phi \) is continuous in the above sense [Lyons 98].

• Various useful extensions ...
  - Error estimates for step-\( N \) Euler schemes [F-Victoir, JDE 07]
  - RDE smoothness and Malliavin calculus [Cass-F, Annals of Math 09]
  - Continuity of \( \Phi \) as flow of diffeomorphisms [Caruana-F, JDE 08]
• Apply Davie’s lemma: get \( \{y_n\} \) with uniform Hölder bound \( c_4 \).

• Arzela–Ascoli \( \implies \{y_n : n \geq 1\} \) has limit points
  ... call them \( RDE \) solutions

• More regularity + a bit work \( \implies \exists \) RDE solution \( y \equiv \Phi(x; y_0) \)
  and write
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  • Rough partial differential equations [Caruana-F-Oberhauser ...]
Examples of RDEs

- **ODEs:** For a smooth driving signal, RDEs are just ODEs. Even here, continuity statements are powerful.
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\[ \dot{y} = V_1(y) + V_2(y) \iff dy = V_1(y) \, dt + V_2(y) \, dt; \]

we immediately get the (splitting) result

\[ e^{\frac{1}{n}V_2} \circ e^{\frac{1}{n}V_1} \circ \cdots \circ e^{\frac{1}{n}V_2} \circ e^{\frac{1}{n}V_1} \rightarrow e^{V_1 + V_2} \]

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This approximation converges with uniform 1-Hölder (i.e. Lipschitz) bounds.
Differential equations driven by pure area:

\[ t \mapsto \mathbf{x}_t \equiv \begin{pmatrix} 0 & 0 \\ 0 & -t \end{pmatrix} \begin{pmatrix} 0 \\ t \end{pmatrix} \]

is the limit (with uniform 1/2-Hölder bounds ...) of the highly oscillatory

\[ x_n(t) = n^{-1} \exp(2\pi in^2 t) \in \mathbb{C} \cong \mathbb{R}^2. \]

Given two vector fields \( V = (V_1, V_2) \) the RDE solution

\[ dy = V(y) \, dx \quad (1) \]

models the effective behaviour of the highly oscillatory ODE

\[ dy^n = V(y^n) \, dx^n \quad \text{as} \ n \to \infty. \]

In fact, the RDE solution of (1) solves the ODE

\[ \dot{y} = [V_1, V_2](y) \]

where \([V_1, V_2]\) is the Lie bracket of \(V_1\) and \(V_2\).
Stochastic differential equations: Let $B$ be $d$-dimensional Brownian motion. Since $B(\omega) \notin C^1$ careful interpretation of the stochastic differential equation

$$dy = V(y) \partial B$$

is necessary (Itô-theory). Define enhanced Brownian motion

$$B_t(\omega) = \left( B_t, \int_0^t B_s \otimes \partial B_s \right)$$

where $\partial$ indicates (Stratonovich) stochastic integration. Then

$$\mathbb{P}[B \text{ is a geometric rough path}] = 1.$$ 

In fact, martingale arguments shows that $B(\omega)$ is the limit of piecewise linear approximations (with uniform $(1/2 - \varepsilon)$-Hölder bounds ...).

RDE solution to $dy = V(y) dB$ is solved for fixed $\omega$, depends continuously on $B$ and yields a (classical) Stratonovich SDE solution ...
Caution: topology matters. Possible that, uniformly in $t$,

$$\left( B_t^{(n)}, \int_0^t B_s^{(n)} \otimes dB_s^{(n)} \right) \to \left( B_t, \int_0^t B_s \otimes \partial B_s \right)$$

while DE solutions converge to the "wrong" limit.
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Key to understanding: view $B$ as level-$N$ rough path; [F-Oberhauser, JFA 09]
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while DE solutions converge to the "wrong" limit.

Key to understanding: view $\mathbb{B}$ as level-$N$ rough path; [F-Oberhauser, JFA 09]

By rough path continuity, this would *not* happen if, for some $\alpha \in (1/3, 1/2]$,

$$\left| \int_s^t dB_t^{(n)} \right| \vee \left| \int_s^t \int_s^r dB_s^{(n)} \otimes dB_s^{(n)} \right|^{1/2} \leq C(\omega) |t - s|^\alpha.$$
Differential equations with non-Brownian noise

- Recall \( BM = \text{martingale, Gaussian, Markov} \)
Differential equations with non-Brownian noise

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- SDE theory with (semi)martingale noise follows Itô’s approach and is well-known

Thanks to rough path theory:
- large and natural classes of the above processes can be lifted to rough paths with resulting path-by-path stochastic differential equations.
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For $x \in C^1([0,1], \mathbb{R}^d)$, $x_0 = 0$, define generalized increments

$$x_{s,t} = \left(1, \int_s^t dx, \int_s^t \int_s^r dx \otimes dx\right) \in \mathbb{R} \oplus \mathbb{R}^d \oplus \mathbb{R}^{d \times d}, \ 0 \leq s \leq t \leq 1$$
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- The (vector) space \( \mathbb{R} \oplus \mathbb{R}^d \oplus \mathbb{R}^{d \times d} \) with basis
  \( (1, b^i, b^{jk}; 1 \leq i, j, k \leq d) \) has (truncated tensor) algebra structure; e.g.
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  2b^1 \otimes (4 - 3b^2) = 8b^1 - 6b^{12}
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- $x_{s,t} \in T_1 := \{1\} \oplus \mathbb{R}^d \oplus \mathbb{R}^{d \times d}$ and $(T_1, \otimes, 1)$ is a Lie group

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- Non-linear key identity [Chen '37]

  $$x_{s,t} \otimes x_{t,u} = x_{s,u}, \quad 0 \leq s \leq t \leq u \leq 1.$$
Actually, $x_{s,t} \in G := \exp (\mathbb{R}^d \oplus so(d))$ since (1st order calculus!)

$$\text{Sym} \left( \int_s^t \int_s^r dx \otimes dx \right) = \frac{1}{2} \left( \int_s^t dx \right) \otimes \left( \int_s^t dx \right)$$

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- $x_t := x_{0,t}$ defines a $G$-valued path (which lifts $x$) and $x_{s,t} = x_s^{-1} \otimes x_t$
Recall our assumption in Davie’s lemma:

$$\left| \int_s^t dx \right| \vee \left| \int_s^t \int_s^r dx \otimes dx \right|^{1/2} \leq c_1 |t - s|^\alpha$$

... this says precisely that $t \mapsto x_t$ is a Hölder continuous path, with exponent $\alpha$, in the space $G$ with Carnot-Caratheodory metric

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The space of all (\( \alpha \)-Hölder, geometric) rough paths [previously introduced as pointwise limits of \( C^1 \)-paths + iterated integrals subject to uniform \( \alpha \)-Hölder bounds] is precisely

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Very convenient! E.g. to show rough path regularity of \( B_t (\omega) \) ...
Theorem [Donsker ’52] Under finite second moment assumptions, renormalized random walk in $\mathbb{R}^d$ converges weakly (in sup-topology) to BM.
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Corollary: Universal limit theorem for Markov chain approximations to stochastic differential equations.
Donsker’s theorem revisited

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Corollary: Universal limit theorem for Markov chain approximations to stochastic differential equations.

P. K. Friz (TU Berlin and WIAS)  
rough paths, gap ODE/SDEs  
December 2009  
20 / 22
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Random walk can be viewed as random walk \((\xi_i)\) on the step-2 free nilpotent group.
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Boils down to showing

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\forall p < \infty : \mathbb{E} \left[ \| \xi_1 \ast \cdots \ast \xi_k \|_{CC}^{4p} \right] = O(k^{2p}).
\]
If you want to read more about this:

- **[Lyons, Qian '02]:** System control and rough paths, Oxford Univ. Press

- [Caruana, Levy, Lyons '05]: Differential equations driven by rough paths, St. Flour lecture...
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