Rough paths and the Gap Between Deterministic and Stochastic Differential Equations

P. K. Friz

TU Berlin and WIAS

December 2009

Outline

I Donsker's invariance principle and Brownian motion

II Itô integration and SDEs

III Doss-Sussman's ODE approach to SDEs

IV More on ODEs: Euler estimates

V Smart ODE limits: rough differential equations

VI Brownian motion as a rough path

VII SDEs driven by non-Brownian noise

VIII Rough path spaces

IX Donsker's principle revisited

Donsker's invariance principle

- Let (ξ_i) be an IID sequence of zero-mean, unit-variance random variables. **[Donsker '52]** shows that the rescaled, piecewise-linearly-connected, random-walk

$$W_t^{(n)} = \frac{1}{n^{1/2}} \left(\xi_1 + \cdots + \xi_{[tn]} + (nt - [nt])\, \xi_{[nt]+1} \right)$$

converges weakly in the space of continuous functions on $[0,1]$.

Donsker's invariance principle

- Let (ξ_i) be an IID sequence of zero-mean, unit-variance random variables. **[Donsker '52]** shows that the rescaled, piecewise-linearly-connected, random-walk

$$W_t^{(n)} = \frac{1}{n^{1/2}} \left(\xi_1 + \cdots + \xi_{[tn]} + (nt - [nt])\, \xi_{[nt]+1} \right)$$

  converges weakly in the space of continuous functions on $[0,1]$.
- Limit is a probability measure on $C[0,1]$, called Wiener measure $\mathbb{W}$.

Donsker's invariance principle

- Let (ξ_i) be an IID sequence of zero-mean, unit-variance random variables. **[Donsker '52]** shows that the rescaled, piecewise-linearly-connected, random-walk

$$W_t^{(n)} = \frac{1}{n^{1/2}} \left(\xi_1 + \cdots + \xi_{[tn]} + (nt - [nt])\, \xi_{[nt]+1} \right)$$

converges weakly in the space of continuous functions on $[0,1]$.

- Limit is a probability measure on $C[0,1]$, called Wiener measure $\mathbb{W}$.
- Brownian motion (BM) (B_t) is a stochastic process with law $\mathbb{W}$.

Donsker's invariance principle

- Let (ξ_i) be an IID sequence of zero-mean, unit-variance random variables. **[Donsker '52]** shows that the rescaled, piecewise-linearly-connected, random-walk

$$W_t^{(n)} = \frac{1}{n^{1/2}} \left(\xi_1 + \cdots + \xi_{[tn]} + (nt - [nt])\, \xi_{[nt]+1} \right)$$

converges weakly in the space of continuous functions on $[0,1]$.

- Limit is a probability measure on $C\,[0,1]$, called Wiener measure $\mathbb{W}$.
- Brownian motion (BM) (B_t) is a stochastic process with law $\mathbb{W}$.
- Straight-forward extension to $\mathbb{R}^d$-valued case

Donsker's invariance principle

- Let (ξ_i) be an IID sequence of zero-mean, unit-variance random variables. **[Donsker '52]** shows that the rescaled, piecewise-linearly-connected, random-walk

$$W_t^{(n)} = \frac{1}{n^{1/2}} \left(\xi_1 + \cdots + \xi_{[tn]} + (nt - [nt])\, \xi_{[nt]+1} \right)$$

converges weakly in the space of continuous functions on $[0,1]$.

- Limit is a probability measure on $C\,[0,1]$, called Wiener measure $\mathbb{W}$.
- Brownian motion (BM) (B_t) is a stochastic process with law $\mathbb{W}$.
- Straight-forward extension to $\mathbb{R}^d$-valued case
- In particular, a d-dimensional Brownian motion is just an ensemble of d independent Brownian motions, say

$$B_t = \left(B_t^1, \ldots, B_t^d \right).$$

Brownian motion: alternative characterizations

- (i): continuous martingale such that $\left(B_t^2 - t\right)$ is also a martingale

Brownian motion: alternative characterizations

- (i): continuous martingale such that $\left(B_t^2 - t\right)$ is also a martingale
- (ii): continuous 0-mean Gaussian process with covariance

$$E(B_s B_t) = \min(s, t) \quad \forall s, t \in [0, 1]$$

Brownian motion: alternative characterizations

- (i): continuous martingale such that $\left(B_t^2 - t\right)$ is also a martingale
- (ii): continuous 0-mean Gaussian process with covariance

$$E(B_s B_t) = \min(s,t) \quad \forall s,t \in [0,1]$$

- (iii): Markov process with generator $L = \frac{1}{2}\frac{\partial^2}{\partial x^2}$ in the sense that

$$\frac{E\left[f\left(x + B_t\right)\right] - x}{t} \to Lf = \frac{1}{2} f'' \quad \forall f \text{ nice}$$

Brownian motion: alternative characterizations

- (i): continuous martingale such that $\left(B_t^2 - t\right)$ is also a martingale
- (ii): continuous 0-mean Gaussian process with covariance

$$E(B_s B_t) = \min(s,t) \quad \forall s,t \in [0,1]$$

- (iii): Markov process with generator $L = \frac{1}{2}\frac{\partial^2}{\partial x^2}$ in the sense that

$$\frac{E\left[f\left(x+B_t\right)\right] - x}{t} \to Lf = \frac{1}{2}f'' \quad \forall f \text{ nice}$$

- Again, straight-forward extension to $\mathbb{R}^d$-valued case

Itô integration

- Fact: Typical sample paths of Brownian motion, $t \mapsto B_t(\omega)$, have infinite variation on every interval.

Itô integration

- Fact: Typical sample paths of Brownian motion, $t \mapsto B_t(\omega)$, have infinite variation on every interval.
- How to define integration against Brownian motion? Itô's brilliant idea: with some help and intuition from martingale theory,

$$\int_0^1 f(t,\omega)\, dB_t(\omega)$$

can be defined for reasonable non-anticipating f: start with simple integrands and complete with isometry

$$E\left[\left(\int_0^1 f(t,\omega)\, dB_t(\omega)\right)^2\right] = E\left[\int_0^1 f^2(t,\omega)\, dt\right].$$

Itô integration

- Fact: Typical sample paths of Brownian motion, $t \mapsto B_t(\omega)$, have infinite variation on every interval.
- How to define integration against Brownian motion? Itô's brilliant idea: with some help and intuition from martingale theory,

$$\int_0^1 f(t,\omega)\, dB_t(\omega)$$

can be defined for reasonable non-anticipating f: start with simple integrands and complete with isometry

$$E\left[\left(\int_0^1 f(t,\omega)\, dB_t(\omega)\right)^2\right] = E\left[\int_0^1 f^2(t,\omega)\, dt\right].$$

- Example: $\int_0^t B_s dB_s = \frac{1}{2}\left(B_t^2 - t\right)$ ... 2nd order calculus!

Itô integration

- Fact: Typical sample paths of Brownian motion, $t \mapsto B_t(\omega)$, have infinite variation on every interval.
- How to define integration against Brownian motion? Itô's brilliant idea: with some help and intuition from martingale theory,

$$\int_0^1 f(t,\omega)\, dB_t(\omega)$$

can be defined for reasonable non-anticipating f: start with simple integrands and complete with isometry

$$E\left[\left(\int_0^1 f(t,\omega)\, dB_t(\omega)\right)^2\right] = E\left[\int_0^1 f^2(t,\omega)\, dt\right].$$

- Example: $\int_0^t B_s dB_s = \frac{1}{2}\left(B_t^2 - t\right)$ ... 2nd order calculus!
- Fact: Itô-integrals have left-point Riemann-sum approximations.

Itô integration

- Fact: Typical sample paths of Brownian motion, $t \mapsto B_t(\omega)$, have infinite variation on every interval.
- How to define integration against Brownian motion? Itô's brilliant idea: with some help and intuition from martingale theory,

$$\int_0^1 f(t,\omega)\, dB_t(\omega)$$

can be defined for reasonable non-anticipating f: start with simple integrands and complete with isometry

$$E\left[\left(\int_0^1 f(t,\omega)\, dB_t(\omega)\right)^2\right] = E\left[\int_0^1 f^2(t,\omega)\, dt\right].$$

- Example: $\int_0^t B_s dB_s = \frac{1}{2}\left(B_t^2 - t\right)$ ... 2nd order calculus!
- Fact: Itô-integrals have left-point Riemann-sum approximations.
- Define Stratonovich-integration via mid-point Riemann-sum approximations $\implies \int_0^t B_s \partial B_s = \frac{1}{2} B_t^2$ (1st order calculus!)

Stochastic differential equations

- Let B be a d-dimensional Brownian motion.

Stochastic differential equations

- Let B be a d-dimensional Brownian motion.
- Let $V_0, \ldots, V_d$ be a collection of nice vector fields on $\mathbb{R}^e$

Stochastic differential equations

- Let B be a d-dimensional Brownian motion.
- Let $V_0, \ldots, V_d$ be a collection of nice vector fields on $\mathbb{R}^e$
- A solution (process) $y = y_t(\omega)$ to

$$dy = V_0(y)\,dt + \sum_{i=1}^{d} V_i(y)\,\partial B^i$$

is, by definition, a solution to corresponding integral equation.

Stochastic differential equations

- Let B be a d-dimensional Brownian motion.
- Let $V_0, \ldots, V_d$ be a collection of nice vector fields on $\mathbb{R}^e$
- A solution (process) $y = y_t(\omega)$ to

$$dy = V_0(y)\,dt + \sum_{i=1}^{d} V_i(y)\,\partial B^i$$

  is, by definition, a solution to corresponding integral equation.
- At the price of modifying the drift vector field V_0 we can switch to Itô formulation $(\partial B \rightarrow dB)$

Stochastic differential equations

- Let B be a d-dimensional Brownian motion.
- Let $V_0, \ldots, V_d$ be a collection of nice vector fields on $\mathbb{R}^e$
- A solution (process) $y = y_t(\omega)$ to

$$dy = V_0(y)\,dt + \sum_{i=1}^{d} V_i(y)\,\partial B^i$$

  is, by definition, a solution to corresponding integral equation.
- At the price of modifying the drift vector field V_0 we can switch to Itô formulation $(\partial B \to dB)$
- Existence, uniqueness by fixpoint arguments.

Stochastic differential equations

- Let B be a d-dimensional Brownian motion.
- Let $V_0, \ldots, V_d$ be a collection of nice vector fields on $\mathbb{R}^e$
- A solution (process) $y = y_t(\omega)$ to

$$dy = V_0(y)\,dt + \sum_{i=1}^{d} V_i(y)\,\partial B^i$$

  is, by definition, a solution to corresponding integral equation.
- At the price of modifying the drift vector field V_0 we can switch to Itô formulation $(\partial B \to dB)$
- Existence, uniqueness by fixpoint arguments.
- For simplicity only: from here on $V_0 = 0$.

The Doss-Sussman approach

- Let B be a d-dimensional Brownian motion, $d = 1$.

The Doss-Sussman approach

- Let B be a d-dimensional Brownian motion, $d = 1$.
- Let V be a nice vector field on $\mathbb{R}^e$

The Doss-Sussman approach

- Let B be a d-dimensional Brownian motion, $d = 1$.
- Let V be a nice vector field on $\mathbb{R}^e$
- Aim: find solution to SDE

$$dy = V(y)\,\partial dB.$$

The Doss-Sussman approach

- Let B be a d-dimensional Brownian motion, $d = 1$.
- Let V be a nice vector field on $\mathbb{R}^e$
- Aim: find solution to SDE

$$dy = V(y) \partial dB.$$

- Let $\left(e^{tV}\right)$ be the solution flow to the ODE $\dot{z} = V(z)$. Then

$$y(t, \omega) := e^{B_t(\omega)V} y_0$$

is the SDE solution. Proof: First order calculus.

The Doss-Sussman approach

- Let B be a d-dimensional Brownian motion, $d = 1$.
- Let V be a nice vector field on $\mathbb{R}^e$
- Aim: find solution to SDE

$$dy = V(y)\,\partial dB.$$

- Let $\left(e^{tV}\right)$ be the solution flow to the ODE $\dot{z} = V(z)$. Then

$$y(t,\omega) := e^{B_t(\omega)V} y_0$$

  is the SDE solution. Proof: First order calculus.
- This is an ODE solution method for SDEs.

The Doss-Sussman approach

- Let B be a d-dimensional Brownian motion, $d = 1$.
- Let V be a nice vector field on $\mathbb{R}^e$
- Aim: find solution to SDE

$$dy = V(y) \partial dB.$$

- Let $\left(e^{tV}\right)$ be the solution flow to the ODE $\dot{z} = V(z)$. Then

$$y(t, \omega) := e^{B_t(\omega) V} y_0$$

  is the SDE solution. Proof: First order calculus.
- This is an ODE solution method for SDEs.
- Benefit: solution depends in a robust way on B and y_0.

The Doss-Sussman approach

- Let B be a d-dimensional Brownian motion, $d = 1$.
- Let V be a nice vector field on $\mathbb{R}^e$
- Aim: find solution to SDE

$$dy = V(y) \partial dB.$$

- Let $\left(e^{tV}\right)$ be the solution flow to the ODE $\dot{z} = V(z)$. Then

$$y(t, \omega) := e^{B_t(\omega)V} y_0$$

  is the SDE solution. Proof: First order calculus.
- This is an ODE solution method for SDEs.
- Benefit: solution depends in a robust way on B and y_0.
- A drift $V_0(y)dt$ can be incorporated (flow decomposition)

The Doss-Sussman approach

- Let B be a d-dimensional Brownian motion, $d = 1$.
- Let V be a nice vector field on $\mathbb{R}^e$
- Aim: find solution to SDE

$$dy = V(y)\,\partial dB.$$

- Let $\left(e^{tV}\right)$ be the solution flow to the ODE $\dot{z} = V(z)$. Then

$$y(t,\omega) := e^{B_t(\omega)V} y_0$$

  is the SDE solution. Proof: First order calculus.
- This is an ODE solution method for SDEs.
- Benefit: solution depends in a robust way on B and y_0.
- A drift $V_0(y)dt$ can be incorporated (flow decomposition)
- ... but this method fails when $d > 1$.

More on ODEs: Euler estimates

- So far, we have been interested in stochastic differential equations of the type

$$dy_t = \sum_{i=1}^{d} V_i\left(y_t\right)\partial B_t^i$$

More on ODEs: Euler estimates

- So far, we have been interested in stochastic differential equations of the type

$$dy_t = \sum_{i=1}^{d} V_i \left(y_t \right) \partial B_t^i$$

- Let us now look at such differential equations when B is replaced by some path $x \in C^1 \left([0,1] , \mathbb{R}^d \right)$; that is

$$(*) : \dot{y}_t = \sum_{i=1}^{d} V_i \left(y_t \right) \dot{x}_t$$

More on ODEs: Euler estimates

- So far, we have been interested in stochastic differential equations of the type

$$dy_t = \sum_{i=1}^{d} V_i\left(y_t\right)\partial B_t^i$$

- Let us now look at such differential equations when B is replaced by some path $x \in C^1\left(\left[0,1\right], \mathbb{R}^d\right)$; that is

$$(*) : \dot{y}_t = \sum_{i=1}^{d} V_i\left(y_t\right)\dot{x}_t$$

- This is a classical setup in system control theory ...

$$\text{input signal } x \quad \Longrightarrow \quad \text{output signal } y$$

More on ODEs: Euler estimates

- So far, we have been interested in stochastic differential equations of the type

$$dy_t = \sum_{i=1}^{d} V_i \left(y_t \right) \partial B_t^i$$

- Let us now look at such differential equations when B is replaced by some path $x \in C^1\left([0,1] , \mathbb{R}^d \right)$; that is

$$(*) : \dot{y}_t = \sum_{i=1}^{d} V_i \left(y_t \right) \dot{x}_t$$

- This is a classical setup in system control theory ...

$$\text{input signal } x \quad \Longrightarrow \quad \text{output signal } y$$

- ... and in our case the system response is modelled by ODE $(*)$.

More on ODEs: Euler estimates

- So far, we have been interested in stochastic differential equations of the type

$$dy_t = \sum_{i=1}^{d} V_i\left(y_t\right) \partial B_t^i$$

- Let us now look at such differential equations when B is replaced by some path $x \in C^1\left(\left[0,1\right], \mathbb{R}^d\right)$; that is

$$(*) : \dot{y}_t = \sum_{i=1}^{d} V_i\left(y_t\right) \dot{x}_t$$

- This is a classical setup in system control theory ...

$$\text{input signal } x \implies \text{output signal } y$$

- ... and in our case the system response is modelled by ODE $(*)$.
- How would one simulate $(*)$ on a computer?

- More precisely: $x \in C^1\left([0,1], \mathbb{R}^d\right)$, $V_1, \ldots, V_d \in C^{2,b}\left(\mathbb{R}^e, \mathbb{R}^e\right)$

$$dy = V(y)\, dx \Longleftrightarrow \dot{y} = V_i(y)\, \dot{x}^i$$

(Summation over repeated indices!) Usual Euler-scheme:

$$y_t - y_s \approx V_i(y_s) \int_s^t dx^i$$

- More precisely: $x \in C^1\left([0,1], \mathbb{R}^d\right)$, $V_1, ..., V_d \in C^{2,b}\left(\mathbb{R}^e, \mathbb{R}^e\right)$

$$dy = V(y)\, dx \Longleftrightarrow \dot{y} = V_i(y)\, \dot{x}^i$$

(Summation over repeated indices!) Usual Euler-scheme:

$$y_t - y_s \approx V_i(y_s) \int_s^t dx^i$$

- Step-2 Euler scheme:

$$y_t - y_s \approx \underbrace{V_i(y_s) \int_s^t dx^i + V_i V_j(y_s) \int_s^t \int_s^r dx^i dx^j}_{=\mathcal{E}\left(y_s, \underline{\mathbf{x}}_{s,t}\right)}$$

with

$$\underline{\mathbf{x}}_{s,t} = \left(\int_s^t dx, \int_s^t \int_s^r dx \otimes dx \right) \in \mathbb{R}^d \oplus \mathbb{R}^{d \times d}.$$

- More precisely: $x \in C^1\left([0,1], \mathbb{R}^d\right)$, $V_1, \dots, V_d \in C^{2,b}\left(\mathbb{R}^e, \mathbb{R}^e\right)$

$$dy = V(y)\, dx \Longleftrightarrow \dot{y} = V_i(y)\, \dot{x}^i$$

(Summation over repeated indices!) Usual Euler-scheme:

$$y_t - y_s \approx V_i(y_s) \int_s^t dx^i$$

- Step-2 Euler scheme:

$$y_t - y_s \approx \underbrace{V_i(y_s) \int_s^t dx^i + V_i V_j(y_s) \int_s^t \int_s^r dx^i dx^j}_{=\mathcal{E}\left(y_s, \underline{\mathbf{x}}_{s,t}\right)}$$

with

$$\underline{\mathbf{x}}_{s,t} = \left(\int_s^t dx, \int_s^t \int_s^r dx \otimes dx \right) \in \mathbb{R}^d \oplus \mathbb{R}^{d \times d}.$$

- Natural scaling assumption. For some $\alpha \in (0,1]$,

$$\left| \int_s^t dx^i \right| \vee \left| \int_s^t \int_s^r dx^i dx^j \right|^{1/2} \leq c_1 |t-s|^\alpha.$$

[Okay for BM with $\alpha < 1/2$ but keep $x \in C^1$ for now ...]

- *Davie's Lemma*: Error estimate on Step-2 Euler scheme

$$\left| y_t - y_s - \mathcal{E}\left(y_s, \underline{\mathbf{x}}_{s,t}\right)\right| \leq c_2 \left|t-s\right|^{\theta}$$

with $\theta = 3\alpha > 1 \implies$ need $\alpha > 1/3$ [Okay for BM ...]. The catch is *uniformity*

$$c_2 = c_2\left(c_1\right) \;\; .... \; \underline{\text{not}} \; c_2\left(\left|\dot{x}\right|_\infty\right) \text{ or } c_2\left(\left|x\right|_{Lip}\right)$$

- *Davie's Lemma*: Error estimate on Step-2 Euler scheme

$$\left| y_t - y_s - \mathcal{E}\left(y_s, \underline{\mathbf{x}}_{s,t}\right) \right| \leq c_2 \left| t-s \right|^{\theta}$$

with $\theta = 3\alpha > 1 \implies$ need $\alpha > 1/3$ [Okay for BM ...]. The catch is *uniformity*

$$c_2 = c_2\left(c_1\right) \quad .... \ \underline{\text{not}}\ c_2\left(|\dot{x}|_{\infty}\right) \text{ or } c_2\left(|x|_{Lip}\right)$$

- Easy to see that

$$\begin{aligned} \mathcal{E}\left(y_s, \underline{\mathbf{x}}_{s,t}\right) &\leq c_3 \left|t-s\right|^{\alpha}, \quad c_3 = c_3\left(c_1\right) \\ \left|y_t - y_s\right| &\leq c_4 \left|t-s\right|^{\alpha}, \quad c_4 = c_4\left(c_1\right). \end{aligned}$$

- *Davie's Lemma*: Error estimate on Step-2 Euler scheme

$$\left|y_t - y_s - \mathcal{E}\left(y_s, \underline{\mathbf{x}}_{s,t}\right)\right| \leq c_2 \left|t-s\right|^{\theta}$$

with $\theta = 3\alpha > 1 \implies$ need $\alpha > 1/3$ [Okay for BM ...]. The catch is *uniformity*

$$c_2 = c_2\left(c_1\right) \;\; .... \;\; \underline{\text{not}} \; c_2\left(\left|\dot{x}\right|_\infty\right) \text{ or } c_2\left(\left|x\right|_{Lip}\right)$$

- Easy to see that

$$\begin{aligned} \mathcal{E}\left(y_s, \underline{\mathbf{x}}_{s,t}\right) &\leq c_3 \left|t-s\right|^{\alpha}, \quad c_3 = c_3\left(c_1\right) \\ \left|y_t - y_s\right| &\leq c_4 \left|t-s\right|^{\alpha}, \quad c_4 = c_4\left(c_1\right). \end{aligned}$$

- Take $x_n \in C^1\left(\left[0,1\right], \mathbb{R}^d\right)$ with *uniform bounds*

$$\sup_n \left|\int_s^t dx_n^i\right| \vee \left|\int_s^t \int_s^r dx_n^i dx_n^j\right|^{1/2} \leq c_1 \left|t-s\right|^{\alpha}$$

s.t. x_n + iterated integrals converge (pointwise) to

$$\underline{\mathbf{x}}_t = \left(\underline{\mathbf{x}}_t^{(1)}, \underline{\mathbf{x}}_t^{(2)}\right) \in \mathbb{R}^d \oplus \mathbb{R}^{d \times d};$$

then call $t \mapsto \underline{\mathbf{x}}_t$ a *(geometric) rough path.*

- Apply Davie's lemma: get $\{y_n\}$ with uniform Hölder bound c_4.

- Apply Davie's lemma: get $\{y_n\}$ with uniform Hölder bound c_4.
- Arzela–Ascoli $\implies$ $\{y_n : n \geq 1\}$ has limit points
  ... call them *RDE solutions*

- Apply Davie's lemma: get $\{y_n\}$ with uniform Hölder bound c_4.
- Arzela–Ascoli $\implies$ $\{y_n : n \geq 1\}$ has limit points
  ... call them *RDE solutions*
- More regularity + a bit work $\implies$ $\exists!$ RDE solution $y \equiv \Phi\left(\underline{\mathbf{x}}; y_0\right)$ and write

$$dy = V\left(y\right) d\underline{\mathbf{x}}$$

... and this "*Itô-Lyons*" map Φ is continuous in the above sense [Lyons 98].

- Apply Davie's lemma: get $\{y_n\}$ with uniform Hölder bound c_4.
- Arzela–Ascoli $\implies$ $\{y_n : n \geq 1\}$ has limit points
  ... call them *RDE solutions*
- More regularity + a bit work $\implies$ $\exists!$ RDE solution $y \equiv \Phi(\underline{\mathbf{x}}; y_0)$ and write

$$dy = V(y)\, d\underline{\mathbf{x}}$$

  ... and this "*Itô-Lyons*" map Φ is continuous in the above sense [Lyons 98].
- Various useful extensions ...

- Apply Davie's lemma: get $\{y_n\}$ with uniform Hölder bound c_4.
- Arzela–Ascoli $\implies \{y_n : n \geq 1\}$ has limit points
  ... call them *RDE solutions*
- More regularity + a bit work $\implies \exists!$ RDE solution $y \equiv \Phi(\underline{\mathbf{x}}; y_0)$ and write

$$dy = V(y)\, d\underline{\mathbf{x}}$$

  ... and this "*Itô-Lyons*" map Φ is continuous in the above sense [Lyons 98].
- Various useful extensions ...
  - Error estimates for step-N Euler schemes [F-Victoir , JDE 07]

- Apply Davie's lemma: get $\{y_n\}$ with uniform Hölder bound c_4.
- Arzela–Ascoli $\implies$ $\{y_n : n \geq 1\}$ has limit points
  ... call them *RDE solutions*
- More regularity + a bit work $\implies$ $\exists!$ RDE solution $y \equiv \Phi(\underline{\mathbf{x}}; y_0)$ and write

$$dy = V(y)\, d\underline{\mathbf{x}}$$

  ... and this "*Itô-Lyons*" map Φ is continuous in the above sense [Lyons 98].
- Various useful extensions ...
  - Error estimates for step-N Euler schemes [F-Victoir , JDE 07]
  - RDE smoothness and Malliavin calculus [Cass-F, Annals of Math 09]

- Apply Davie's lemma: get $\{y_n\}$ with uniform Hölder bound c_4.
- Arzela–Ascoli $\implies \{y_n : n \geq 1\}$ has limit points
  ... call them *RDE solutions*
- More regularity + a bit work $\implies \exists!$ RDE solution $y \equiv \Phi(\underline{\mathbf{x}}; y_0)$ and write

$$dy = V(y)\, d\underline{\mathbf{x}}$$

  ... and this "*Itô-Lyons*" map Φ is continuous in the above sense [Lyons 98].
- Various useful extensions ...
  - Error estimates for step-N Euler schemes [F-Victoir , JDE 07]
  - RDE smoothness and Malliavin calculus [Cass-F, Annals of Math 09]
  - Continuity of Φ as flow of diffeomorphisms [Caruana-F, JDE 08]

- Apply Davie's lemma: get $\{y_n\}$ with uniform Hölder bound c_4.
- Arzela–Ascoli $\implies$ $\{y_n : n \geq 1\}$ has limit points
  ... call them *RDE solutions*
- More regularity + a bit work $\implies$ $\exists!$ RDE solution $y \equiv \Phi(\underline{\mathbf{x}}; y_0)$ and write

$$dy = V(y)\, d\underline{\mathbf{x}}$$

  ... and this "*Itô-Lyons*" map Φ is continuous in the above sense [Lyons 98].
- Various useful extensions ...
  - Error estimates for step-N Euler schemes [F-Victoir , JDE 07]
  - RDE smoothness and Malliavin calculus [Cass-F, Annals of Math 09]
  - Continuity of Φ as flow of diffeomorphisms [Caruana-F, JDE 08]
  - Rough partial differential equations [Caruana-F-Oberhauser ...]

Examples of RDEs

- **ODEs:** For a smooth driving signal, RDEs are just ODEs. Even here, continuity statements are powerful.

Examples of RDEs

- **ODEs:** For a smooth driving signal, RDEs are just ODEs. Even here, continuity statements are powerful.
- As example, consider

$$\dot{y} = V_1(y) + V_2(y) \Longleftrightarrow dy = V_1(y)\,dt + V_2(y)\,dt;$$

we immediately get the (splitting) result

$$e^{\frac{1}{n}V_2} \circ e^{\frac{1}{n}V_1} \circ \cdots \circ e^{\frac{1}{n}V_2} \circ e^{\frac{1}{n}V_1} \longrightarrow e^{V_1+V_2}$$

where e^{tW} denotes the solution flow to $\dot{z} = W(z)$.

Examples of RDEs

- **ODEs:** For a smooth driving signal, RDEs are just ODEs. Even here, continuity statements are powerful.
- As example, consider

$$\dot{y} = V_1(y) + V_2(y) \Longleftrightarrow dy = V_1(y)\,dt + V_2(y)\,dt;$$

  we immediately get the (splitting) result

$$e^{\frac{1}{n}V_2} \circ e^{\frac{1}{n}V_1} \circ \cdots \circ e^{\frac{1}{n}V_2} \circ e^{\frac{1}{n}V_1} \longrightarrow e^{V_1+V_2}$$

  where e^{tW} denotes the solution flow to $\dot{z} = W(z)$.
- Indeed, it suffices to approximation the diagonal $t \mapsto (t,t)$ by a $1/n$ - step function

Examples of RDEs

- **ODEs:** For a smooth driving signal, RDEs are just ODEs. Even here, continuity statements are powerful.
- As example, consider

$$\dot{y} = V_1(y) + V_2(y) \Longleftrightarrow dy = V_1(y)\,dt + V_2(y)\,dt;$$

we immediately get the (splitting) result

$$e^{\frac{1}{n}V_2} \circ e^{\frac{1}{n}V_1} \circ \cdots \circ e^{\frac{1}{n}V_2} \circ e^{\frac{1}{n}V_1} \longrightarrow e^{V_1+V_2}$$

where e^{tW} denotes the solution flow to $\dot{z} = W(z)$.

- Indeed, it suffices to approximation the diagonal $t \mapsto (t,t)$ by a $1/n$ - step function
- This approximation converges with uniform 1-Hölder (i.e. Lipschitz) bounds

- **Differential equations driven by pure area:**

$$t \mapsto \underline{\mathbf{x}}_t \equiv \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix} \right)$$

is the limit (with uniform 1/2-Hölder bounds ...) of the highly oscillatory

$$x_n(t) = n^{-1} \exp\left(2\pi i n^2 t\right) \in \mathbb{C} \cong \mathbb{R}^2.$$

Given two vector fields $V = (V_1, V_2)$ the RDE solution

$$dy = V(y)\, d\underline{\mathbf{x}} \tag{1}$$

models the effective behaviour of the highly oscillatory ODE

$$dy^n = V(y^n)\, dx^n \quad \text{as } n \to \infty.$$

In fact, the RDE solution of (1) solves the ODE

$$\dot{y} = [V_1, V_2](y)$$

where $[V_1, V_2]$ is the Lie bracket of V_1 and V_2.

- **Stochastic differential equations:** Let B be d-dimensional Brownian motion. Since $B(\omega) \notin C^1$ careful interpretation of the *stochastic* differential equation

$$dy = V(y)\,\partial B$$

is necessary (Itô-theory). Define *enhanced Brownian motion*

$$\underline{\mathbf{B}}_t(\omega) = \left(B_t, \int_0^t B_s \otimes \partial B_s \right)$$

where ∂ indicates (Stratonovich) stochastic integration. Then

$$\mathbb{P}\left[\underline{\mathbf{B}} \text{ is a geometric rough path}\right] = 1.$$

In fact, martingale arguments shows that $\underline{\mathbf{B}}(\omega)$ is the limit of piecewise linear approximations (with uniform $(1/2-\varepsilon)$-Hölder bounds ...).

- RDE solution to $dy = V(y)\,d\underline{\mathbf{B}}$ is solved for fixed ω, depends continuously on $\underline{\mathbf{B}}$ and yields a (classical) Stratonovich SDE solution ...

- **Caution:** topology matters. Possible that, uniformly in t,

$$\left(B_t^{(n)}, \int_0^t B_s^{(n)} \otimes dB_s^{(n)} \right) \to \left(B_t, \int_0^t B_s \otimes \partial B_s \right)$$

while DE solutions converge to the "wrong" limit.

- **Caution:** topology matters. Possible that, uniformly in t,

$$\left(B_t^{(n)}, \int_0^t B_s^{(n)} \otimes dB_s^{(n)}\right) \to \left(B_t, \int_0^t B_s \otimes \partial B_s\right)$$

while DE solutions converge to the "wrong" limit.

- Key to understanding: view $\underline{\mathbf{B}}$ as level-N rough path; [F-Oberhauser, JFA 09]

- **Caution:** topology matters. Possible that, uniformly in t,

$$\left(B_t^{(n)}, \int_0^t B_s^{(n)} \otimes dB_s^{(n)} \right) \rightarrow \left(B_t, \int_0^t B_s \otimes \partial B_s \right)$$

while DE solutions converge to the "wrong" limit.

- Key to understanding: view $\underline{\mathbf{B}}$ as level-N rough path; [F-Oberhauser, JFA 09]
- By rough path continuity, this would *not* happen if, for some $\alpha \in (1/3, 1/2]$,

$$\left| \int_s^t dB_t^{(n)} \right| \vee \left| \int_s^t \int_s^r dB_s^{(n)} \otimes dB_s^{(n)} \right|^{1/2} \leq C(\omega) |t-s|^{\alpha}.$$

Differential equations with non-Brownian noise

- Recall BM = martingale, Gaussian, Markov

Differential equations with non-Brownian noise

- Recall BM = martingale, Gaussian, Markov
- SDE theory with (semi)martingale noise follows Itô's approach and is well-known

Differential equations with non-Brownian noise

- Recall BM = martingale, Gaussian, Markov
- SDE theory with (semi)martingale noise follows Itô's approach and is well-known
- SDE theory with Gaussian noise is: previous to rough path theory restricted to special examples (e.g. fractional Brownian motion)

Differential equations with non-Brownian noise

- Recall BM = martingale, Gaussian, Markov
- SDE theory with (semi)martingale noise follows Itô's approach and is well-known
- SDE theory with Gaussian noise is: previous to rough path theory restricted to special examples (e.g. fractional Brownian motion)
- SDE theory with Markovian noise: previous to rough path theory, hardly anything

Differential equations with non-Brownian noise

- Recall BM = martingale, Gaussian, Markov
- SDE theory with (semi)martingale noise follows Itô's approach and is well-known
- SDE theory with Gaussian noise is: previous to rough path theory restricted to special examples (e.g. fractional Brownian motion)
- SDE theory with Markovian noise: previous to rough path theory, hardly anything
- **Thanks to rough path theory:** large and natural classes of the above processes can be lifted to rough paths with resulting path-by-path stochastic differential equations.

Rough path spaces

- For $x \in C^1([0,1], \mathbb{R}^d)$, $x_0 = 0$, define generalized increments

$$\underline{\mathbf{x}}_{s,t} = \left(1, \int_s^t dx, \int_s^t \int_s^r dx \otimes dx\right) \in \mathbb{R} \oplus \mathbb{R}^d \oplus \mathbb{R}^{d \times d}, \quad 0 \leq s \leq t \leq 1$$

Rough path spaces

- For $x \in C^1\left([0,1], \mathbb{R}^d\right)$, $x_0 = 0$, define generalized increments

$$\underline{\mathbf{x}}_{s,t} = \left(1, \int_s^t dx, \int_s^t \int_s^r dx \otimes dx\right) \in \mathbb{R} \oplus \mathbb{R}^d \oplus \mathbb{R}^{d \times d}, \quad 0 \leq s \leq t \leq 1$$

- The (vector) space $\mathbb{R} \oplus \mathbb{R}^d \oplus \mathbb{R}^{d \times d}$ with basis $\left(1, b^i, b^{jk}; 1 \leq i,j,k \leq d\right)$ has (truncated tensor) algebra structure; e.g.

$$2b^1 \otimes (4 - 3b^2) = 8b^1 - 6b^{12}$$

Rough path spaces

- For $x \in C^1([0,1], \mathbb{R}^d)$, $x_0 = 0$, define generalized increments

$$\underline{\mathbf{x}}_{s,t} = \left(1, \int_s^t dx, \int_s^t \int_s^r dx \otimes dx\right) \in \mathbb{R} \oplus \mathbb{R}^d \oplus \mathbb{R}^{d \times d}, \quad 0 \leq s \leq t \leq 1$$

- The (vector) space $\mathbb{R} \oplus \mathbb{R}^d \oplus \mathbb{R}^{d \times d}$ with basis $\left(1, b^i, b^{jk}; 1 \leq i, j, k \leq d\right)$ has (truncated tensor) algebra structure; e.g.

$$2b^1 \otimes (4 - 3b^2) = 8b^1 - 6b^{12}$$

- $\underline{\mathbf{x}}_{s,t} \in T_1 := \{1\} \oplus \mathbb{R}^d \oplus \mathbb{R}^{d \times d}$ and $(T_1, \otimes, 1)$ is a Lie group

$$T_1 = \exp\left(\mathbb{R}^d \oplus \mathbb{R}^{d \times d}\right)$$

Rough path spaces

- For $x \in C^1\left([0,1], \mathbb{R}^d\right)$, $x_0 = 0$, define generalized increments

$$\underline{\mathbf{x}}_{s,t} = \left(1, \int_s^t dx, \int_s^t \int_s^r dx \otimes dx\right) \in \mathbb{R} \oplus \mathbb{R}^d \oplus \mathbb{R}^{d \times d}, \quad 0 \leq s \leq t \leq 1$$

- The (vector) space $\mathbb{R} \oplus \mathbb{R}^d \oplus \mathbb{R}^{d \times d}$ with basis $\left(1, b^i, b^{jk}; 1 \leq i, j, k \leq d\right)$ has (truncated tensor) algebra structure; e.g.

$$2b^1 \otimes (4 - 3b^2) = 8b^1 - 6b^{12}$$

- $\underline{\mathbf{x}}_{s,t} \in T_1 := \{1\} \oplus \mathbb{R}^d \oplus \mathbb{R}^{d \times d}$ and $(T_1, \otimes, 1)$ is a Lie group

$$T_1 = \exp\left(\mathbb{R}^d \oplus \mathbb{R}^{d \times d}\right)$$

- Non-linear key identity **[Chen '37]**

$$\underline{\mathbf{x}}_{s,t} \otimes \underline{\mathbf{x}}_{t,u} = \underline{\mathbf{x}}_{s,u}, \quad 0 \leq s \leq t \leq u \leq 1.$$

- Actually, $\underline{\mathbf{x}}_{s,t} \in G := \exp\left(\mathbb{R}^d \oplus so\left(d\right)\right)$ since (1st order calculus!)

$$\mathrm{Sym}\left(\int_s^t \int_s^r dx \otimes dx\right) = \frac{1}{2}\left(\int_s^t dx\right) \otimes \left(\int_s^t dx\right)$$

$G = \exp\left(\mathbb{R}^d \oplus so\left(d\right)\right)$ is (a realization of) step-2 nilpotent Lie group with d generators

- Actually, $\underline{\mathbf{x}}_{s,t} \in G := \exp\left(\mathbb{R}^d \oplus so(d)\right)$ since (1st order calculus!)

$$\mathrm{Sym}\left(\int_s^t \int_s^r dx \otimes dx\right) = \frac{1}{2}\left(\int_s^t dx\right) \otimes \left(\int_s^t dx\right)$$

  $G = \exp\left(\mathbb{R}^d \oplus so(d)\right)$ is (a realization of) step-2 nilpotent Lie group with d generators
- geometric interpretation of $\log\left(\underline{\mathbf{x}}_{s,t}\right)$: path- and area-increment

- Actually, $\underline{\mathbf{x}}_{s,t} \in G := \exp\left(\mathbb{R}^d \oplus so(d)\right)$ since (1st order calculus!)

$$\mathrm{Sym}\left(\int_s^t \int_s^r dx \otimes dx\right) = \frac{1}{2}\left(\int_s^t dx\right) \otimes \left(\int_s^t dx\right)$$

  $G = \exp\left(\mathbb{R}^d \oplus so(d)\right)$ is (a realization of) step-2 nilpotent Lie group with d generators
- geometric interpretation of $\log\left(\underline{\mathbf{x}}_{s,t}\right)$: path- and area-increment
- For $d = 2$, G isomorphic to the 3-dimensional Heisenberg group

- Actually, $\underline{\mathbf{x}}_{s,t} \in G := \exp\left(\mathbb{R}^d \oplus so\left(d\right)\right)$ since (1st order calculus!)

$$\mathrm{Sym}\left(\int_s^t \int_s^r dx \otimes dx\right) = \frac{1}{2}\left(\int_s^t dx\right) \otimes \left(\int_s^t dx\right)$$

  $G = \exp\left(\mathbb{R}^d \oplus so\left(d\right)\right)$ is (a realization of) step-2 nilpotent Lie group with d generators
- geometric interpretation of $\log\left(\underline{\mathbf{x}}_{s,t}\right)$: path- and area-increment
- For $d = 2$, G isomorphic to the 3-dimensional Heisenberg group
- Familiar concepts (scalar product, norm) generalize

- Actually, $\underline{\mathbf{x}}_{s,t} \in G := \exp\left(\mathbb{R}^d \oplus so(d)\right)$ since (1st order calculus!)

$$\mathrm{Sym}\left(\int_s^t \int_s^r dx \otimes dx\right) = \frac{1}{2}\left(\int_s^t dx\right) \otimes \left(\int_s^t dx\right)$$

  $G = \exp\left(\mathbb{R}^d \oplus so(d)\right)$ is (a realization of) step-2 nilpotent Lie group with d generators
- geometric interpretation of $\log\left(\underline{\mathbf{x}}_{s,t}\right)$: path- and area-increment
- For $d = 2$, G isomorphic to the 3-dimensional Heisenberg group
- Familiar concepts (scalar product, norm) generalize
  - Dilation $\delta_\lambda (1 + v + M) = 1 + \lambda v + \lambda^2 M$

- Actually, $\underline{\mathbf{x}}_{s,t} \in G := \exp\left(\mathbb{R}^d \oplus so\left(d\right)\right)$ since (1st order calculus!)

$$\mathrm{Sym}\left(\int_s^t \int_s^r dx \otimes dx\right) = \frac{1}{2}\left(\int_s^t dx\right) \otimes \left(\int_s^t dx\right)$$

  $G = \exp\left(\mathbb{R}^d \oplus so\left(d\right)\right)$ is (a realization of) step-2 nilpotent Lie group with d generators
- geometric interpretation of $\log\left(\underline{\mathbf{x}}_{s,t}\right)$: path- and area-increment
- For $d = 2$, G isomorphic to the 3-dimensional Heisenberg group
- Familiar concepts (scalar product, norm) generalize
  - Dilation $\delta_\lambda\left(1 + v + M\right) = 1 + \lambda v + \lambda^2 M$
  - Carnot-Carathoedory norm:
    $\left\|1 + v + M\right\|_{CC} \sim |v| \vee |M|^{1/2} \sim |v| \vee \left|\mathrm{Anti}\left(M\right)\right|^{1/2}$

- Actually, $\underline{\mathbf{x}}_{s,t} \in G := \exp\left(\mathbb{R}^d \oplus so(d)\right)$ since (1st order calculus!)

$$\mathrm{Sym}\left(\int_s^t \int_s^r dx \otimes dx\right) = \frac{1}{2}\left(\int_s^t dx\right) \otimes \left(\int_s^t dx\right)$$

  $G = \exp\left(\mathbb{R}^d \oplus so(d)\right)$ is (a realization of) step-2 nilpotent Lie group with d generators
- geometric interpretation of $\log\left(\underline{\mathbf{x}}_{s,t}\right)$: path- and area-increment
- For $d = 2$, G isomorphic to the 3-dimensional Heisenberg group
- Familiar concepts (scalar product, norm) generalize
  - Dilation $\delta_\lambda (1 + v + M) = 1 + \lambda v + \lambda^2 M$
  - Carnot-Carathoedory norm:
    $\|1 + v + M\|_{CC} \sim |v| \vee |M|^{1/2} \sim |v| \vee |\mathrm{Anti}(M)|^{1/2}$
- $\underline{\mathbf{x}}_t := \underline{\mathbf{x}}_{0,t}$ defines a G-valued path (which lifts x) and $\underline{\mathbf{x}}_{s,t} = \underline{\mathbf{x}}_s^{-1} \otimes \underline{\mathbf{x}}_t$

- Recall our assumption in Davie's lemma:

$$\left| \int_s^t dx \right| \vee \left| \int_s^t \int_s^r dx \otimes dx \right|^{1/2} \leq c_1 \left| t - s \right|^{\alpha}$$

... this says precisely that $t \mapsto \underline{\mathbf{x}}_t$ is a Hölder continuous path, with exponent α, in the space G with Carnot-Caratheodory metric

$$d_{CC} \left(\underline{\mathbf{x}}_s, \underline{\mathbf{x}}_t \right) := \left\| \underline{\mathbf{x}}_s^{-1} \otimes \underline{\mathbf{x}}_t \right\|_{CC} = \left\| \underline{\mathbf{x}}_{s,t} \right\|_{CC}.$$

- Recall our assumption in Davie's lemma:

$$\left| \int_s^t dx \right| \vee \left| \int_s^t \int_s^r dx \otimes dx \right|^{1/2} \leq c_1 \left| t-s \right|^{\alpha}$$

... this says precisely that $t \mapsto \underline{\mathbf{x}}_t$ is a Hölder continuous path, with exponent α, in the space G with Carnot-Caratheodory metric

$$d_{CC}\left(\underline{\mathbf{x}}_s, \underline{\mathbf{x}}_t\right) := \left\| \underline{\mathbf{x}}_s^{-1} \otimes \underline{\mathbf{x}}_t \right\|_{CC} = \left\| \underline{\mathbf{x}}_{s,t} \right\|_{CC}.$$

- The space of all (α-Hölder, geometric) rough paths [previously introduced as pointwise limits of C^1-paths + iterated integrals subject to uniform α-Hölder bounds] is precisely

$$\left\{ \underline{\mathbf{x}} \in C\left([0,1], G\right) : \sup_{0 \leq s < t \leq 1} \frac{d_{CC}\left(\underline{\mathbf{x}}_s, \underline{\mathbf{x}}_t\right)}{\left|t-s\right|^{\alpha}} < \infty \right\}$$

- Recall our assumption in Davie's lemma:

$$\left|\int_s^t dx\right| \vee \left|\int_s^t \int_s^r dx \otimes dx\right|^{1/2} \leq c_1 |t-s|^\alpha$$

... this says precisely that $t \mapsto \underline{\mathbf{x}}_t$ is a Hölder continuous path, with exponent α, in the space G with Carnot-Caratheodory metric

$$d_{CC}(\underline{\mathbf{x}}_s, \underline{\mathbf{x}}_t) := \left\|\underline{\mathbf{x}}_s^{-1} \otimes \underline{\mathbf{x}}_t\right\|_{CC} = \left\|\underline{\mathbf{x}}_{s,t}\right\|_{CC}.$$

- The space of all (α-Hölder, geometric) rough paths [previously introduced as pointwise limits of C^1-paths + iterated integrals subject to uniform α-Hölder bounds] is precisely

$$\left\{\underline{\mathbf{x}} \in C([0,1], G) : \sup_{0 \leq s < t \leq 1} \frac{d_{CC}(\underline{\mathbf{x}}_s, \underline{\mathbf{x}}_t)}{|t-s|^\alpha} < \infty\right\}$$

- Very convenient! E.g. to show rough path regularity of $\underline{\mathbf{B}}_t(\omega)$ ...

Donsker's theorem revisited

- **Theorem [Donsker '52]** Under finite second moment assumptions, renormalized random walk in $\mathbb{R}^d$ converges weakly (in sup-topology) to BM.

Donsker's theorem revisited

- **Theorem [Donsker '52]** Under finite second moment assumptions, renormalized random walk in $\mathbb{R}^d$ converges weakly (in sup-topology) to BM.
- **Theorem [Lamperti '62]** Assuming finite moments of all order, convergence holds in α-Hölder topology, any $\alpha < 1/2$.

Donsker's theorem revisited

- **Theorem [Donsker '52]** Under finite second moment assumptions, renormalized random walk in $\mathbb{R}^d$ converges weakly (in sup-topology) to BM.
- **Theorem [Lamperti '62]** Assuming finite moments of all order, convergence holds in α-Hölder topology, any $\alpha < 1/2$.
- **Theorem [Breuillard-F-Huessman, Proc. AMS '09]** Under previous assumptions, convergence holds in α-Hölder rough path topology, $\alpha < 1/2$.

Donsker's theorem revisited

- **Theorem [Donsker '52]** Under finite second moment assumptions, renormalized random walk in $\mathbb{R}^d$ converges weakly (in sup-topology) to BM.
- **Theorem [Lamperti '62]** Assuming finite moments of all order, convergence holds in α-Hölder topology, any $\alpha < 1/2$.
- **Theorem [Breuillard-F-Huessman, Proc. AMS '09]** Under previous assumptions, convergence holds in α-Hölder rough path topology, $\alpha < 1/2$.
- **Corollary:** Universal limit theorem for Markov chain approximations to stochastic differential equations.

- Sketch of proof:
- Random walk can be viewed as random walk (ξ_i) on the step-2 free nilpotent group .

- Sketch of proof:

- Random walk can be viewed as random walk (ξ_i) on the step-2 free nilpotent group .
- Convergence of f.d.d. follows from a CLT on free nilpotent groups.

- Sketch of proof:
- Random walk can be viewed as random walk (ξ_i) on the step-2 free nilpotent group .
- Convergence of f.d.d. follows from a CLT on free nilpotent groups.
- Remains to establish tightness in α-Hölder rough path topology:

- Sketch of proof:
- Random walk can be viewed as random walk (ξ_i) on the step-2 free nilpotent group .
- Convergence of f.d.d. follows from a CLT on free nilpotent groups.
- Remains to establish tightness in α-Hölder rough path topology:
- Boils down to showing

$$\forall p < \infty : \mathbb{E}\left[\|\xi_1 * \cdots * \xi_k\|_{CC}^{4p}\right] = O\left(k^{2p}\right).$$

References

- If you want to read more about this:
- **[Lyons, Qian ’02]:** System control and rough paths, Oxford Univ. Press

References

- If you want to read more about this:
- **[Lyons, Qian ’02]:** System control and rough paths, Oxford Univ. Press
- **[Caruana, Levy, Lyons ’05]:** Differential equations driven by rough paths, St. Flours lecture

References

- If you want to read more about this:
- **[Lyons, Qian '02]:** System control and rough paths, Oxford Univ. Press
- **[Caruana, Levy, Lyons '05]:** Differential equations driven by rough paths, St. Flours lecture
- ... google "st_flour" and find Lyons' handwritten St. Flour notes

References

- If you want to read more about this:
- **[Lyons, Qian '02]:** System control and rough paths, Oxford Univ. Press
- **[Caruana, Levy, Lyons '05]:** Differential equations driven by rough paths, St. Flours lecture
- ... google "st_flour" and find Lyons' handwritten St. Flour notes
- **[Friz, Victoir '10]:** Multidimensional stochastic processes as rough paths, Cambridge Univ. Press

References

- If you want to read more about this:
- **[Lyons, Qian '02]:** System control and rough paths, Oxford Univ. Press
- **[Caruana, Levy, Lyons '05]:** Differential equations driven by rough paths, St. Flours lecture
- ... google "st_flour" and find Lyons' handwritten St. Flour notes
- **[Friz, Victoir '10]:** Multidimensional stochastic processes as rough paths, Cambridge Univ. Press
- ... download the pdf from my homepage