Knots, Maps, and Tiles

Three Mathematical Visualization Puzzles

Jack van Wijk

TU Eindhoven

BMS colloquium, December 4, Berlin


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useful techniques

big challenges

Overview

- Knots

  What does a surface bounded by a knot look like?

- Maps

  How to map the earth without distortion?

- Tiles

  How to tile a closed surface symmetrically?

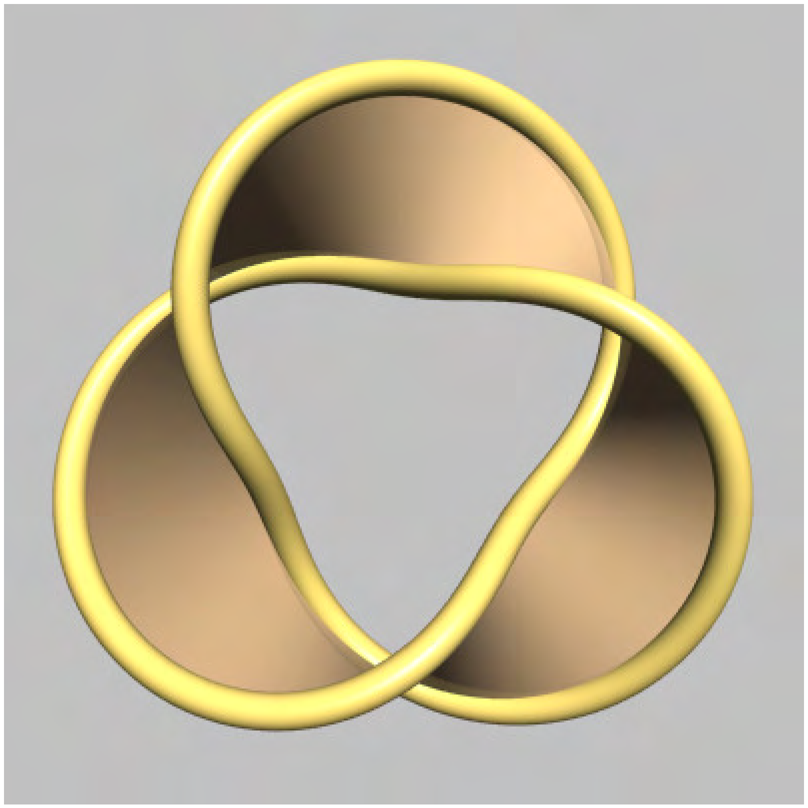
Knots

J.J. van Wijk & A.M. Cohen, Visualization of the Genus of Knots. IEEE Visualization 2005.
J.J. van Wijk & A.M. Cohen, Visualization of Seifert Surfaces . IEEE TVCG 12(4), p. 485-496, 2006.

Puzzle 0

- Given a trefoil, find a surface that is bounded by this knot.

Puzzle 0 – solution



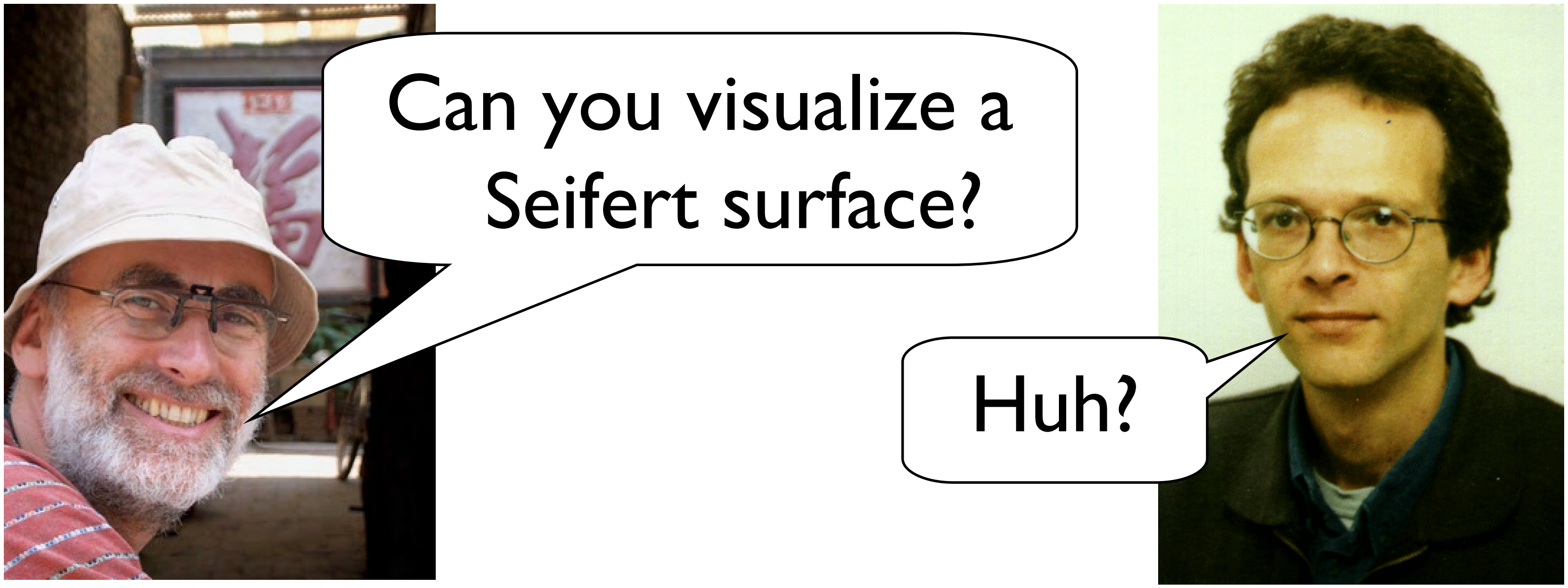
- Given a trefoil, find a surface that is bounded by this knot.
- Möbius strip. Not orientable!

Puzzle I

- Given a trefoil, find an *orientable* surface that is bounded by this knot.

15 minutes

How this started



Arjeh Cohen

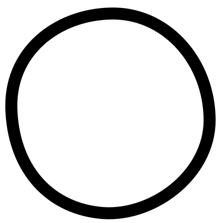
discrete algebra and geometry

Jack van Wijk

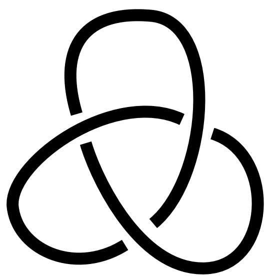
visualization

Mathematical knots and links

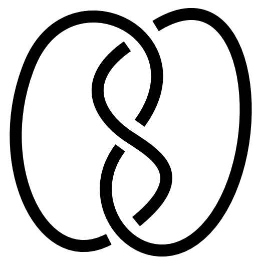
Knots and links: closed curves



unknot



trefoil



trefoil

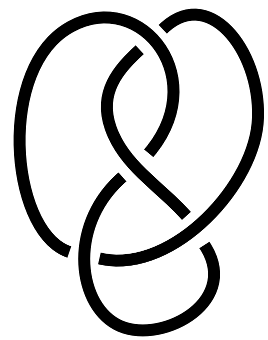
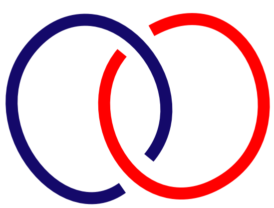
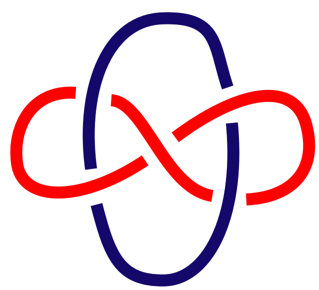


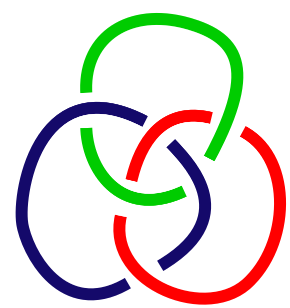
figure 8



Hopf link



Whitehead link



Borromean rings

Knot theory

Typical questions:

- *Are two knots identical?*
- *Is a knot equal to the unknot?*
- *How many different knots do exist?*

Categorize knots via *invariants*:

- *Minimal #crossings*
- *Polynomials*
- *Genus*

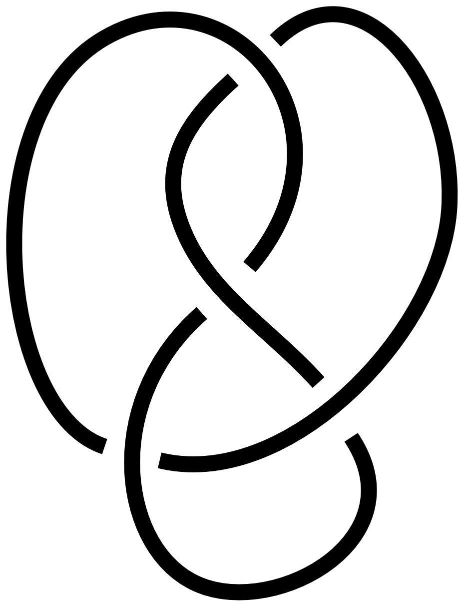
Genus of knot

- Genus ≅ number of holes in object
- Sphere: genus 0; torus: genus 1
- Closed curve: always genus 0

Genus of knot: *Minimal genus of orientable surface that is bounded by the knot*

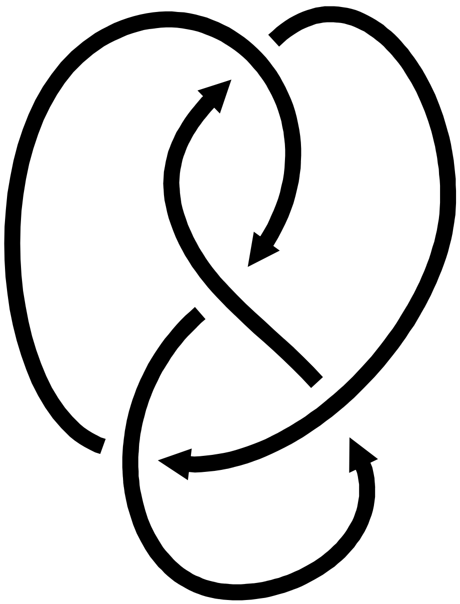
Orientable surface: Seifert's algorithm (1934)

Seifert's algorithm

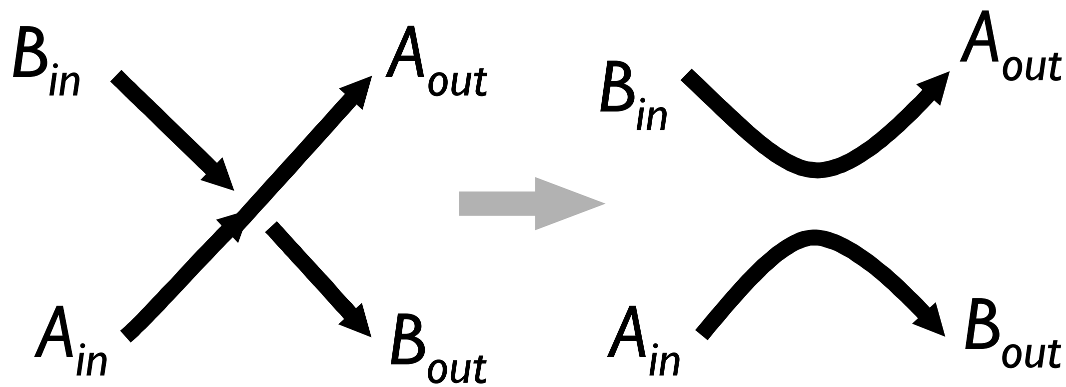


Orient knot...

Seifert's algorithm



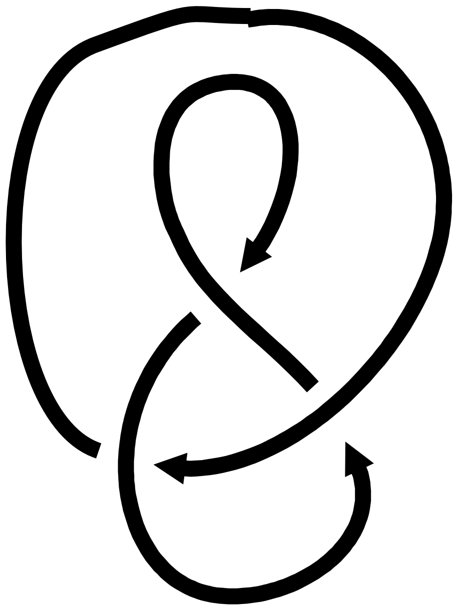
Oriented knot



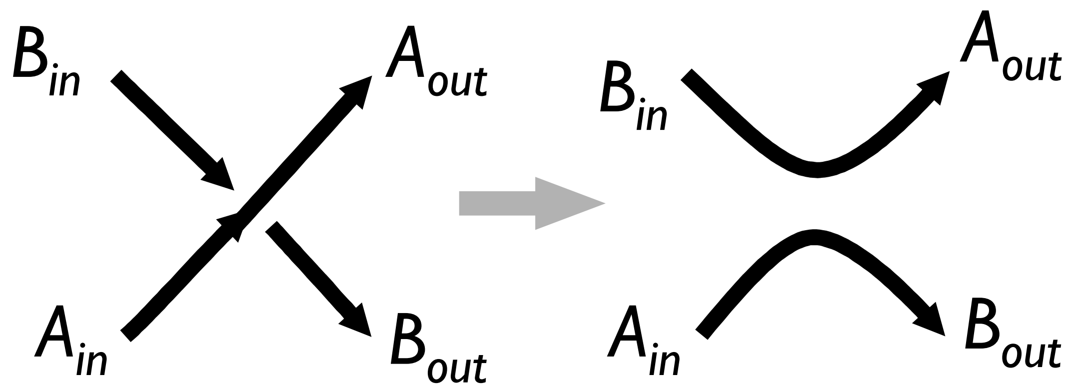
Eliminate crossings:

Connect incoming strand with outcoming

Seifert's algorithm



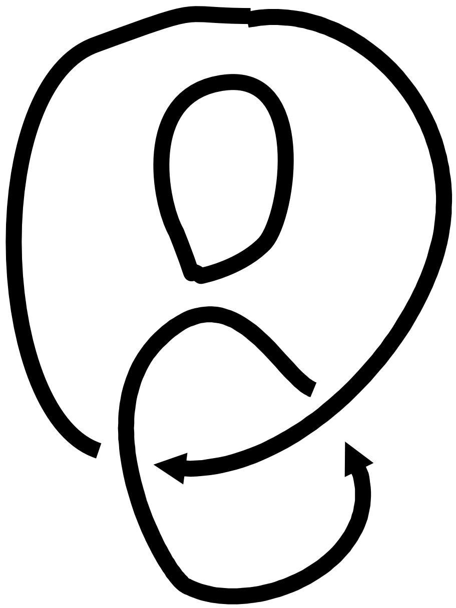
I removed...



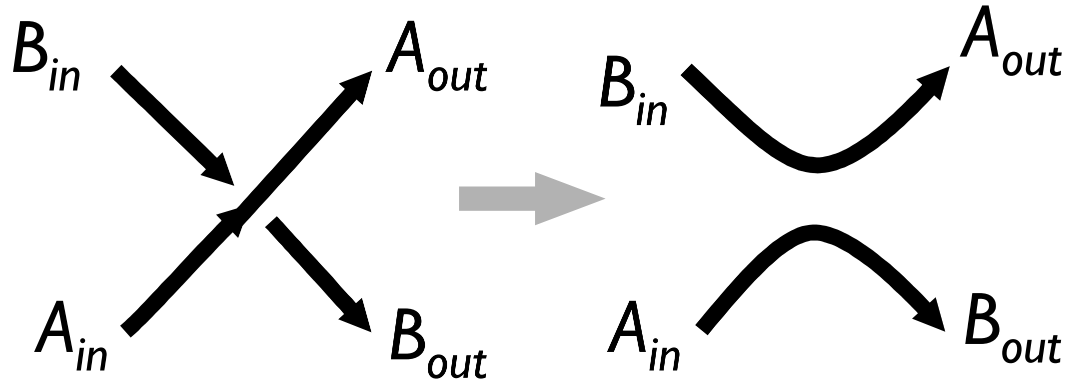
Eliminate crossings:

Connect incoming strand with outcoming

Seifert's algorithm



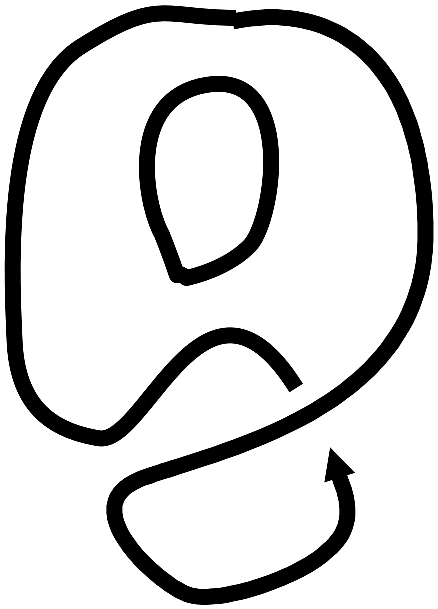
2 removed…



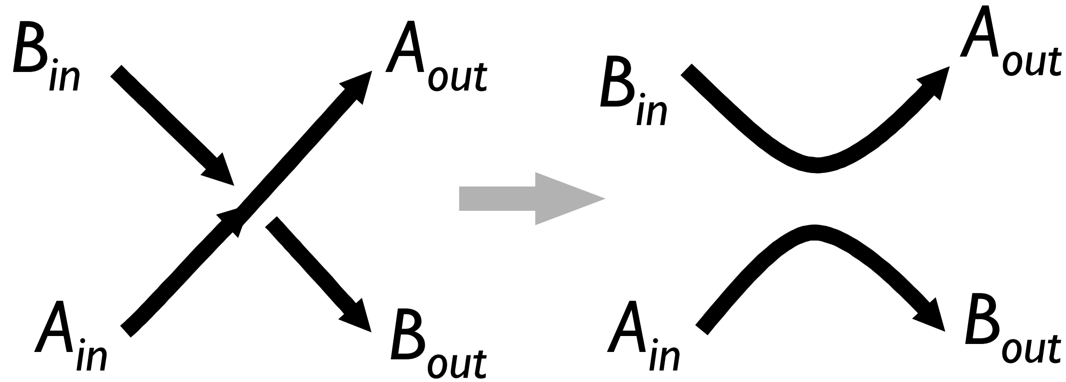
Eliminate crossings:

Connect incoming strand with outcoming

Seifert's algorithm



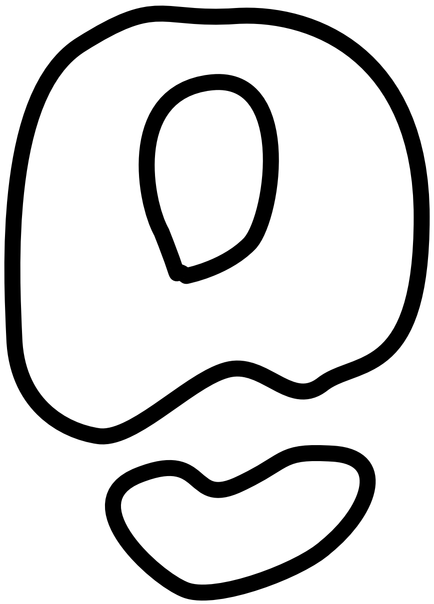
3 removed…

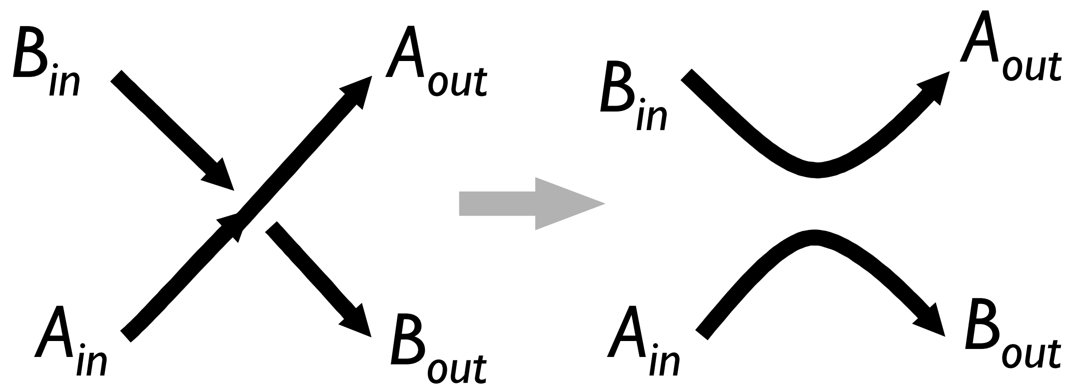


Eliminate crossings:

Connect incoming strand with outcoming

Seifert's algorithm

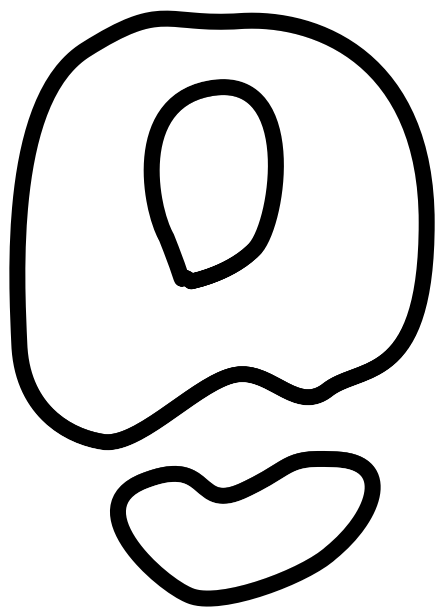


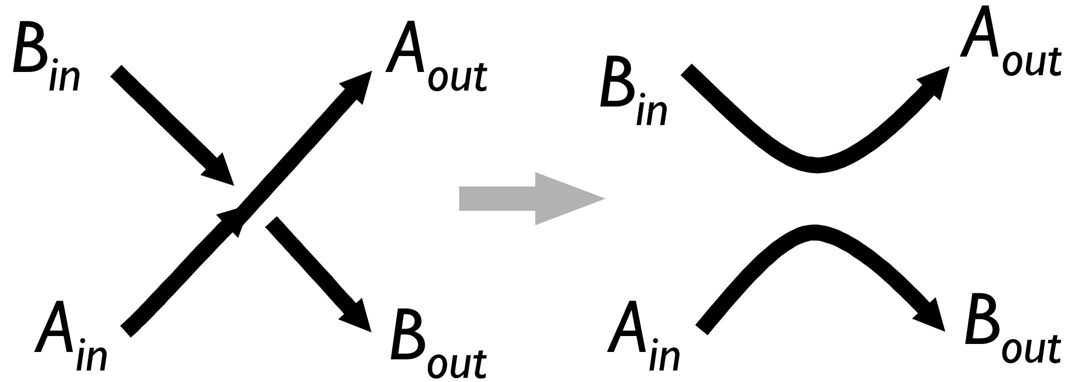


All removed:

3 (topological) circles remain

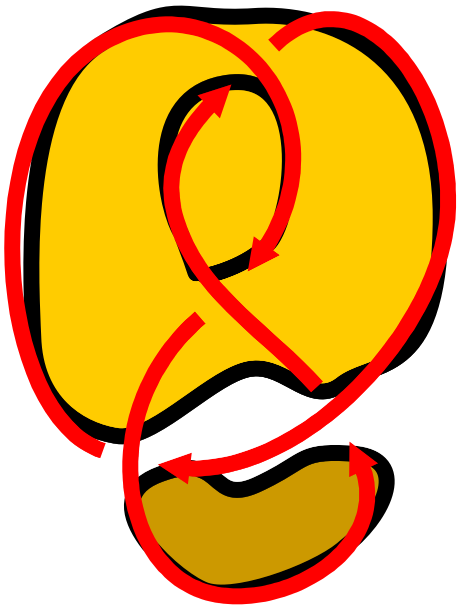
Seifert's algorithm

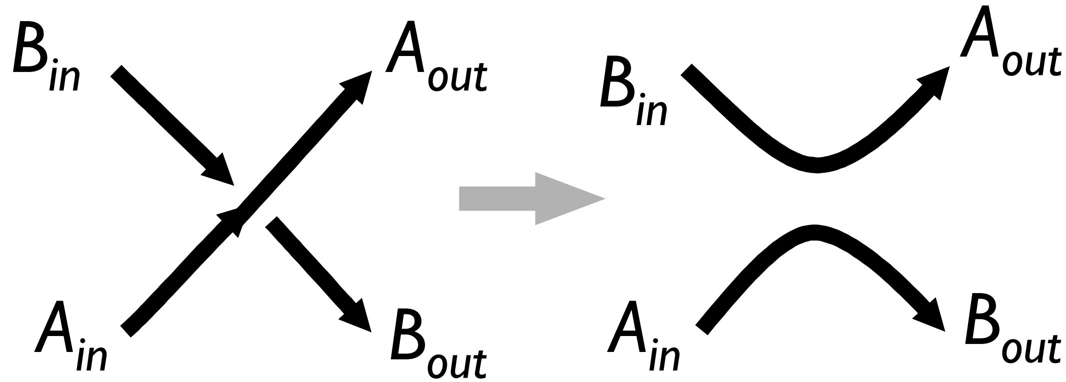




Fill in circles…

Seifert’s algorithm

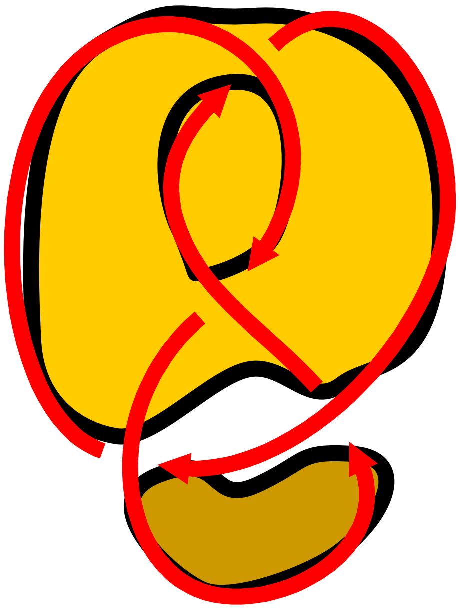


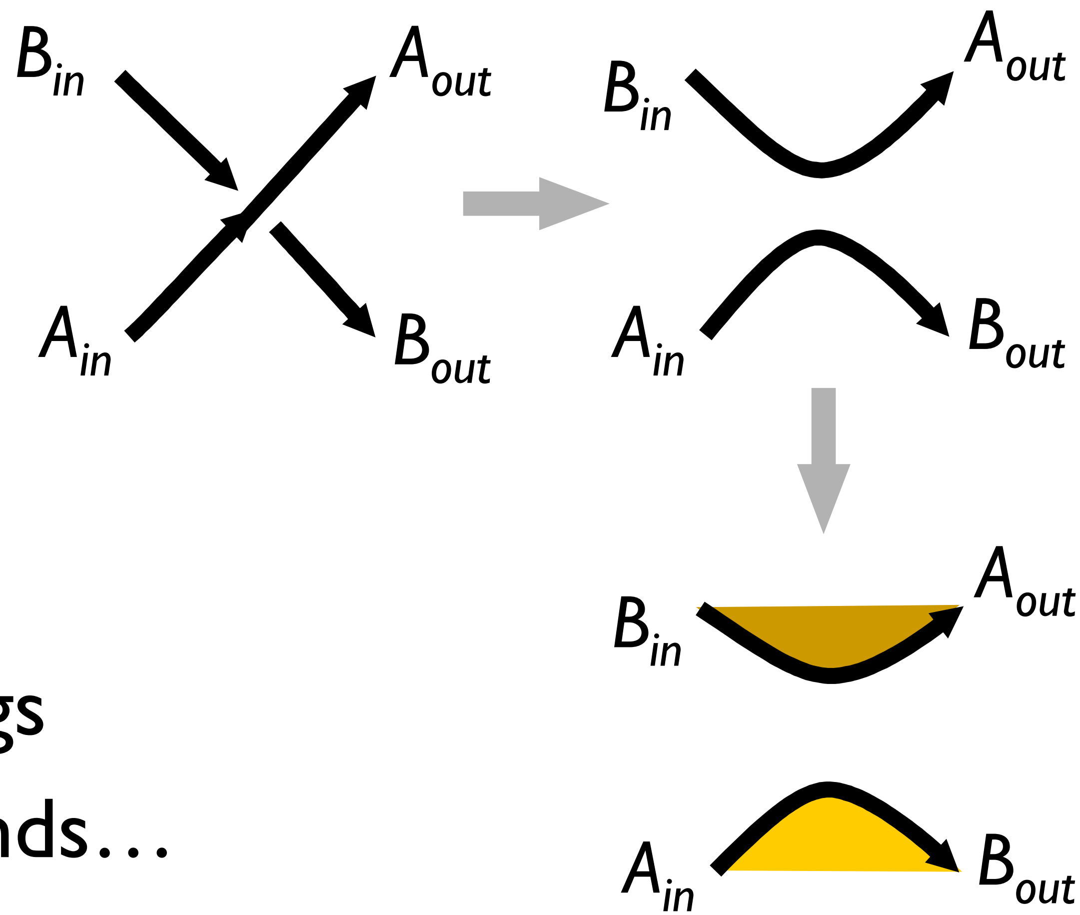


Fill in circles,

get 3 disks

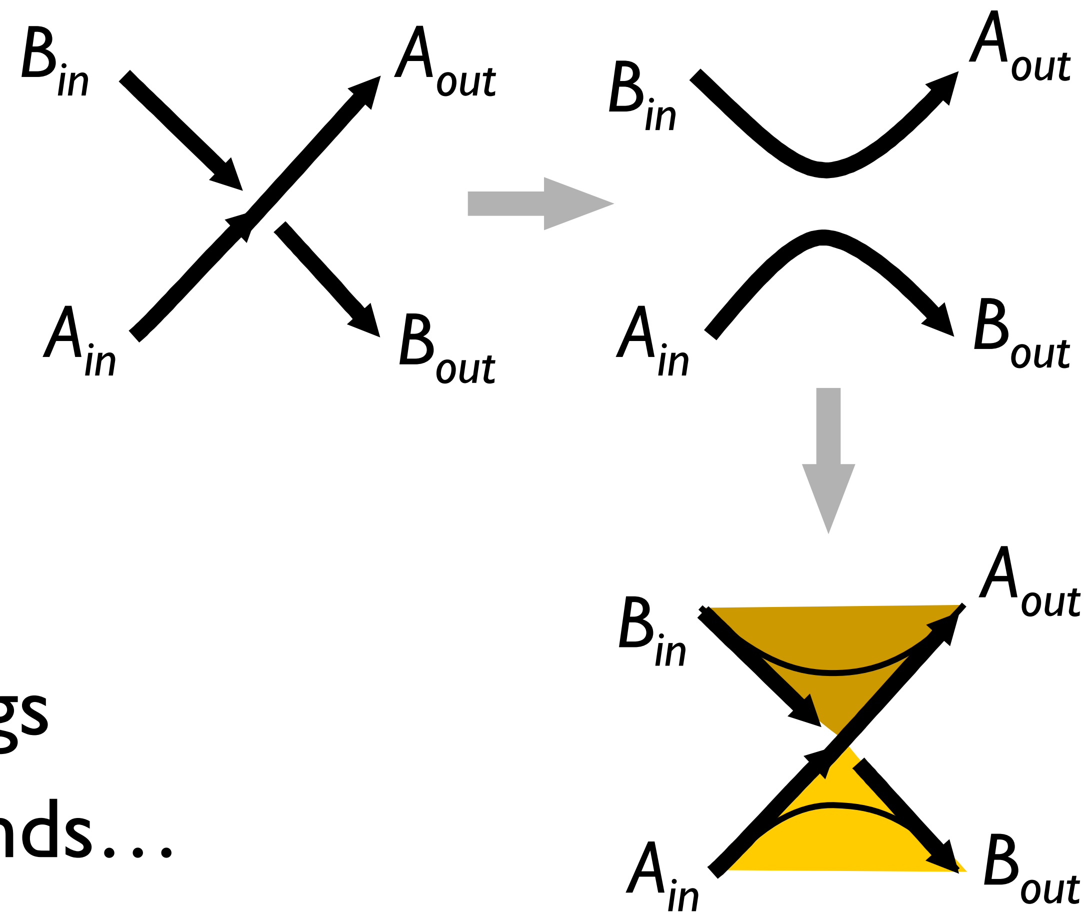
Seifert's algorithm





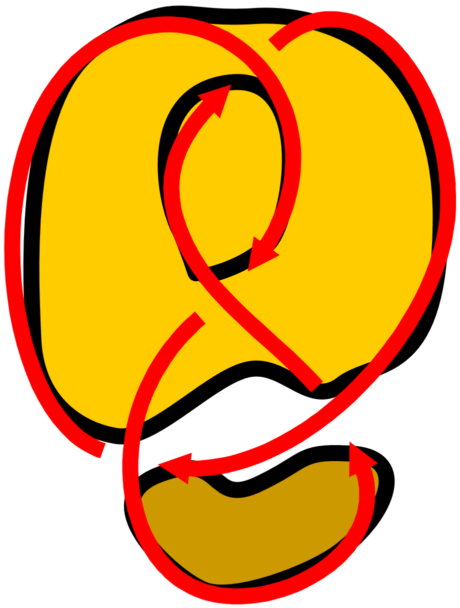
Replace crossings
with twisted bands…

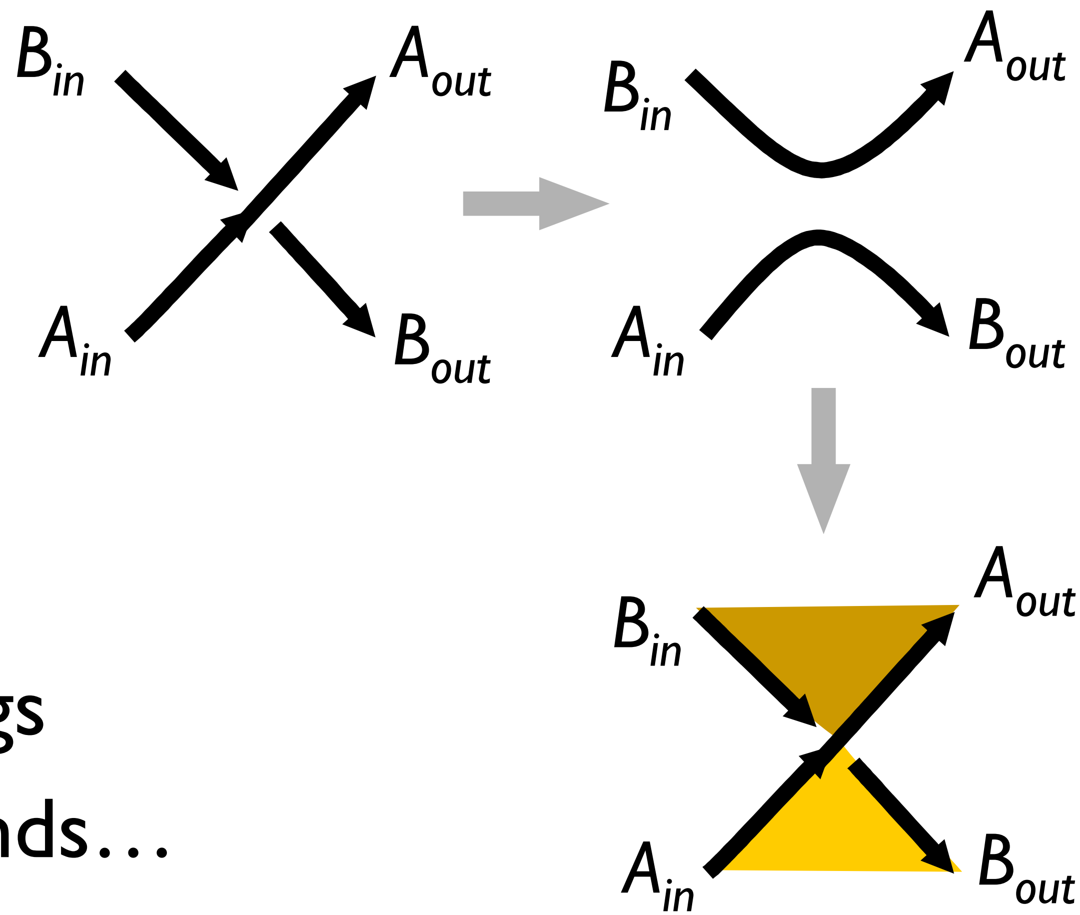
Seifert's algorithm



Replace crossings
with twisted bands…

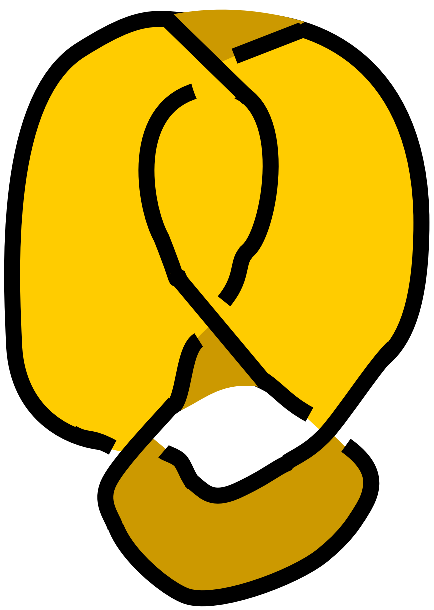
Seifert’s algorithm

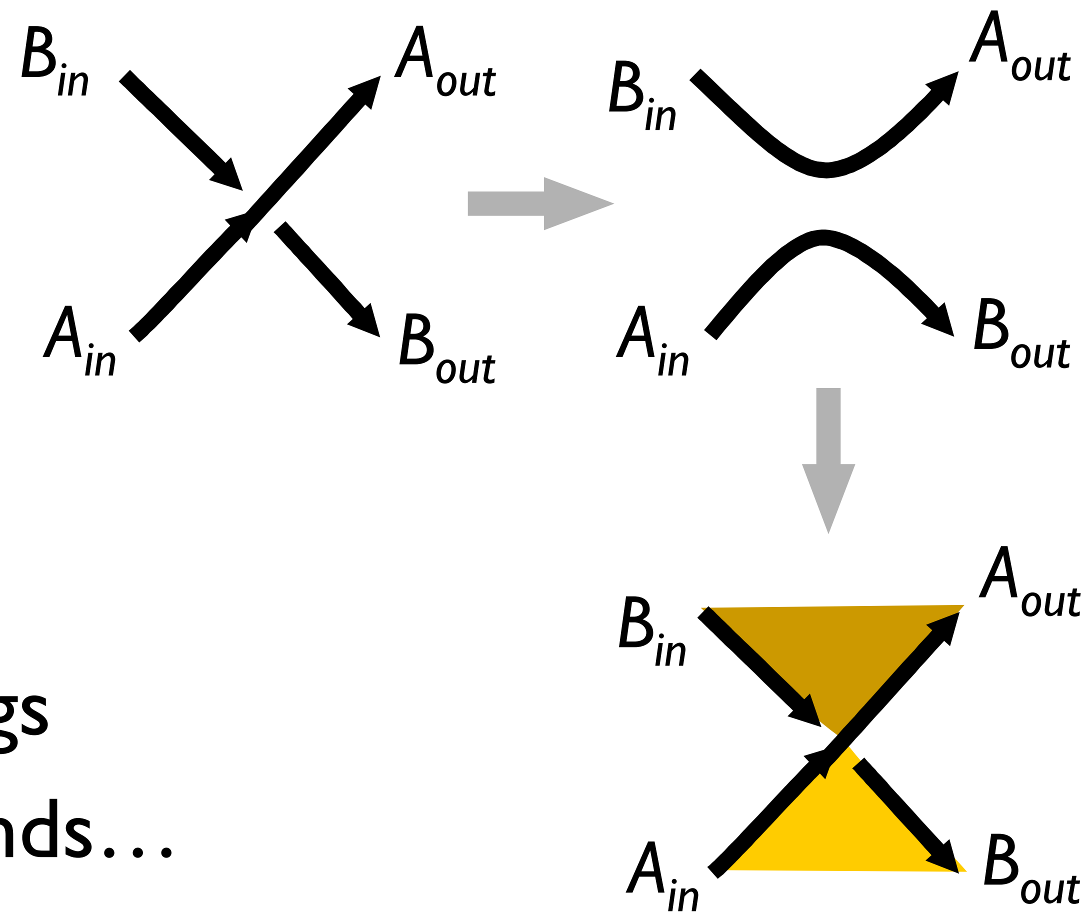




Replace crossings with twisted bands…

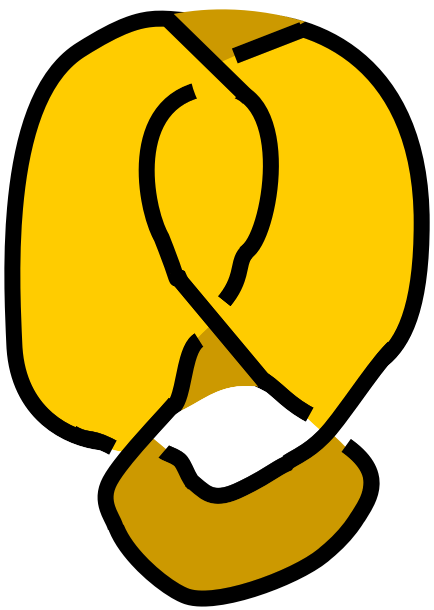
Seifert's algorithm

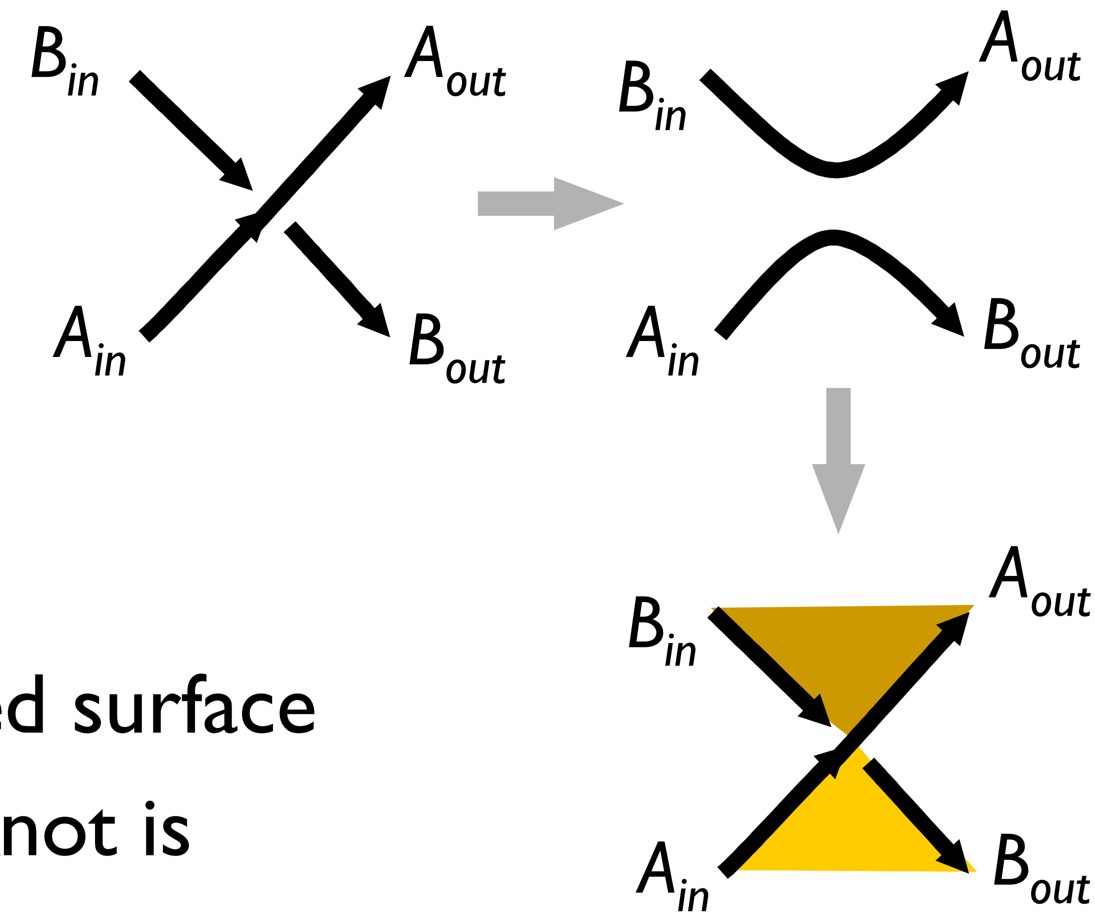




Replace crossings
with twisted bands…

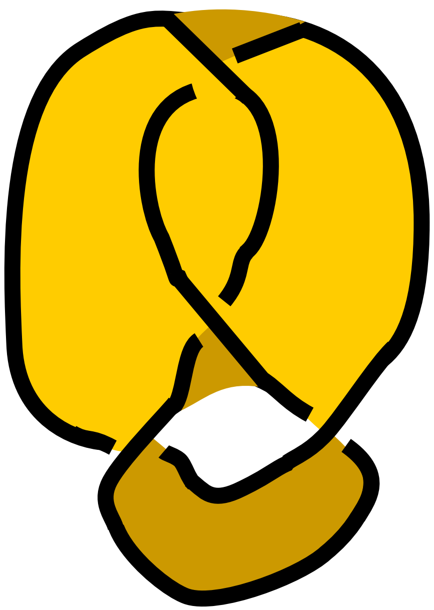
Seifert's algorithm



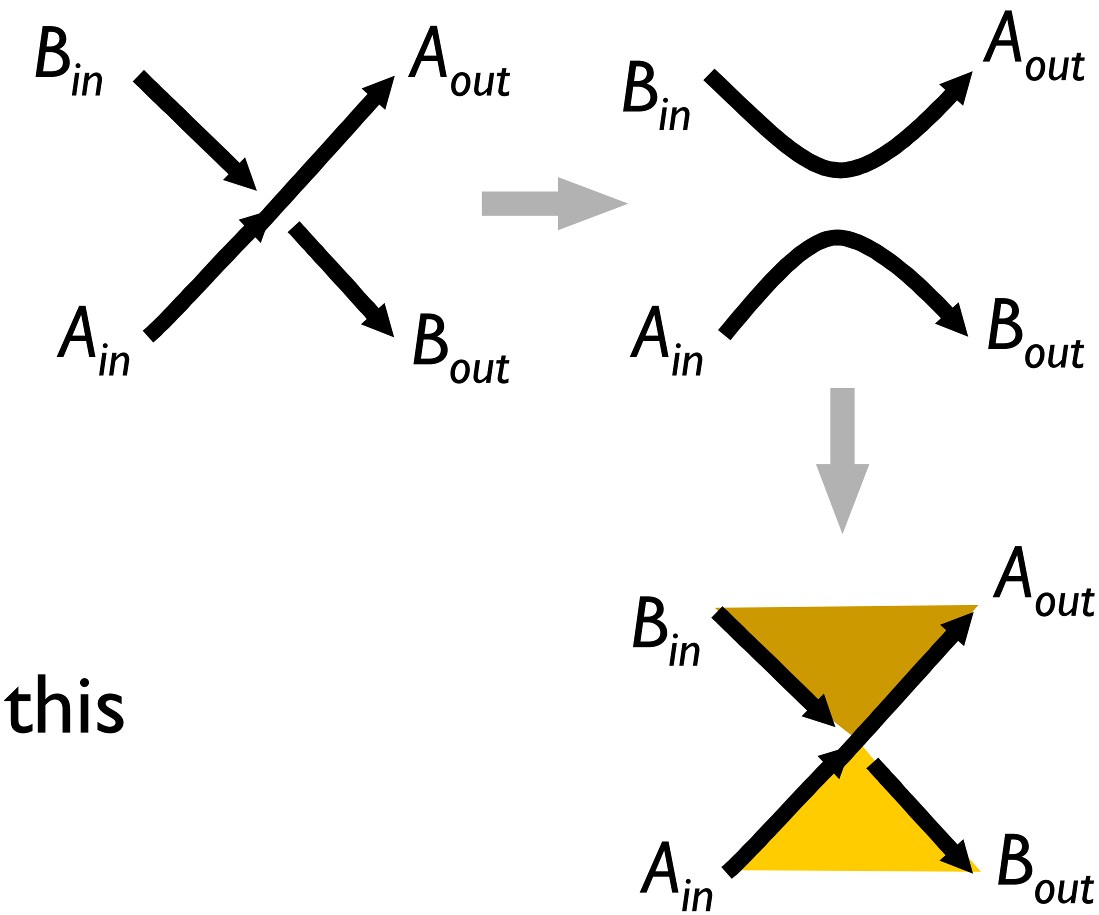


Done. An oriented surface bounded by the knot is obtained. Always works!

Seifert's algorithm



But… what does this look like?



Challenge

Can we make something, such that

- *Arbitrary knots and links can be defined;*
- *Seifert surfaces are generated;*
- *Seifert surfaces can be viewed and inspected?*

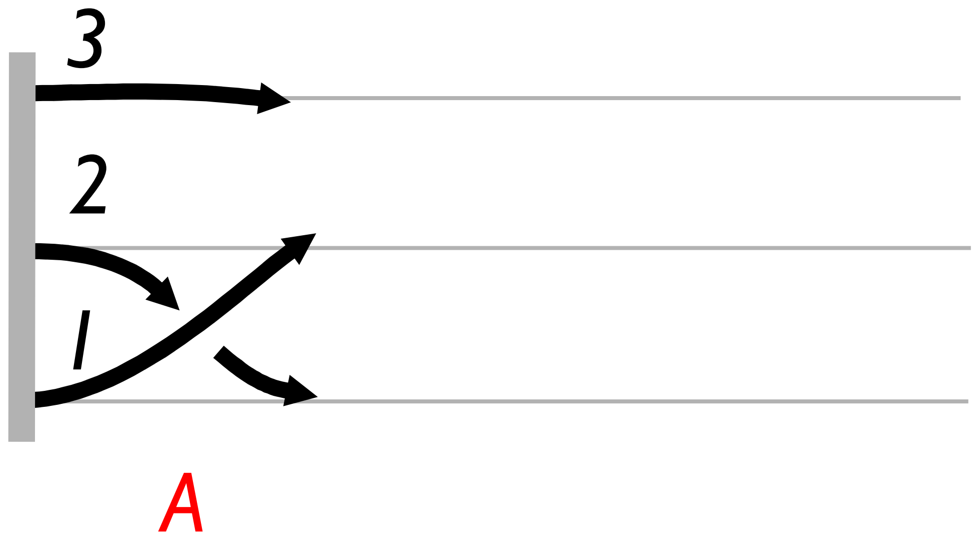
Knot notation

- 3D shape
- Gauss notation
- Conway notation
- Dowker-Thistlethwaite notation
- *Braid representation*

Braid representation

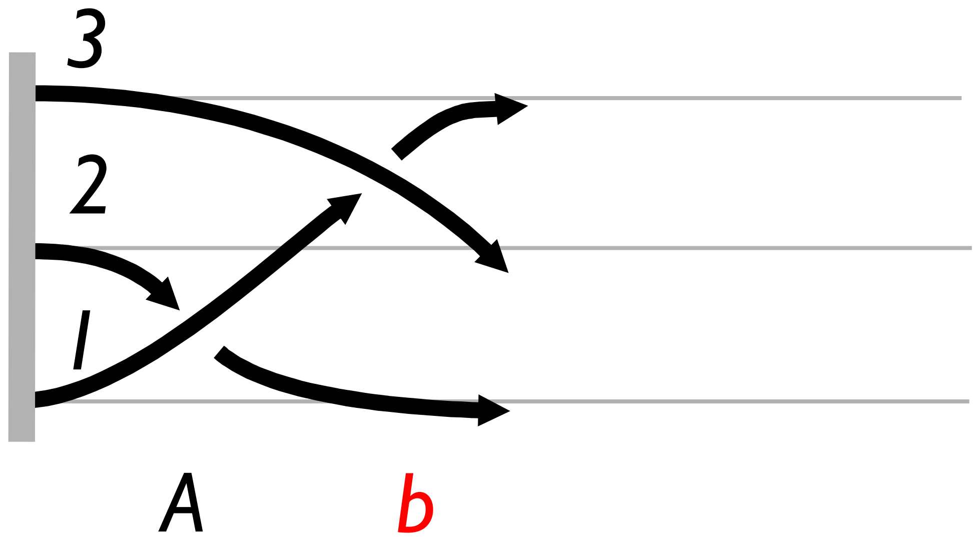
Take some strands

Braid representation



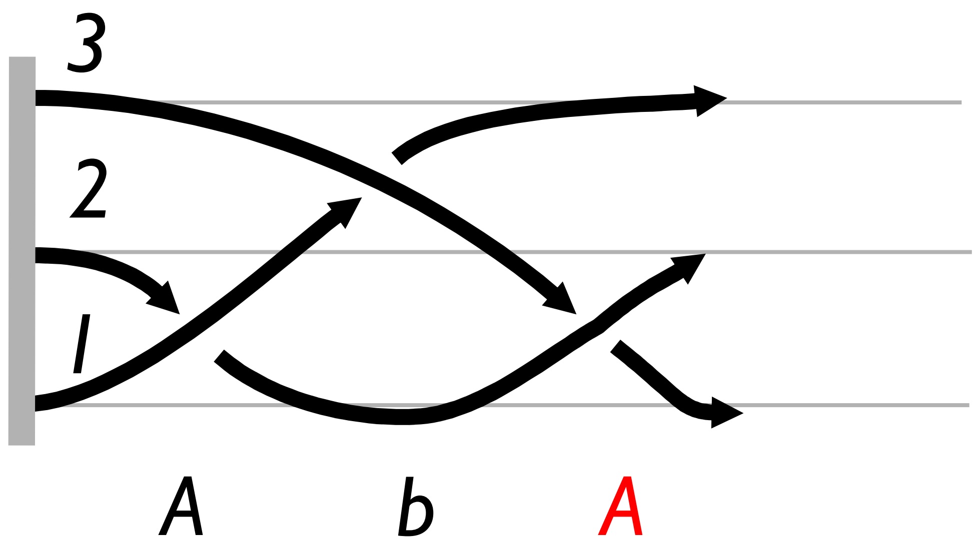
A: move 1 over 2

Braid representation



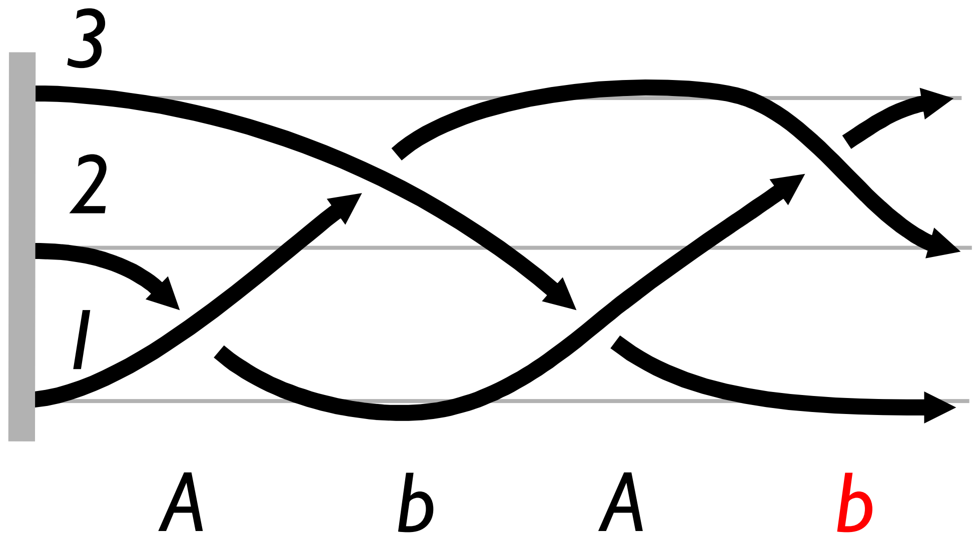
b: move 3 over 2

Braid representation



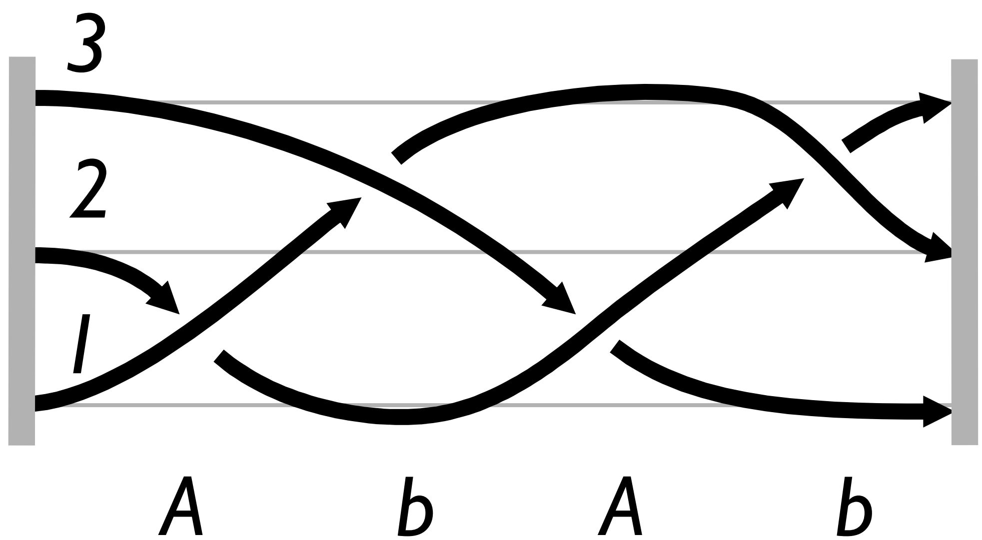
A: move 1 over 2 again

Braid representation



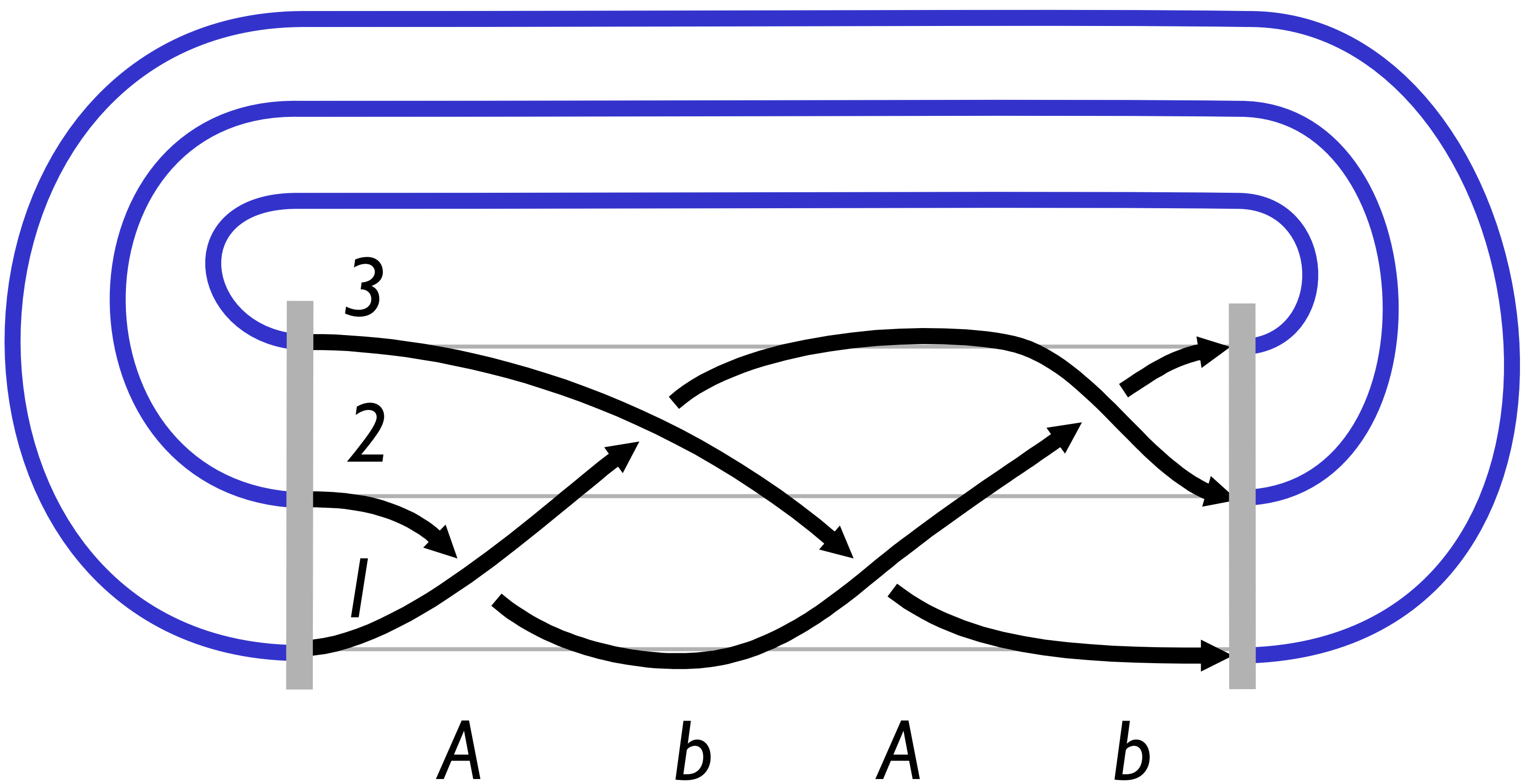
b: move 3 over 2 again

Braid representation



Braid rep: *AbAb*

Braid representation



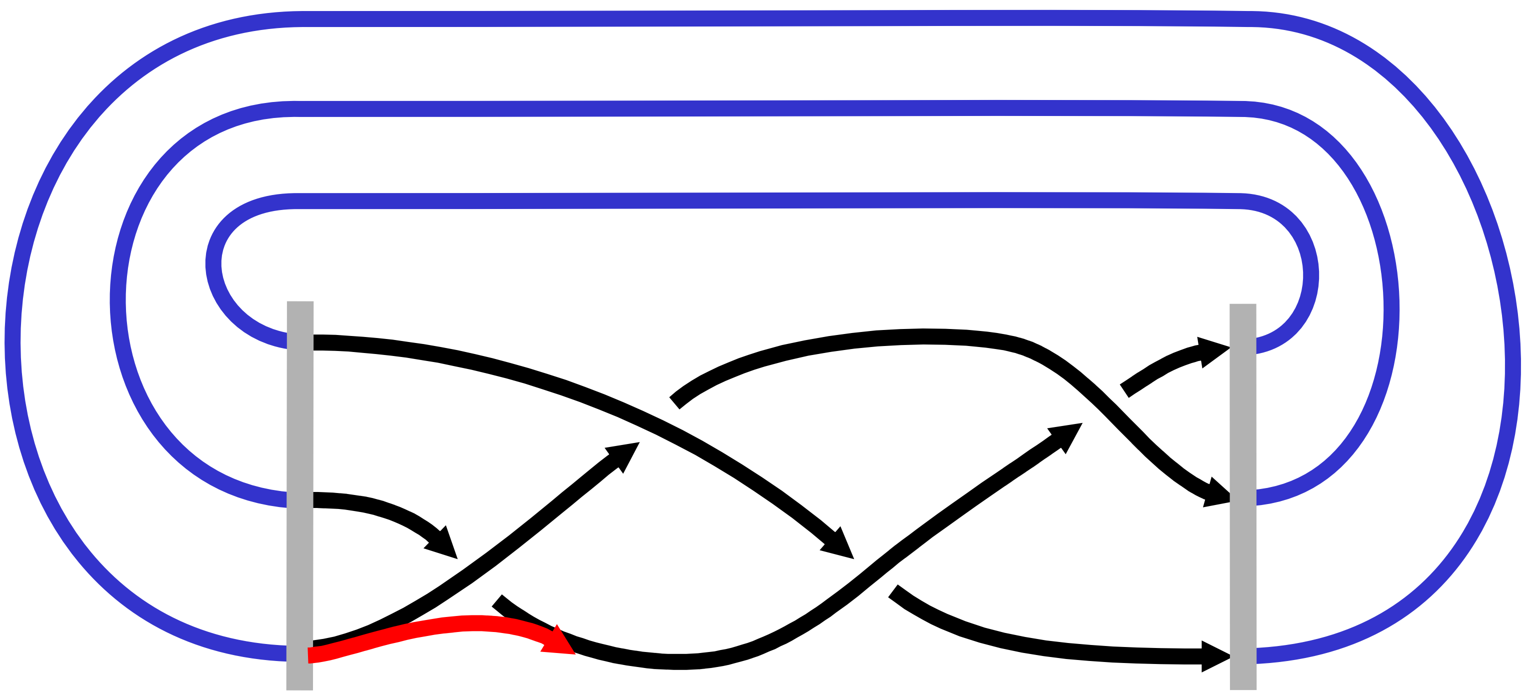
Close without further crossings

Braid representation

- Can be used to define any knot or link
  - Though not always with the minimal number of crossings

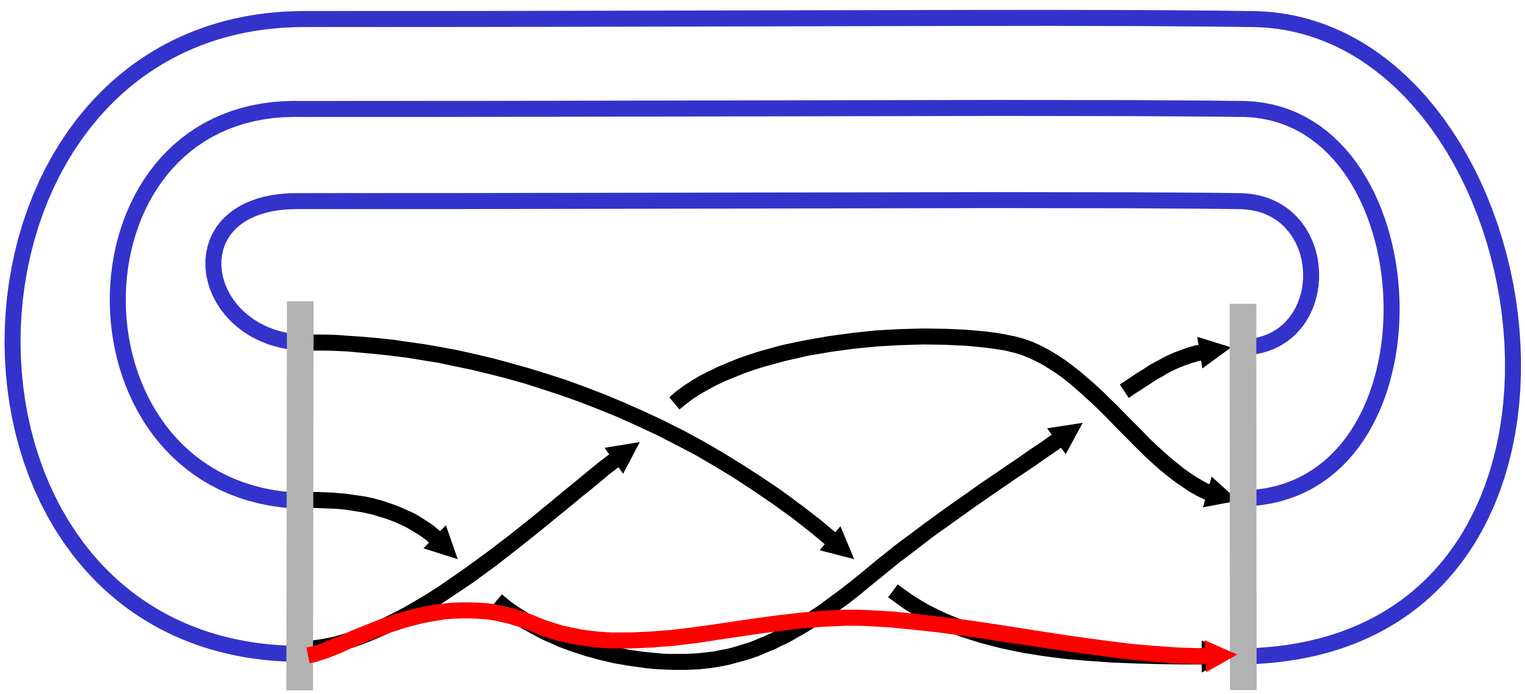
- Finding Seifert surfaces is trivial

Find disks and bands



Trace outgoing edges...

Find disks and bands



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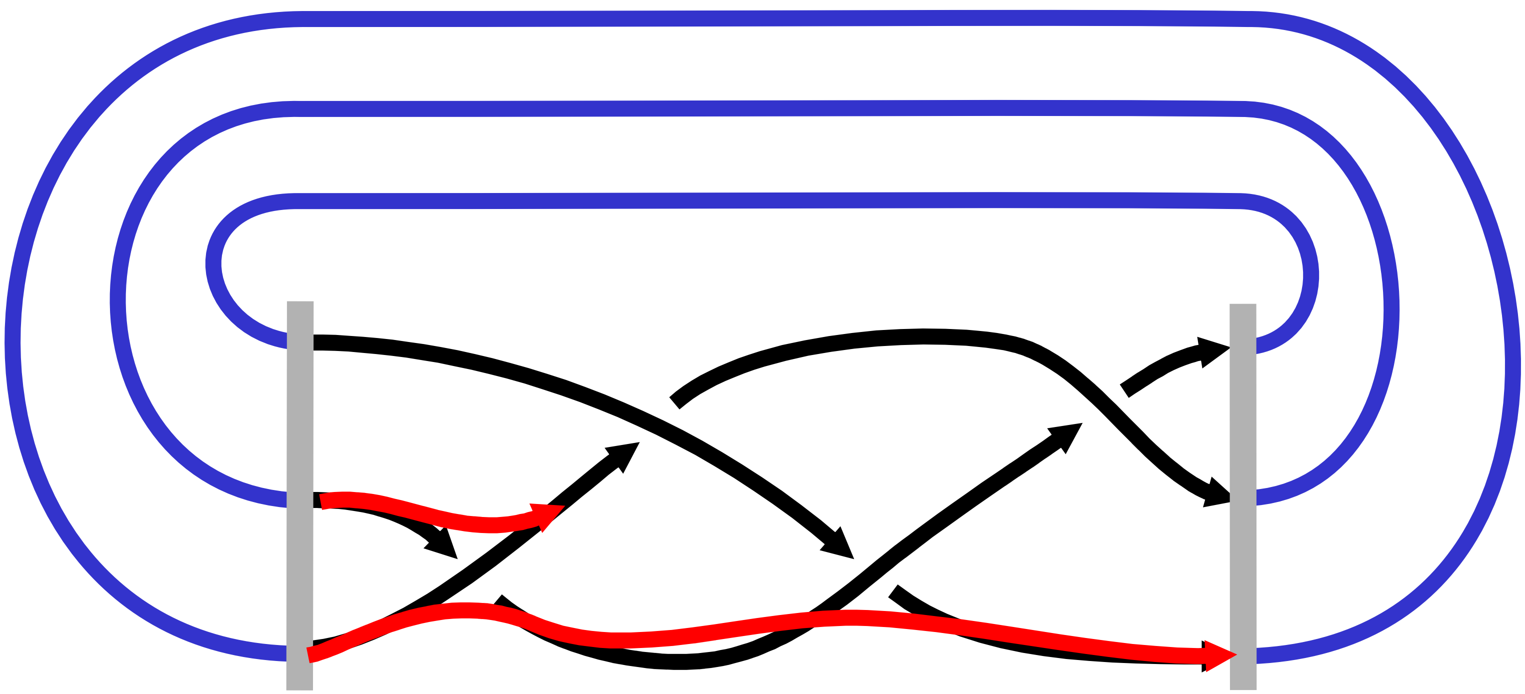
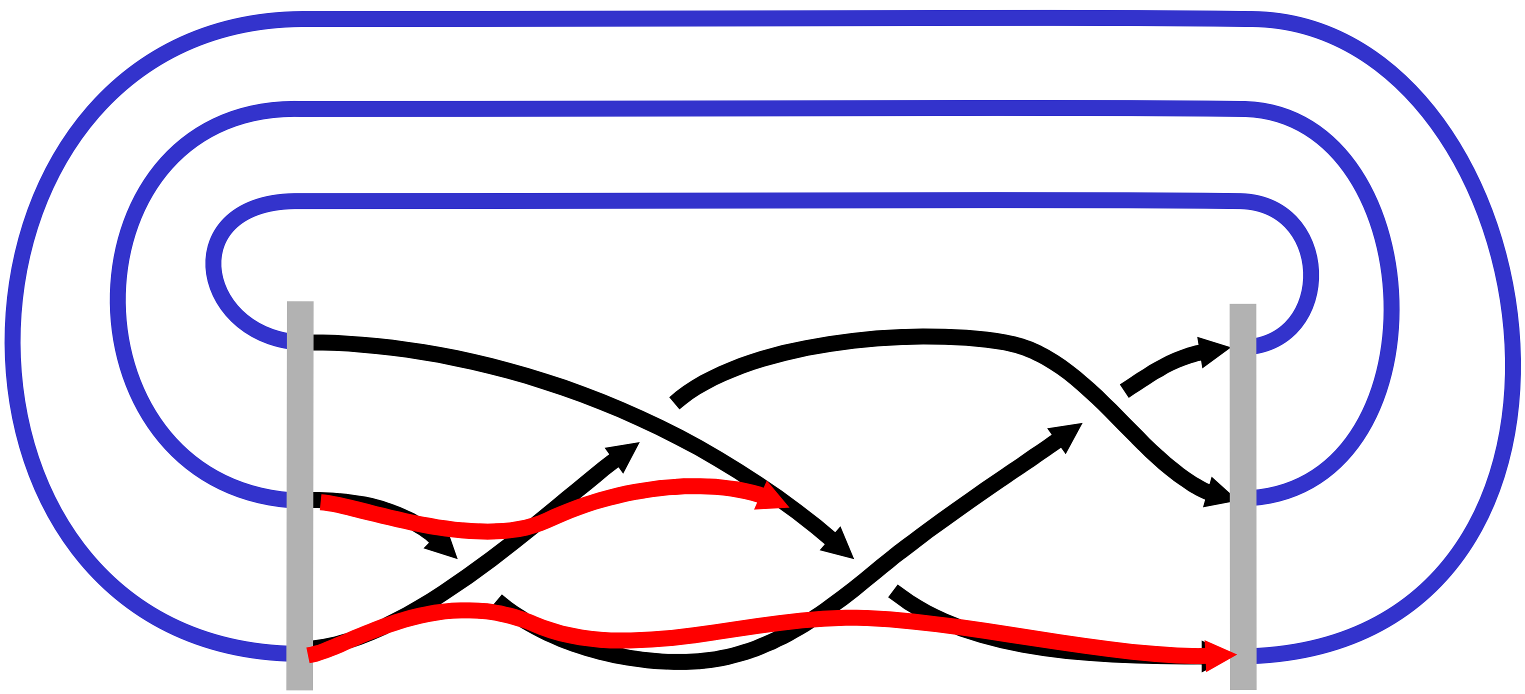


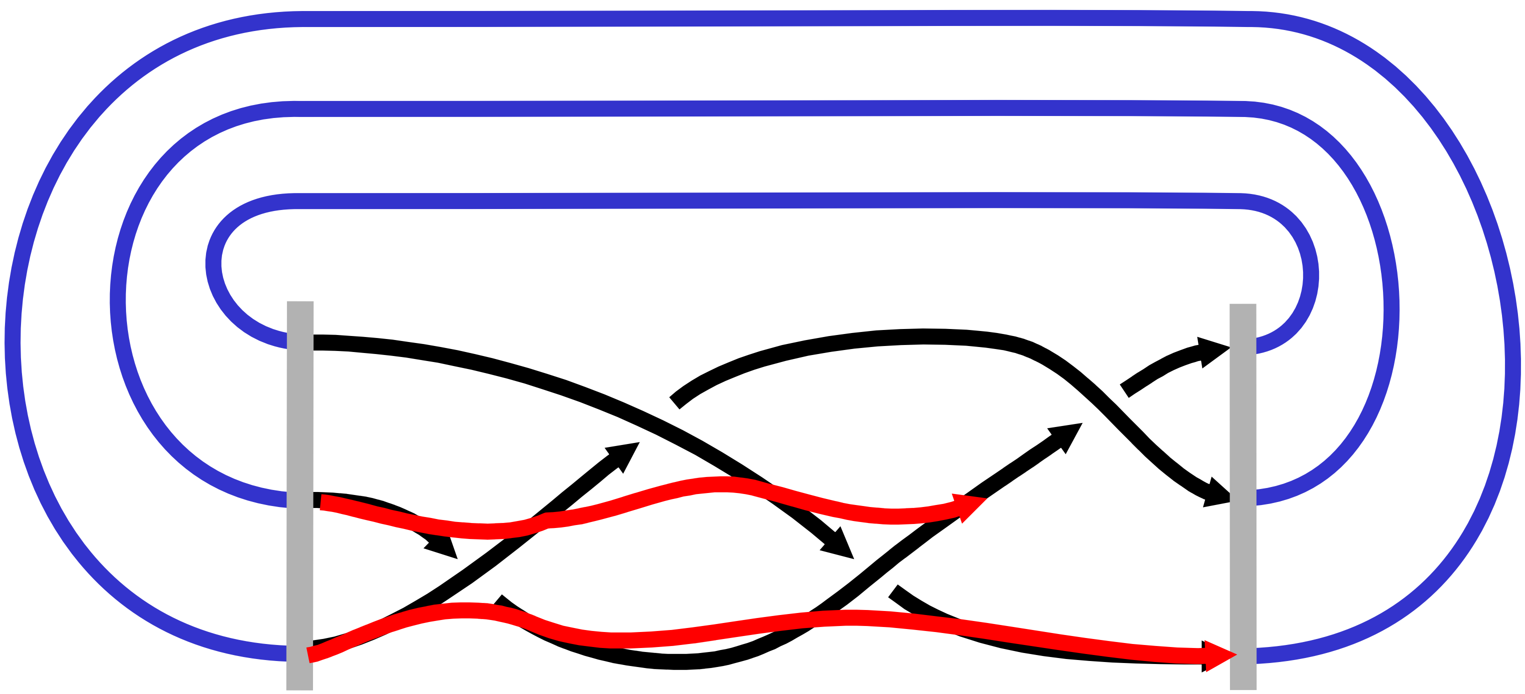
Trace outgoing edges...

Find disks and bands



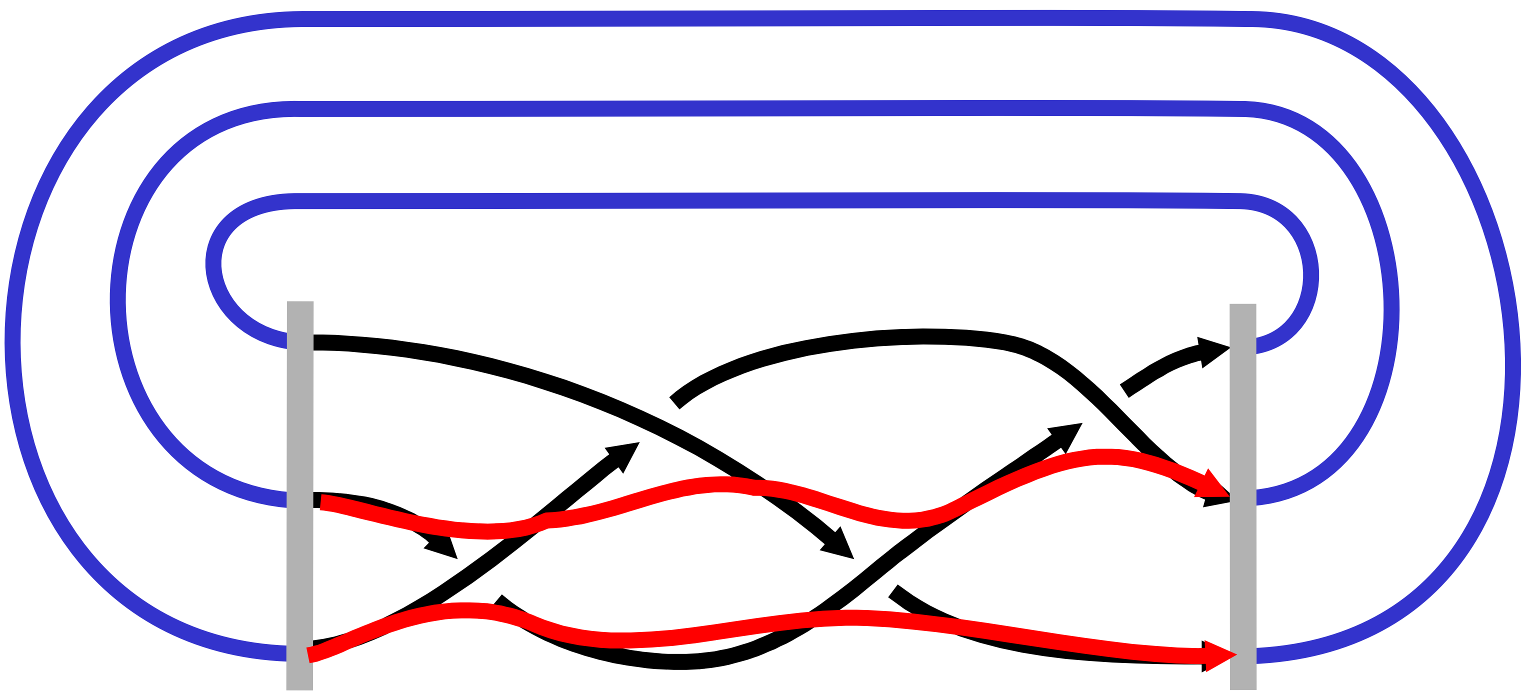
Trace outgoing edges...

Find disks and bands



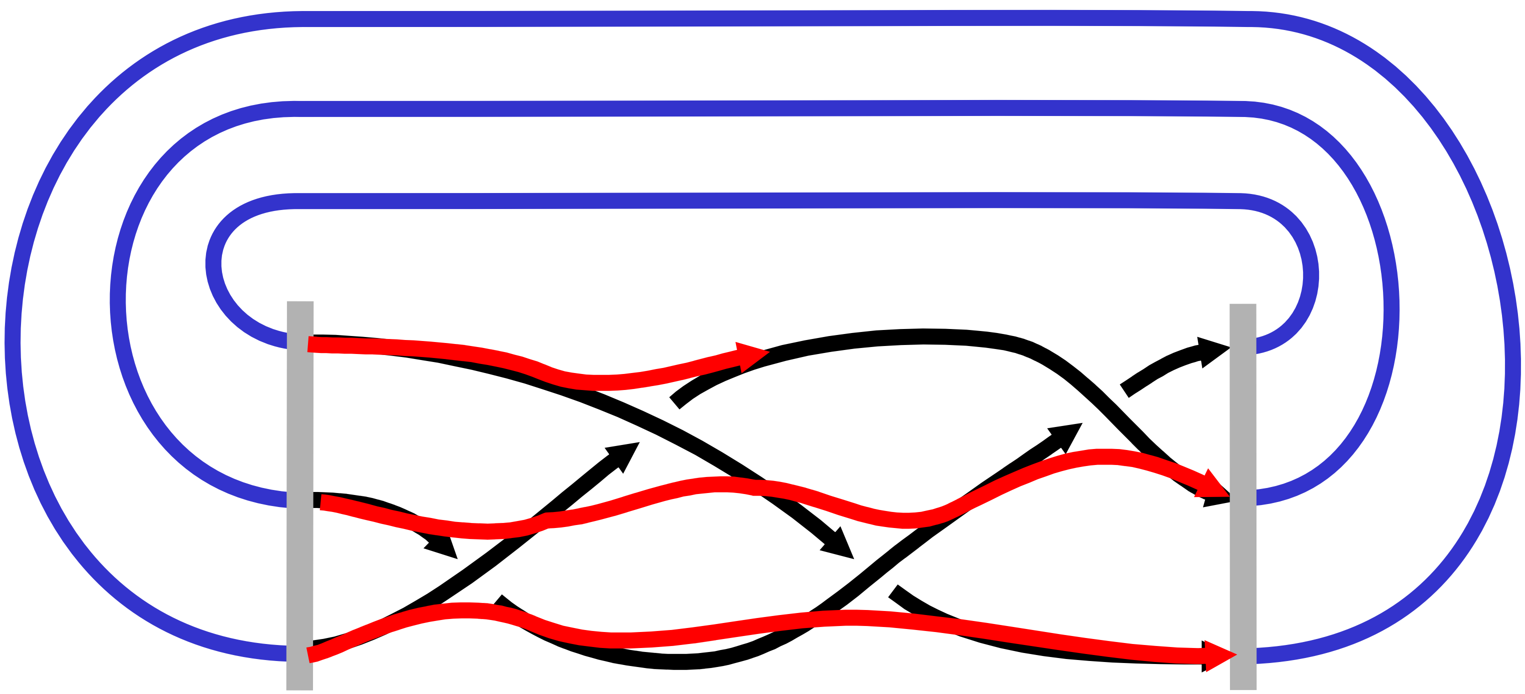
Trace outgoing edges...

Find disks and bands



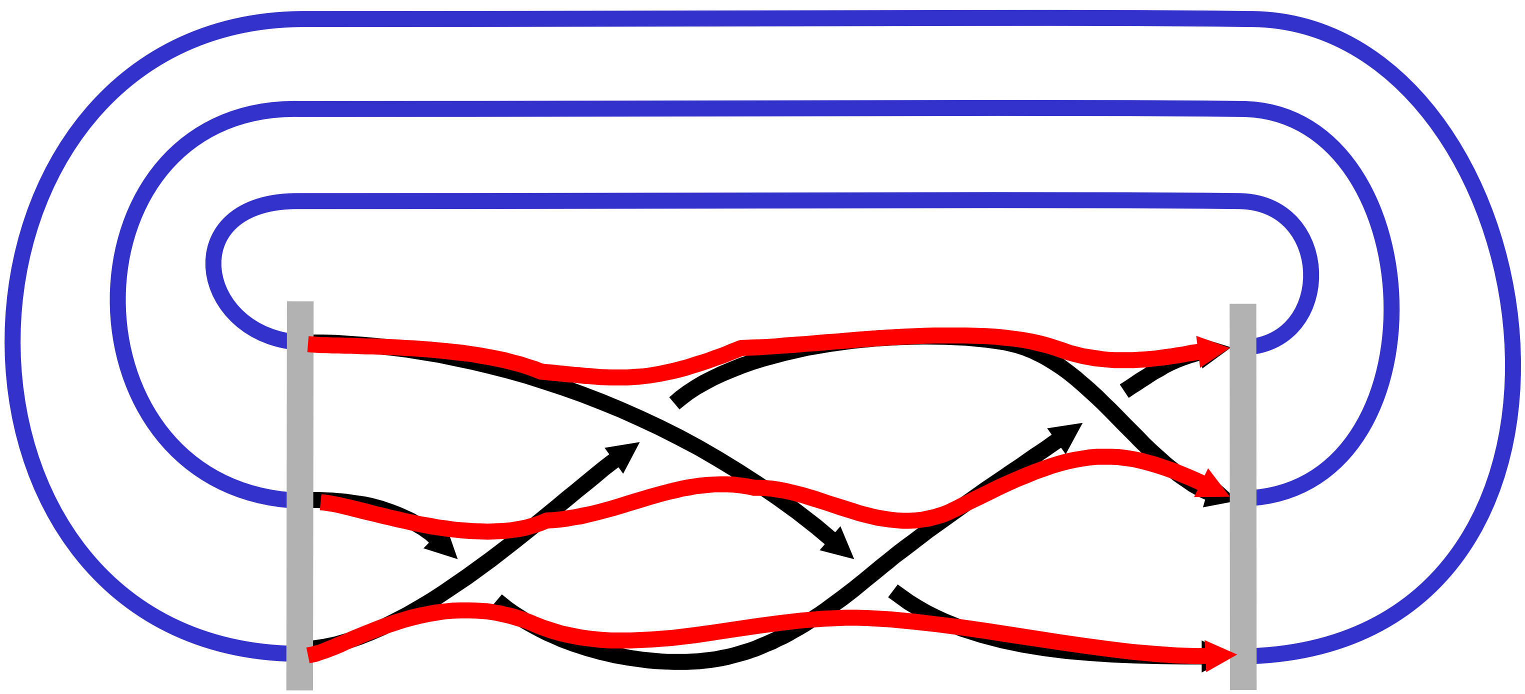
Trace outgoing edges...

Find disks and bands



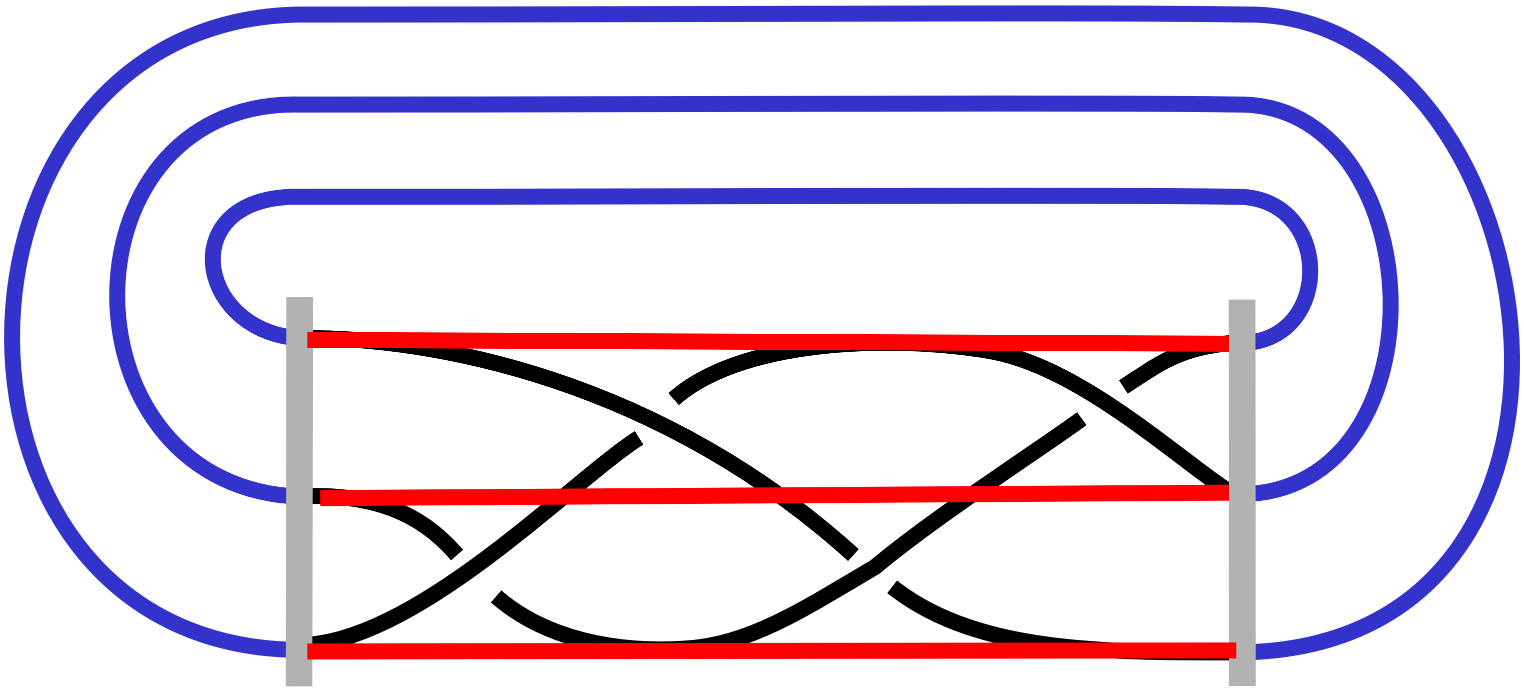
Trace outgoing edges...

Find disks and bands



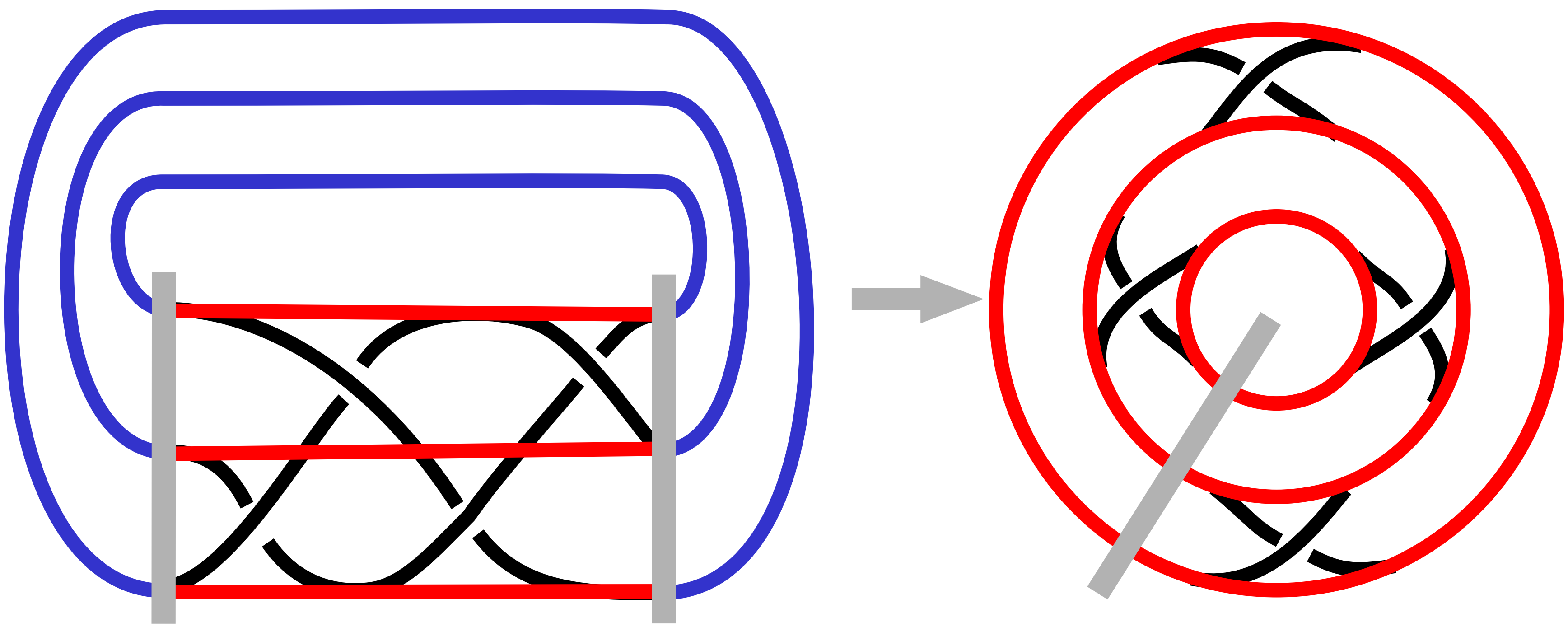
Trace outgoing edges...

Find disks and bands



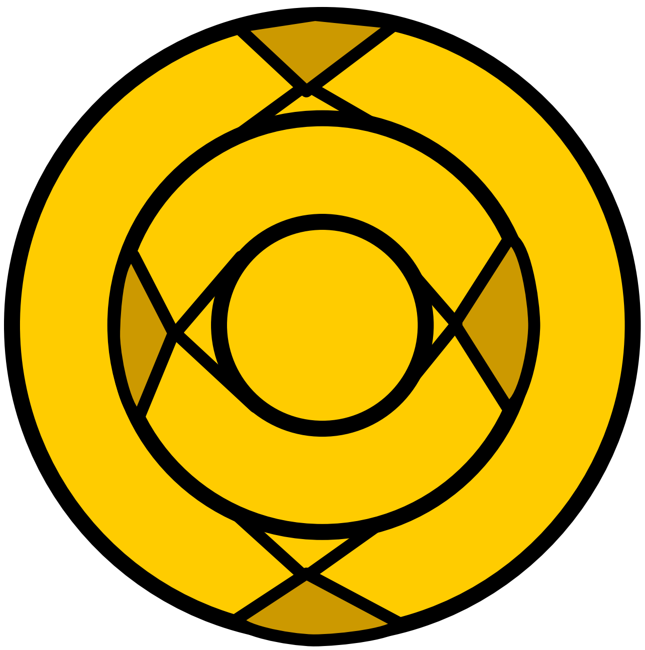
Straighten...

Find disks and bands



Distribute on circle...

Find disks and bands



- Each strand: disk
- Disks are stacked
- Bands connect disks

SeifertView

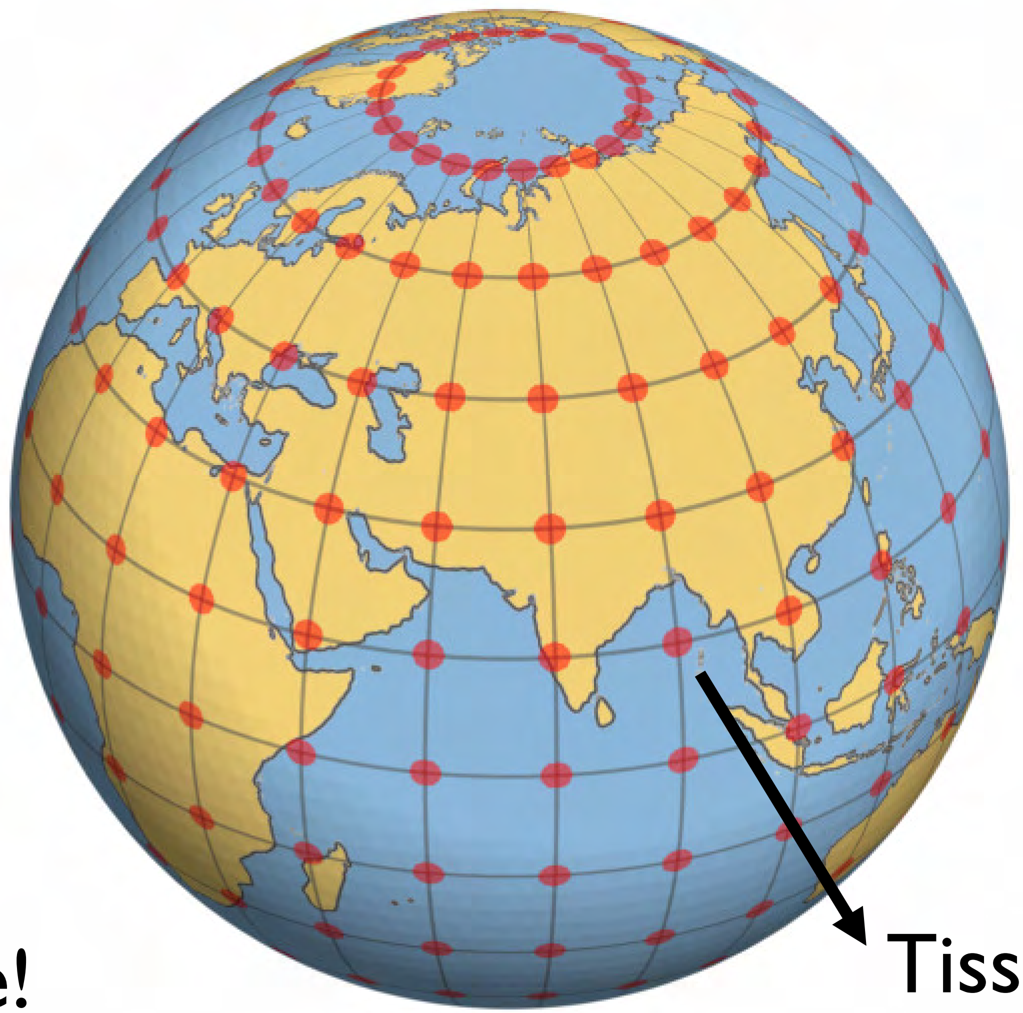
Demo

- Downloadable from www.win.tue.nl/~vanwijk/seifertview

Maps

J.J. van Wijk, Unfolding the Earth: Myriahedral Projections.
The Cartographic Journal, 45(1), p. 32-42, February 2008.

How to map the world?



Globe!

Tissot's indicatrix: ellipse of distortion

How to map the world?

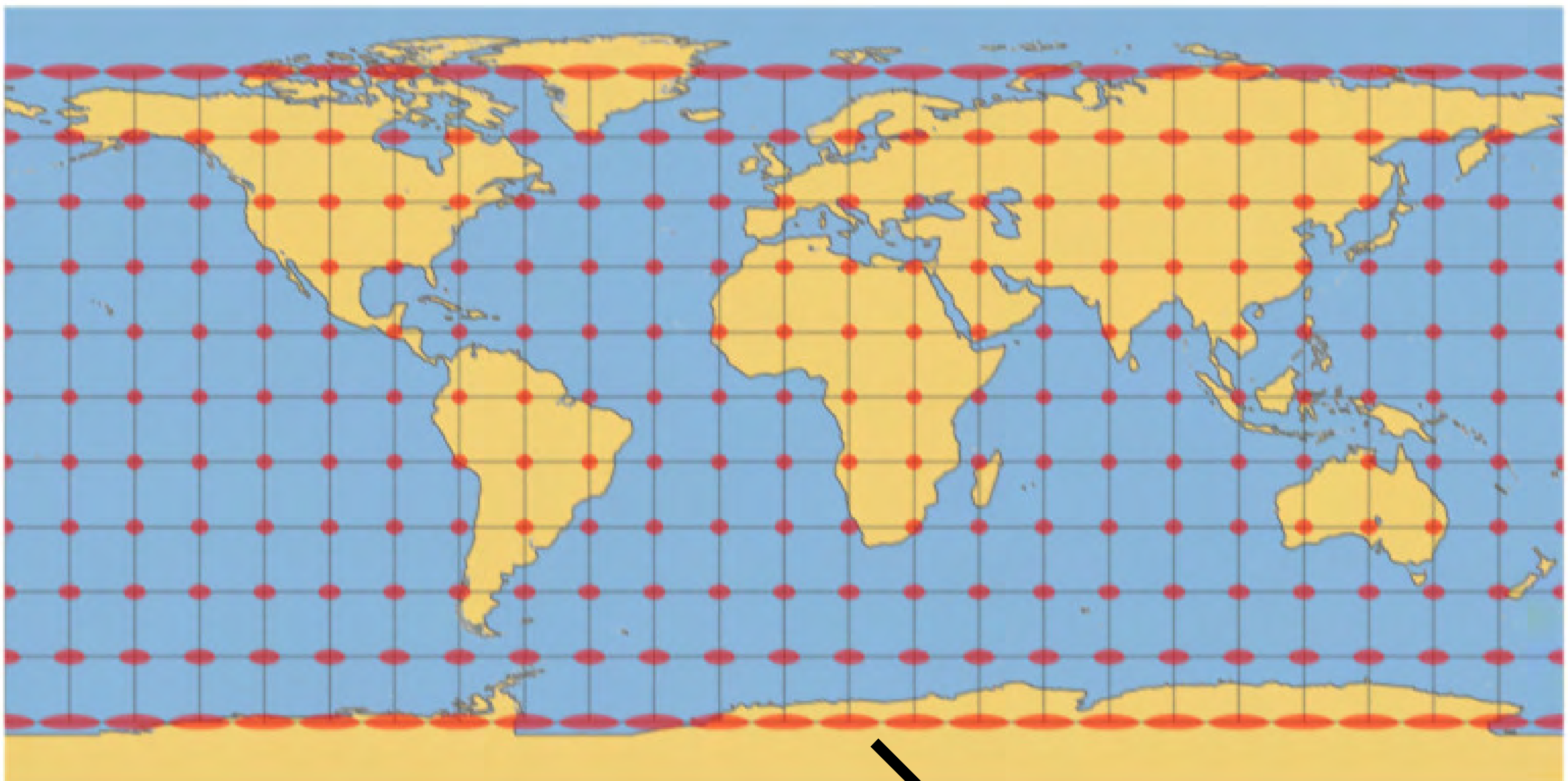
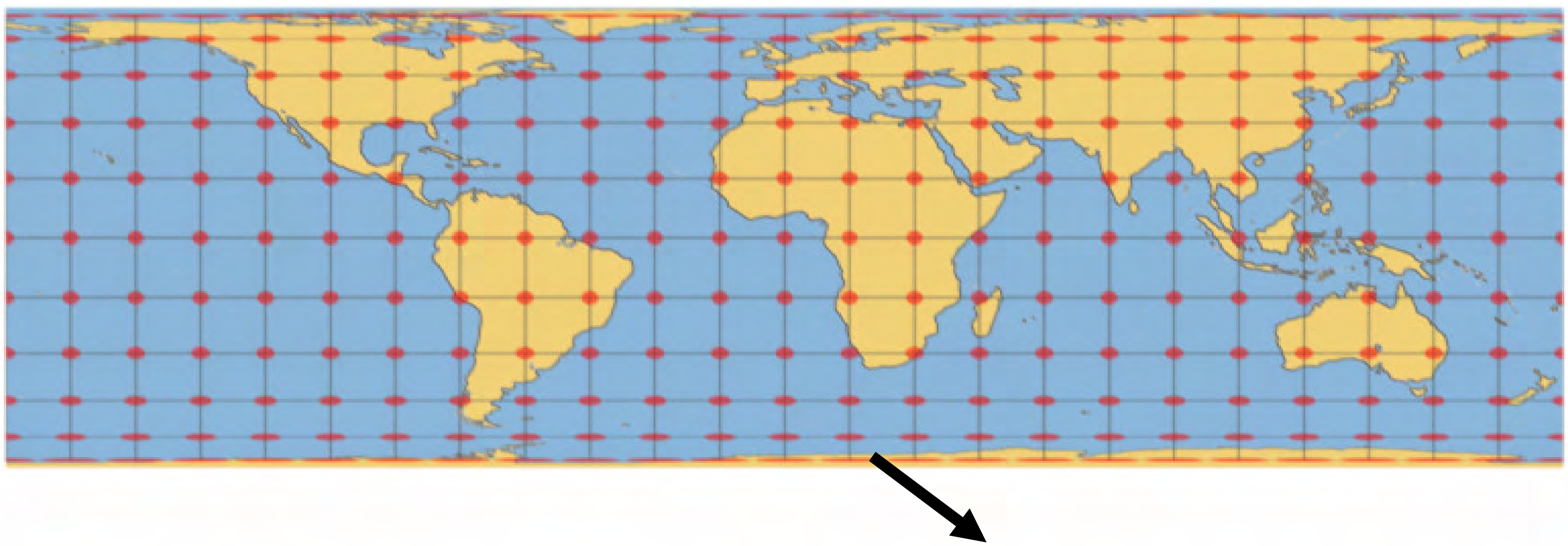


Plate Carrée

Area and shape distortion

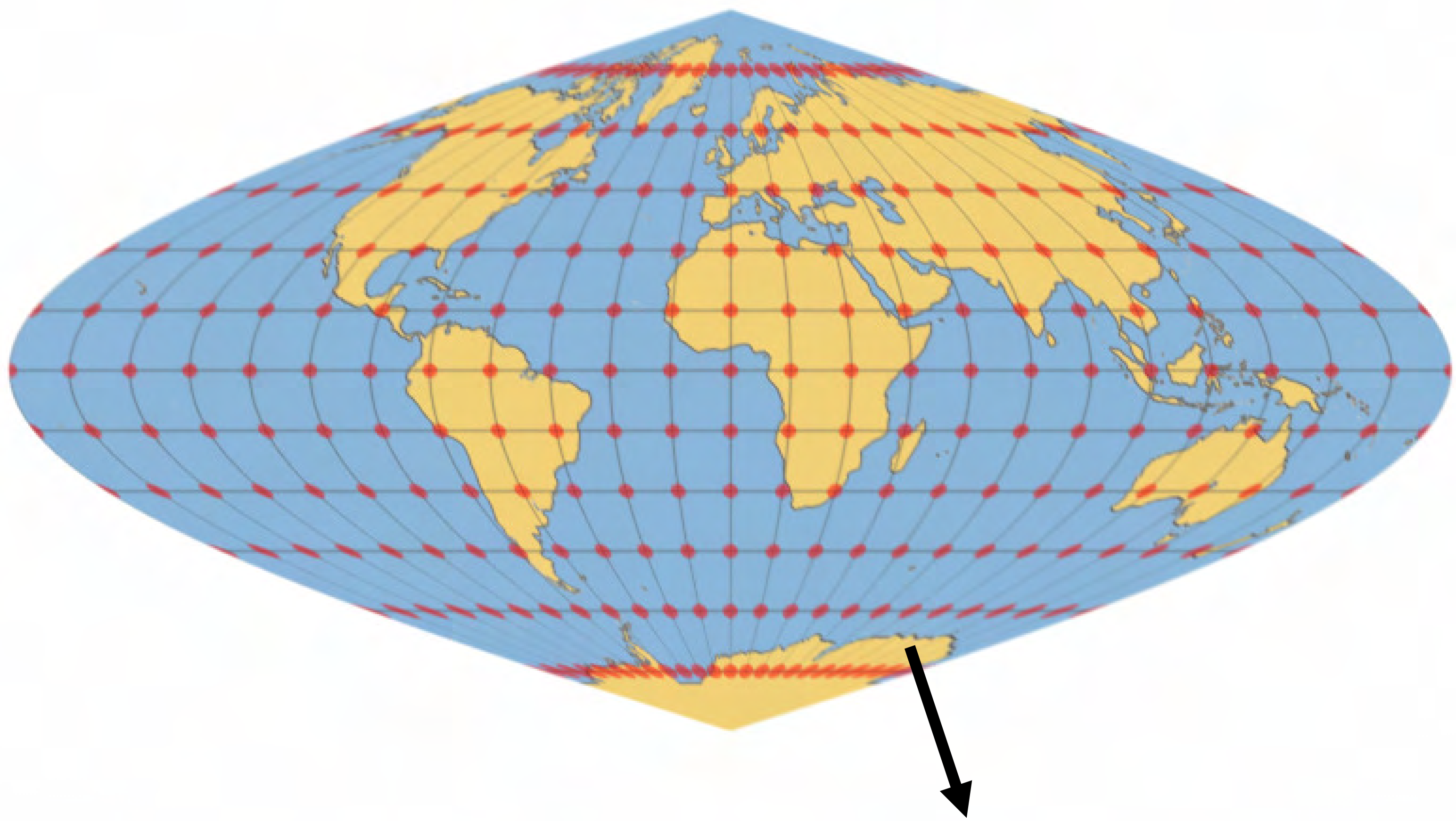
How to map the world?



Shape distortion

Lambert cylindrical equal area

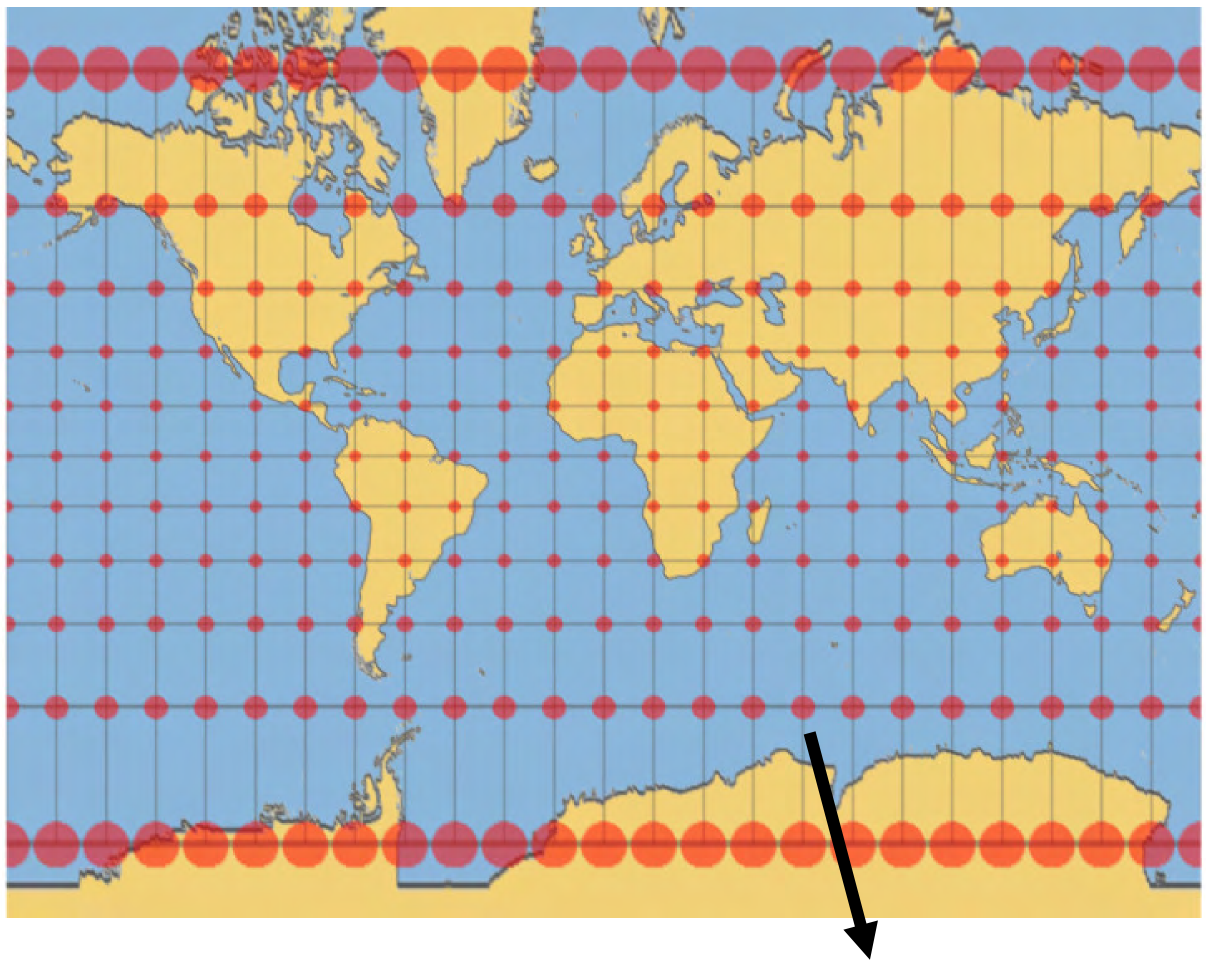
How to map the world?



Sinusoidal projection

Shape distortion

How to map the world?



Mercator projection

Area distortion

How to map the world?

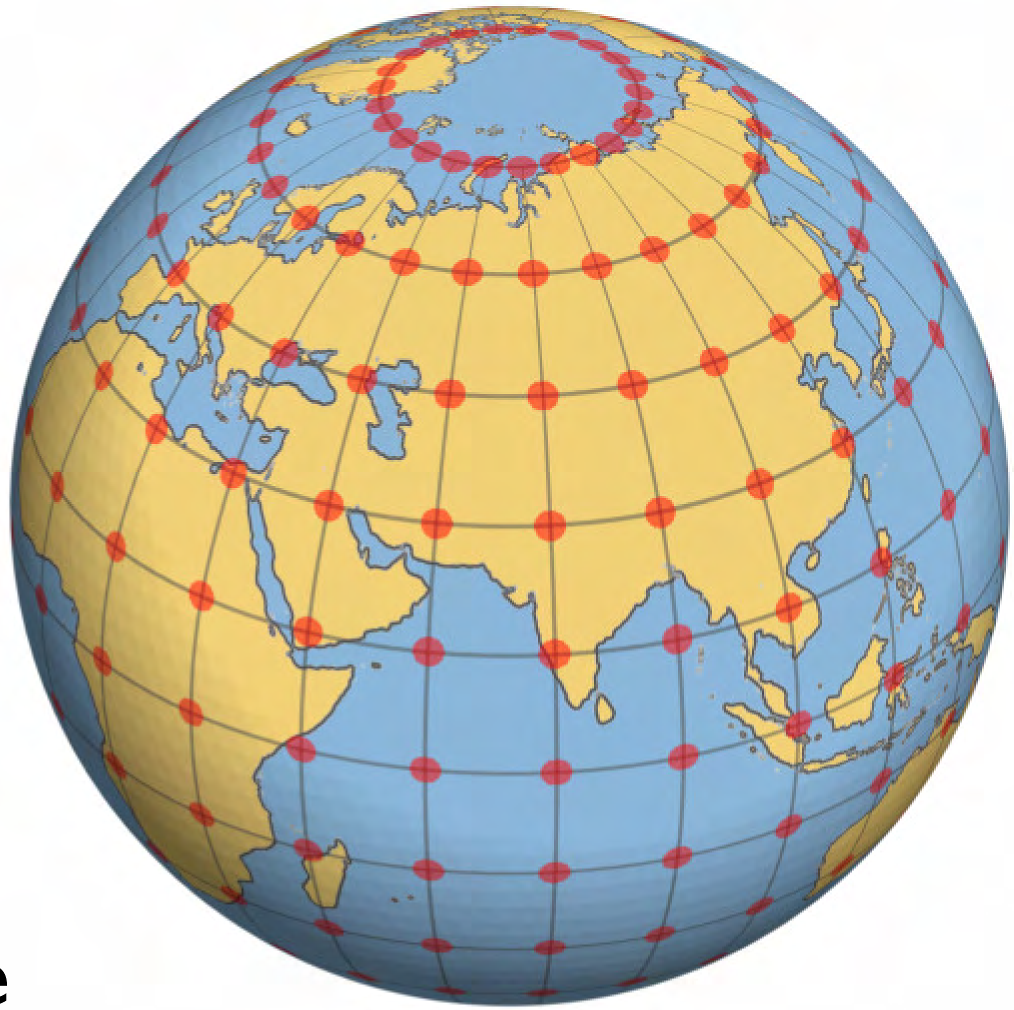
Projecting a curved surface on a plane gives:

- area distortion;
- shape distortion;
- or both.

Can we do better?

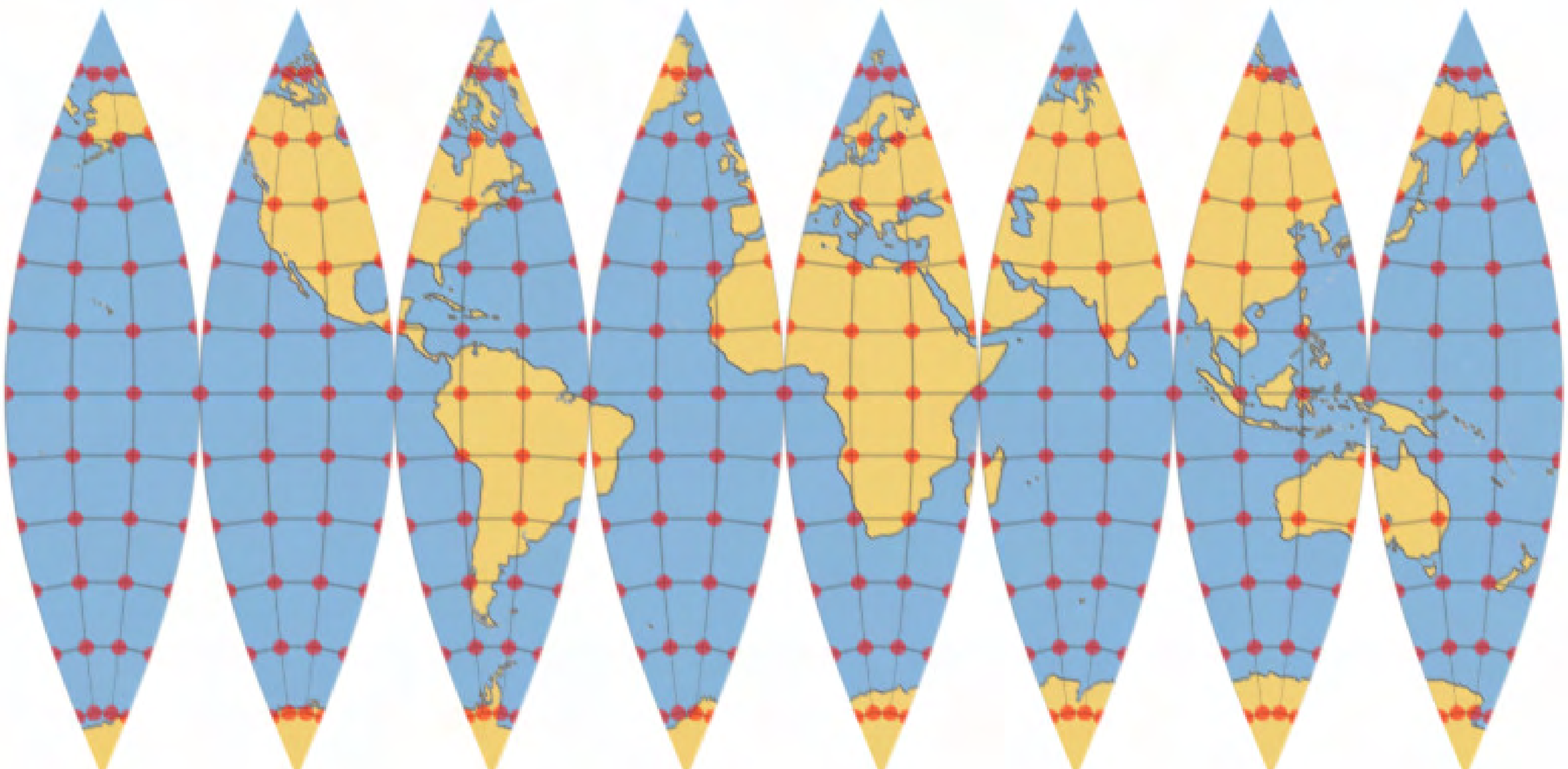
- Yes we can!

How to make a globe?



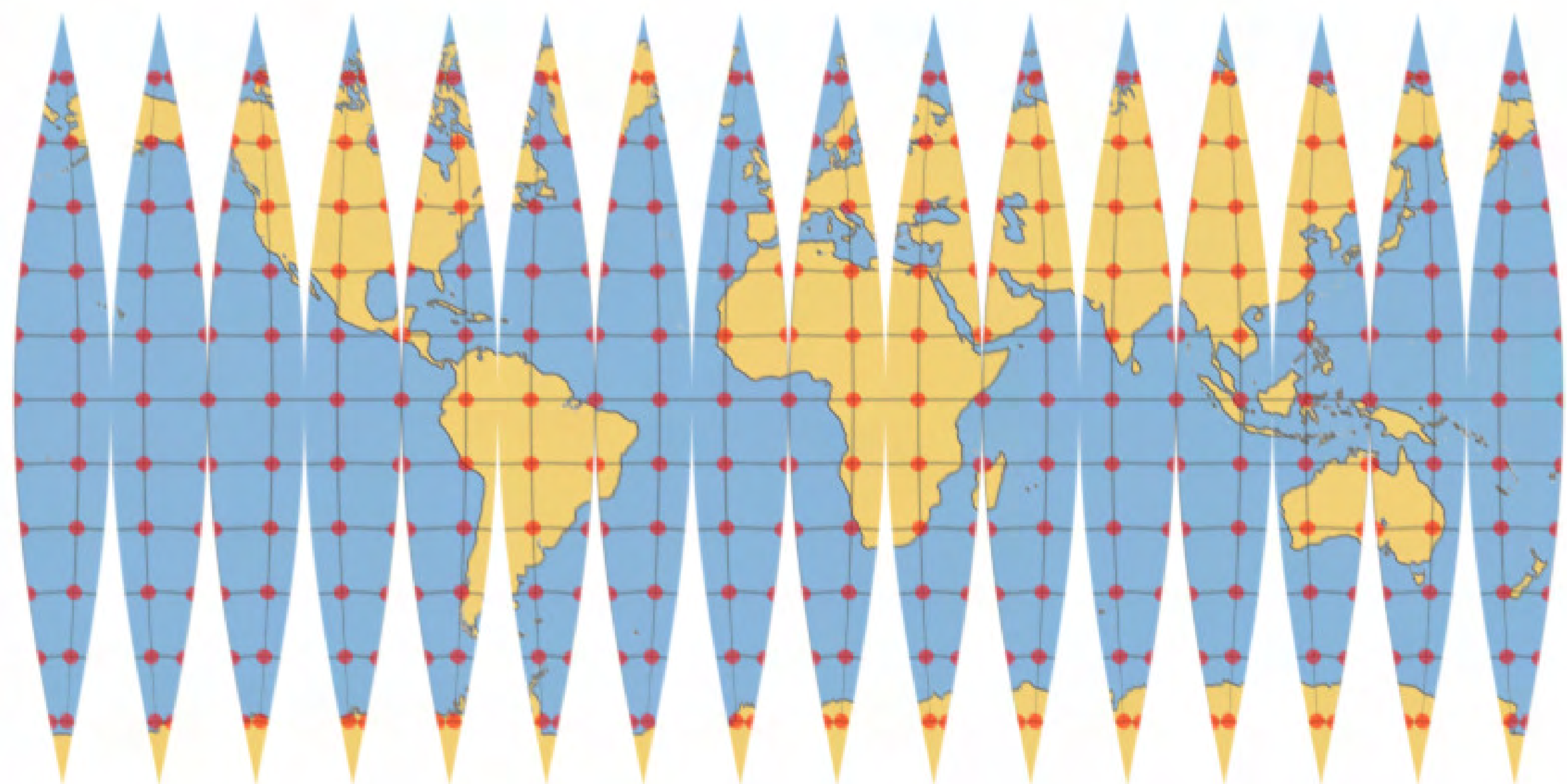
Globe

How to make a globe?



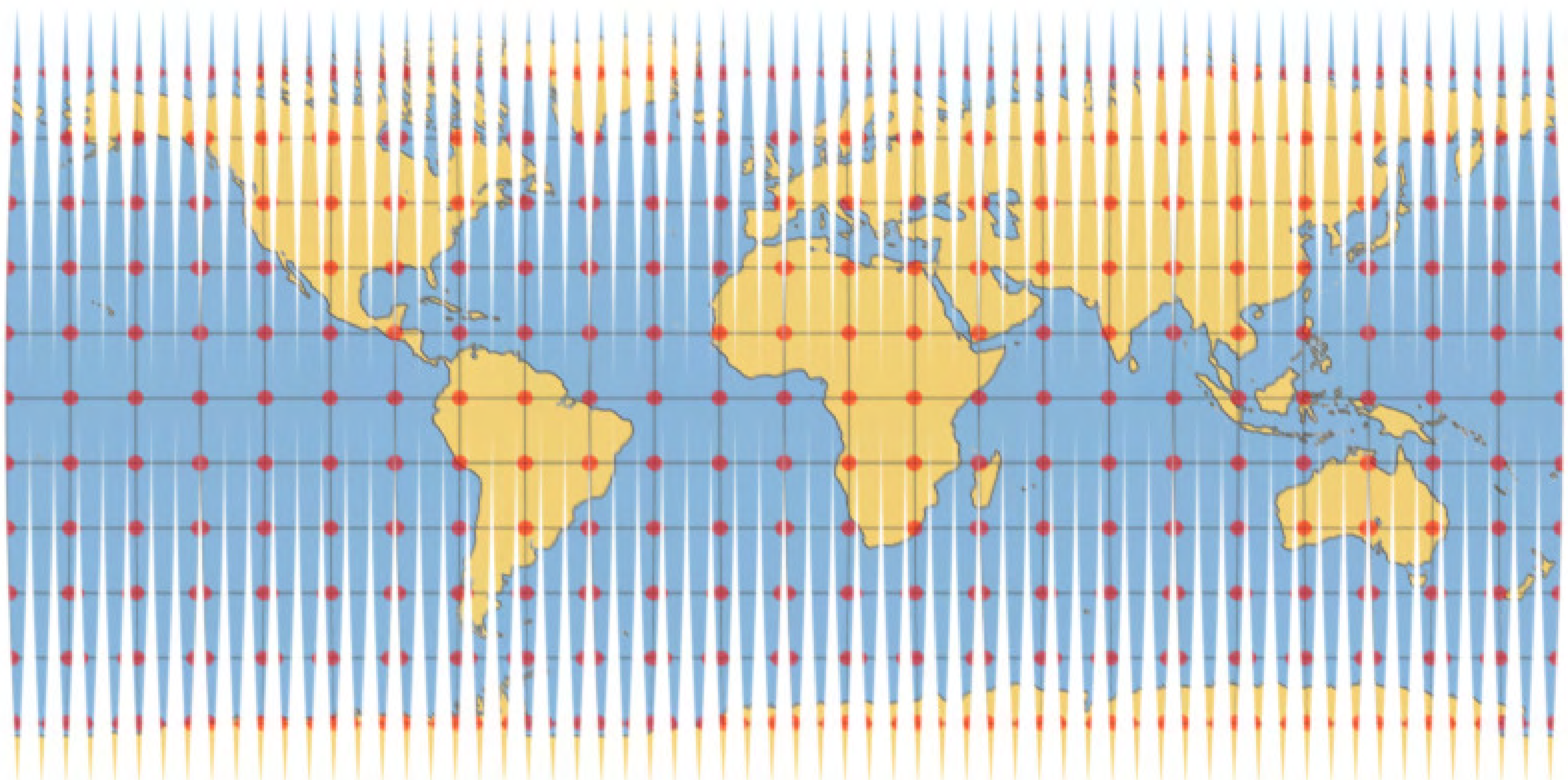
Gore map (with 8 gores)

How to make a globe?



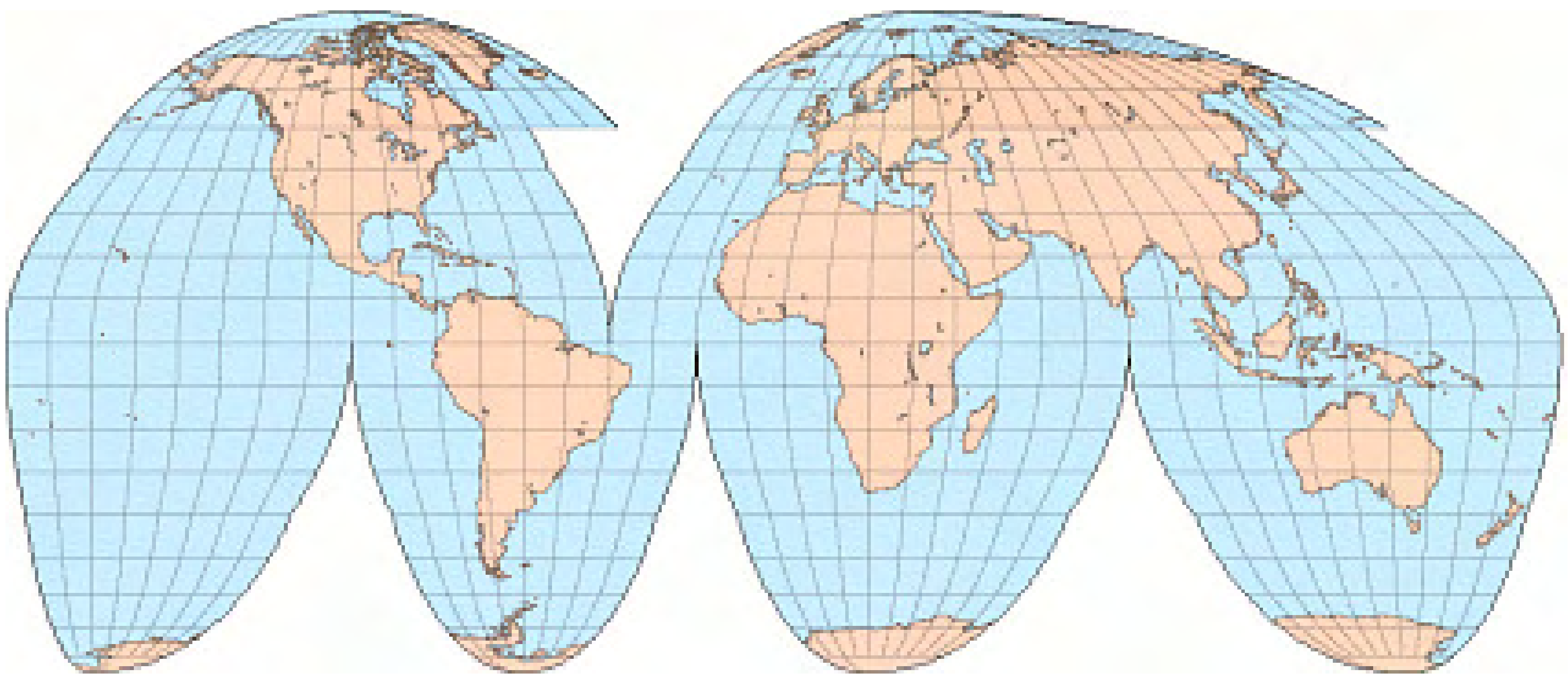
Gore map (with 16 gores)

How to make a globe?



Gore map (with 64 gores)

Interrupted maps



Goode’s homolosine projection, 1946

Interrupted maps

Buckminster Fuller's Dymaxion map, 1946

Interrupted maps

- What if we allow *many* interrupts?
- What choices can we make?
- How to control these?

Interrupted maps

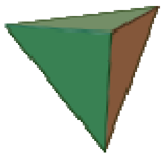
Top down view:

- Given a globe, where to cut?

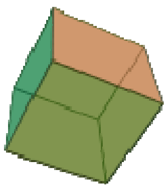
Bottom up view:

- Given many small maps, how to glue these into one big, interrupted flat map?

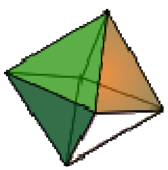
Myriahedron



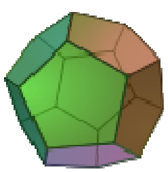
tetrahedron (4 faces)



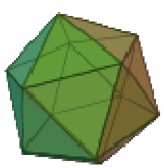
hexahedron (6)



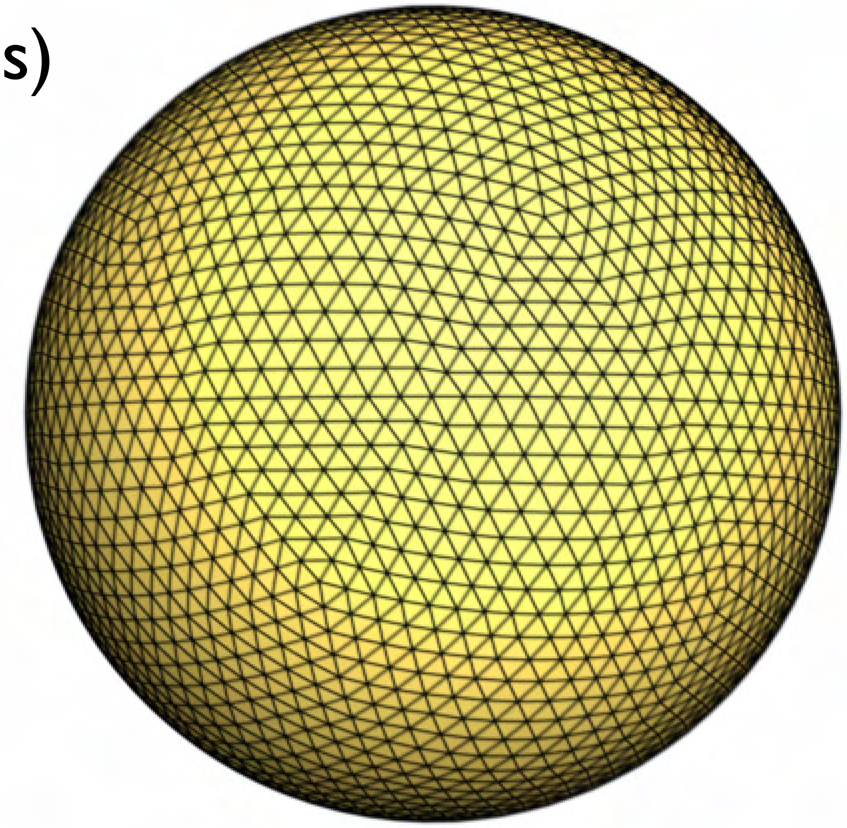
octahedron (8)



dodecahedron (12)



icosahedron (20)



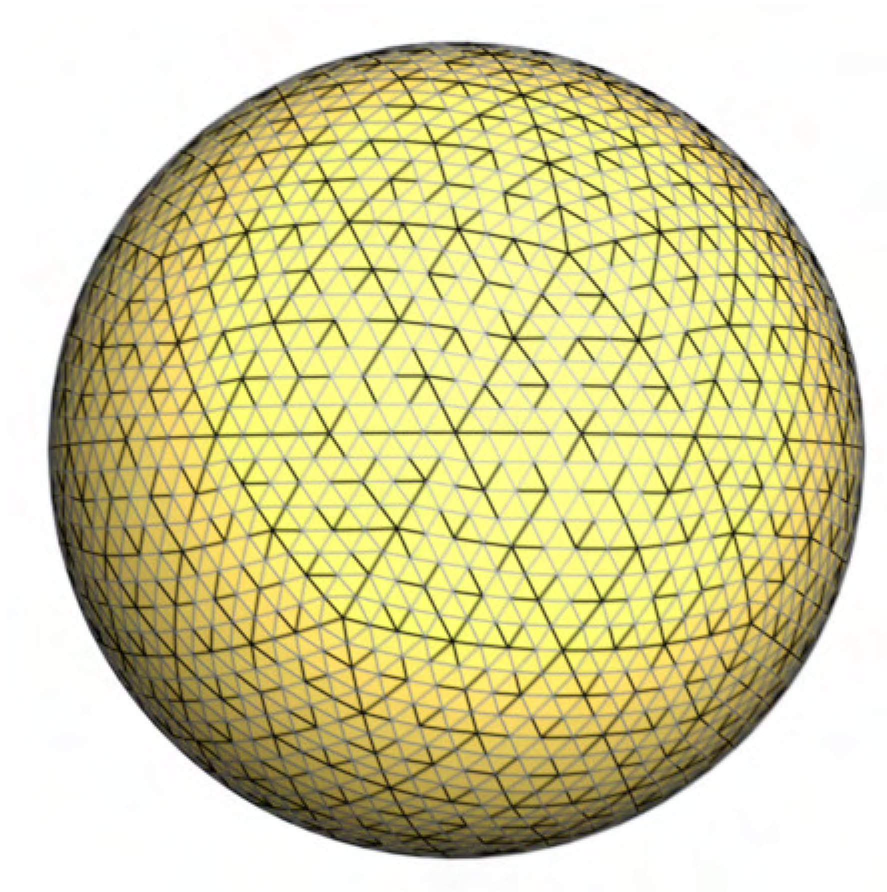
myriahedron (myriads)

Myriahedral projections

1. Define a mesh on a sphere, giving a myriahedron

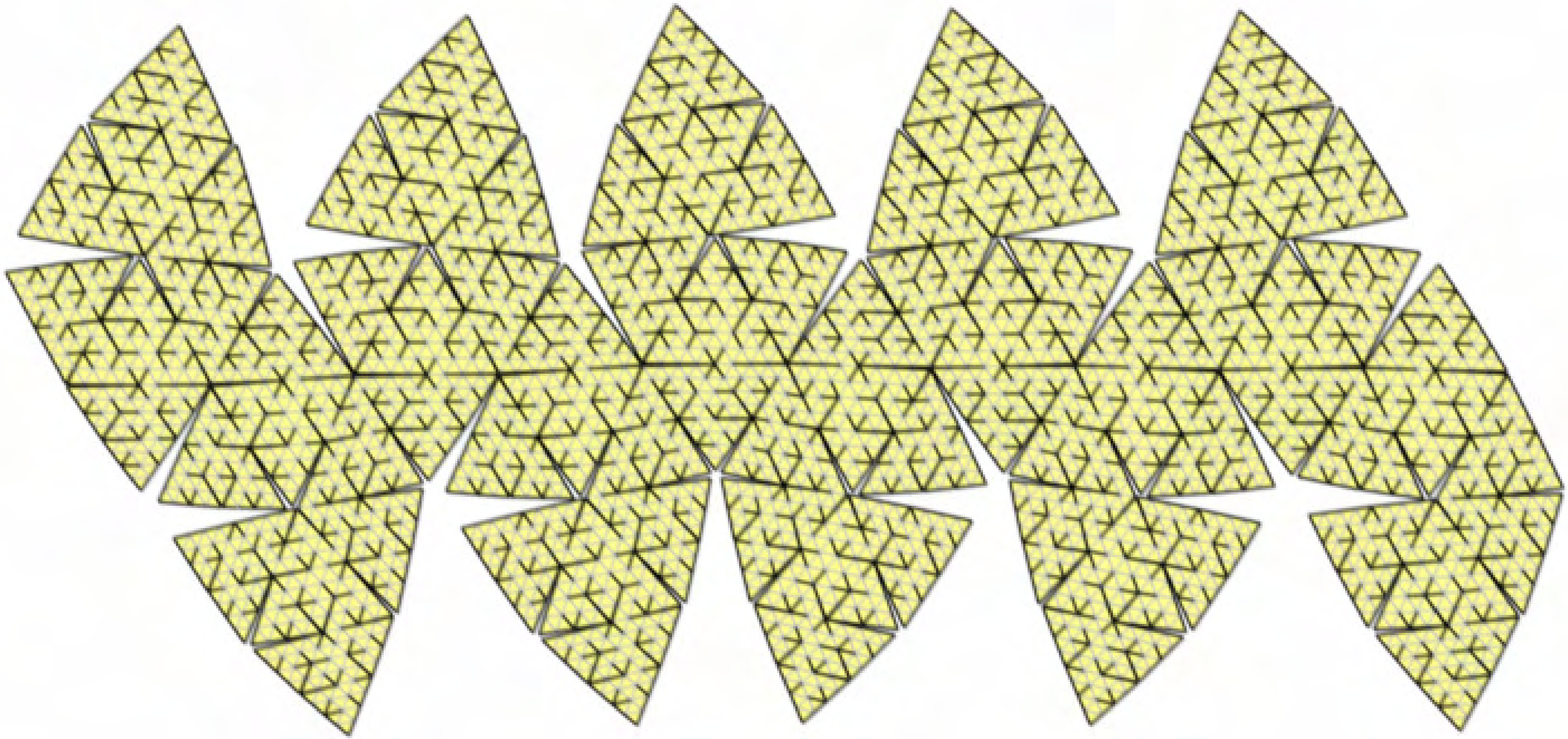
2. Decide which edges are cuts, which are folds

3. Unfold the myriahedron



myriahedron (myriads)

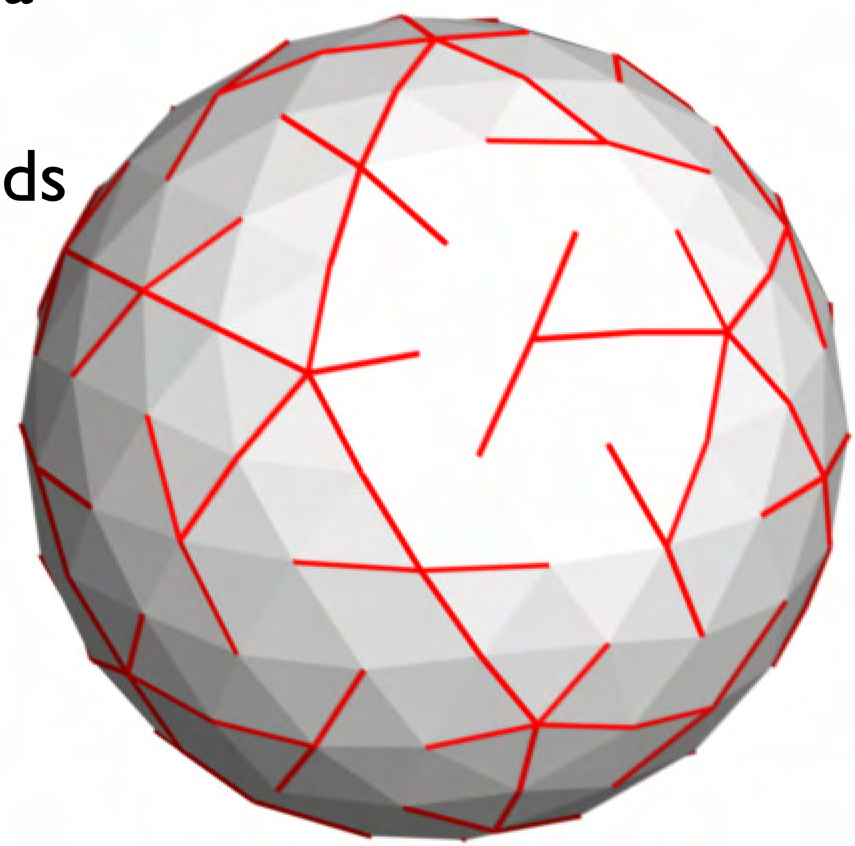
Myriahedral projections



3. Unfold the myriahedron…

Where to cut?

- vertices and edges form a graph
- label edges as cuts or folds



Required:

- around each vertex: at least one cut, to enable flattening
- no cycles in cuts

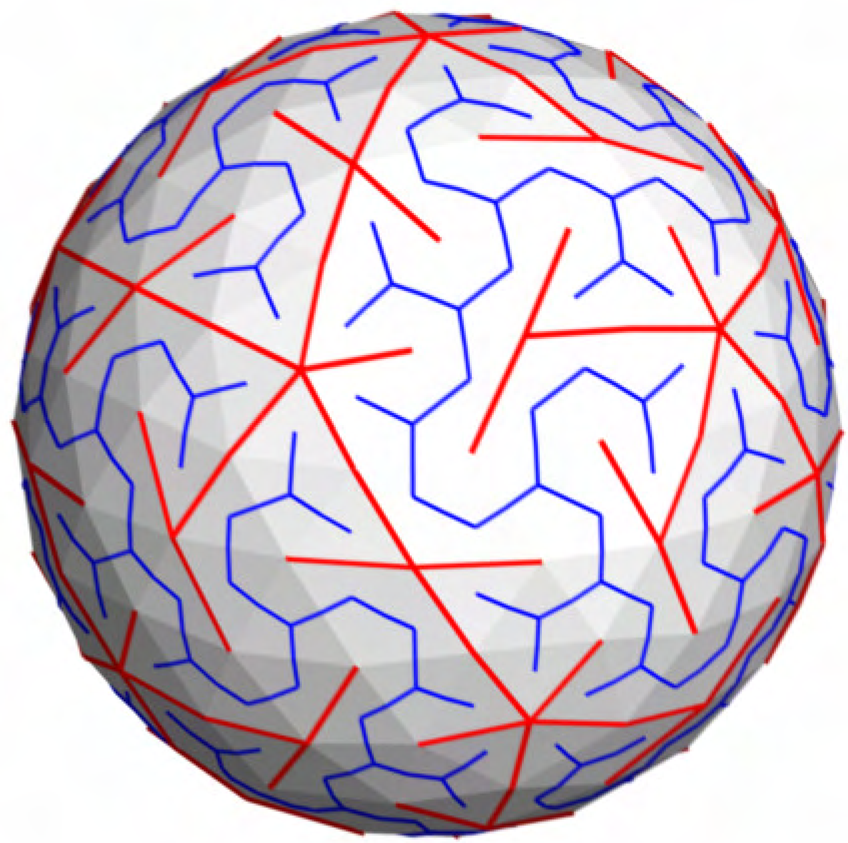
$\rightarrow$ cuts: *spanning tree*

Where to fold?

- similar reasoning, on dual of edge graph

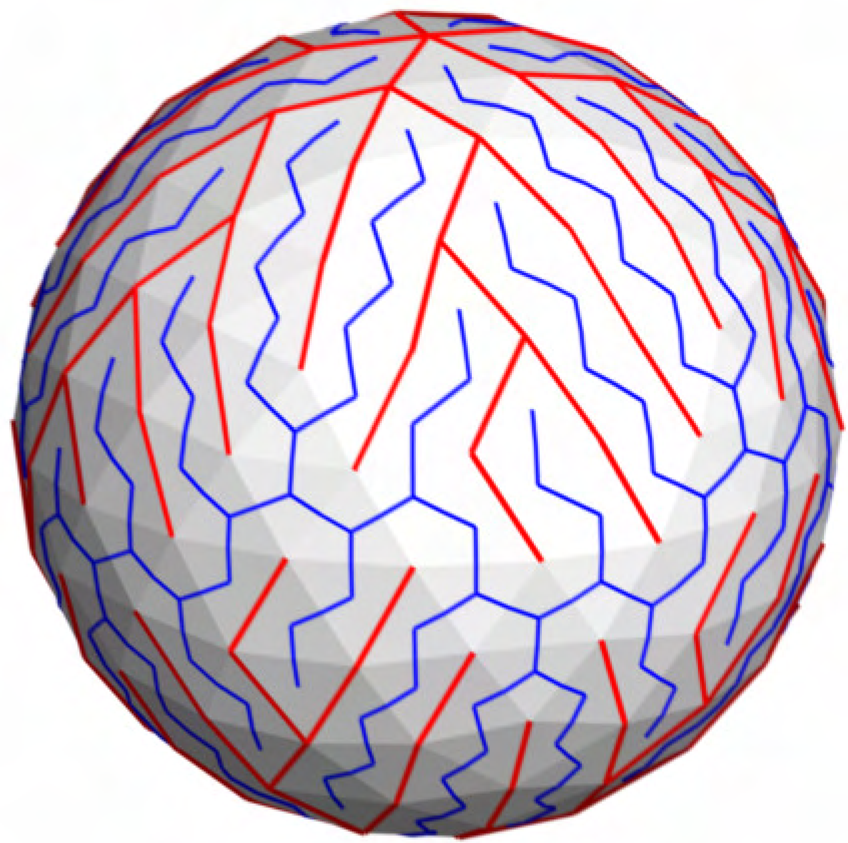
- No cycle of folds around vertex
- All faces connected

→ folds: spanning tree



More control

- assign weights to edges, indicating their ‘strength’
- determine *minimal spanning tree* for cuts: turn weak edges into cuts



Mesh styles

- Use parallels and meridians
- Use Platonic solids with recursive subdivision
- Use image driven mesh

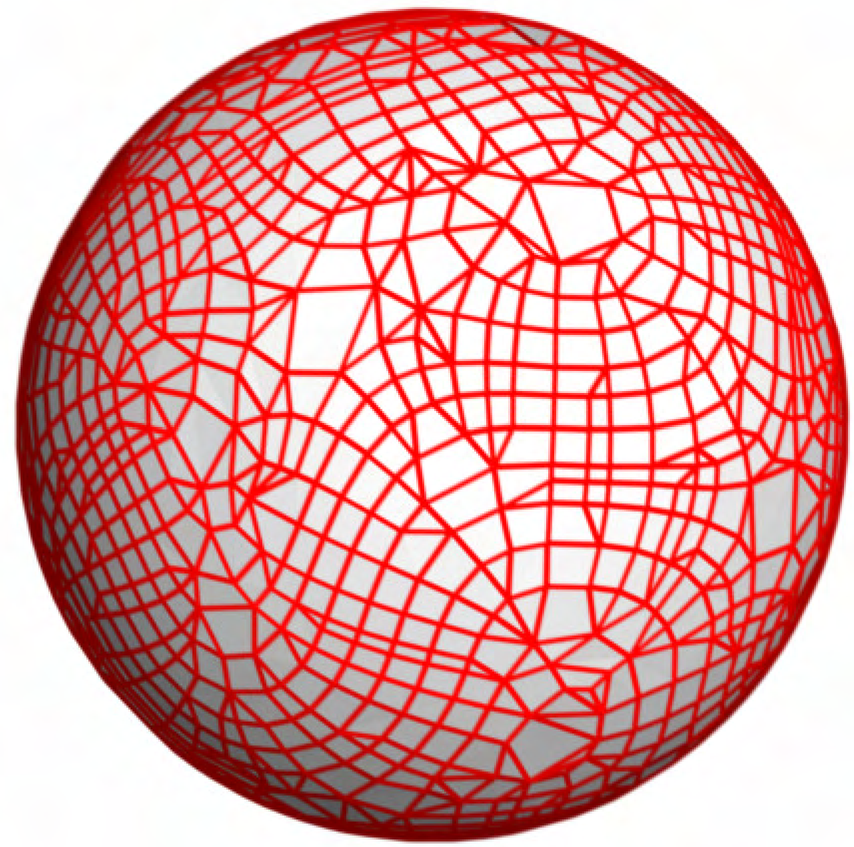


Image driven mesh

- Edges: contours and descent lines of grey shade image *F*

- Use *F* as weight

→ Avoid cutting through continents

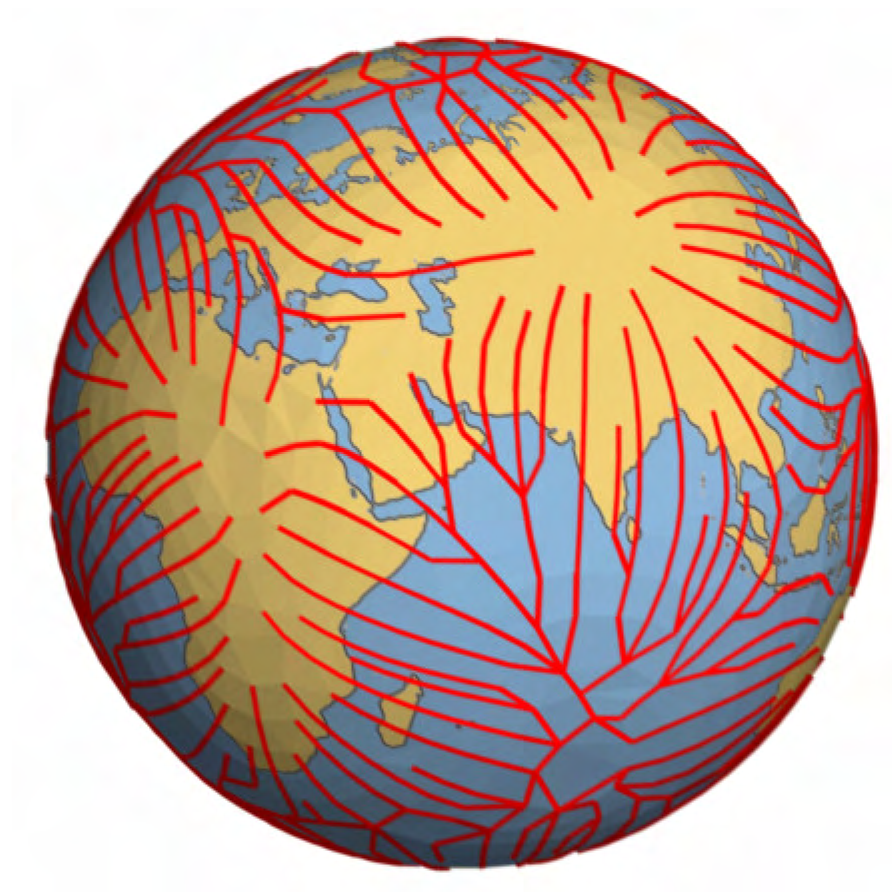


Image driven mesh unfolded

Video

- http://www.win.tue.nl/~vanwijk/myriahedral
- google: myriahedral

Unfolding the Earth: Myriahedral Projections

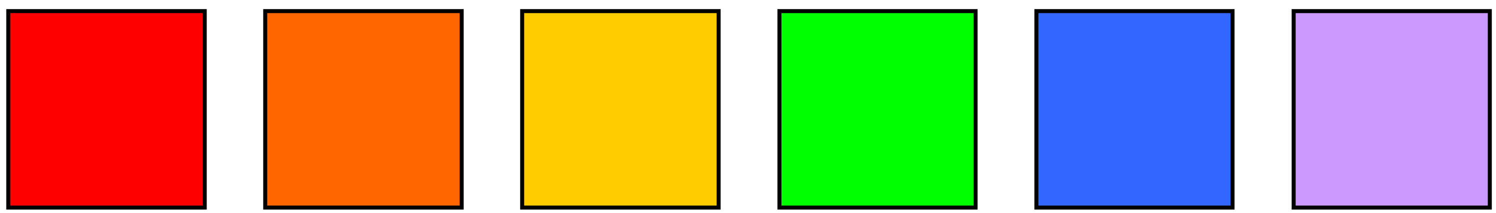
Jarke J. van Wijk
TU Eindhoven

Tiles

J.J. van Wijk, Symmetric Tiling of Closed Surfaces: Visualization of Regular Maps.
ACM Transactions on Graphics, 28(3), (proceedings SIGGRAPH 2009), 12p.

Puzzle: 6 squares

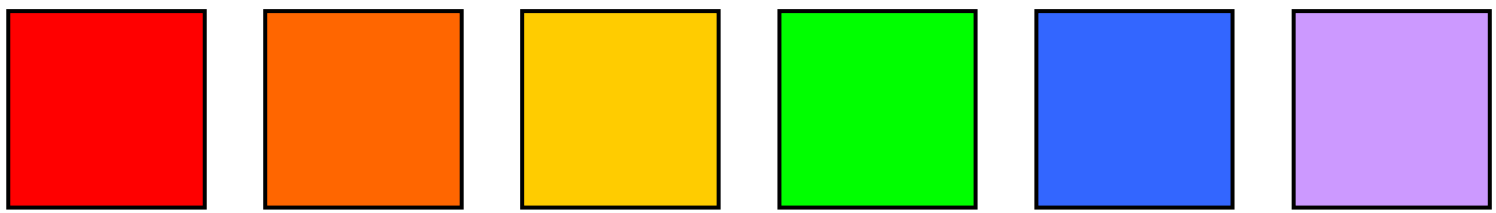
- Take a highly elastic, colorful fabric
- Cut out 6 squares



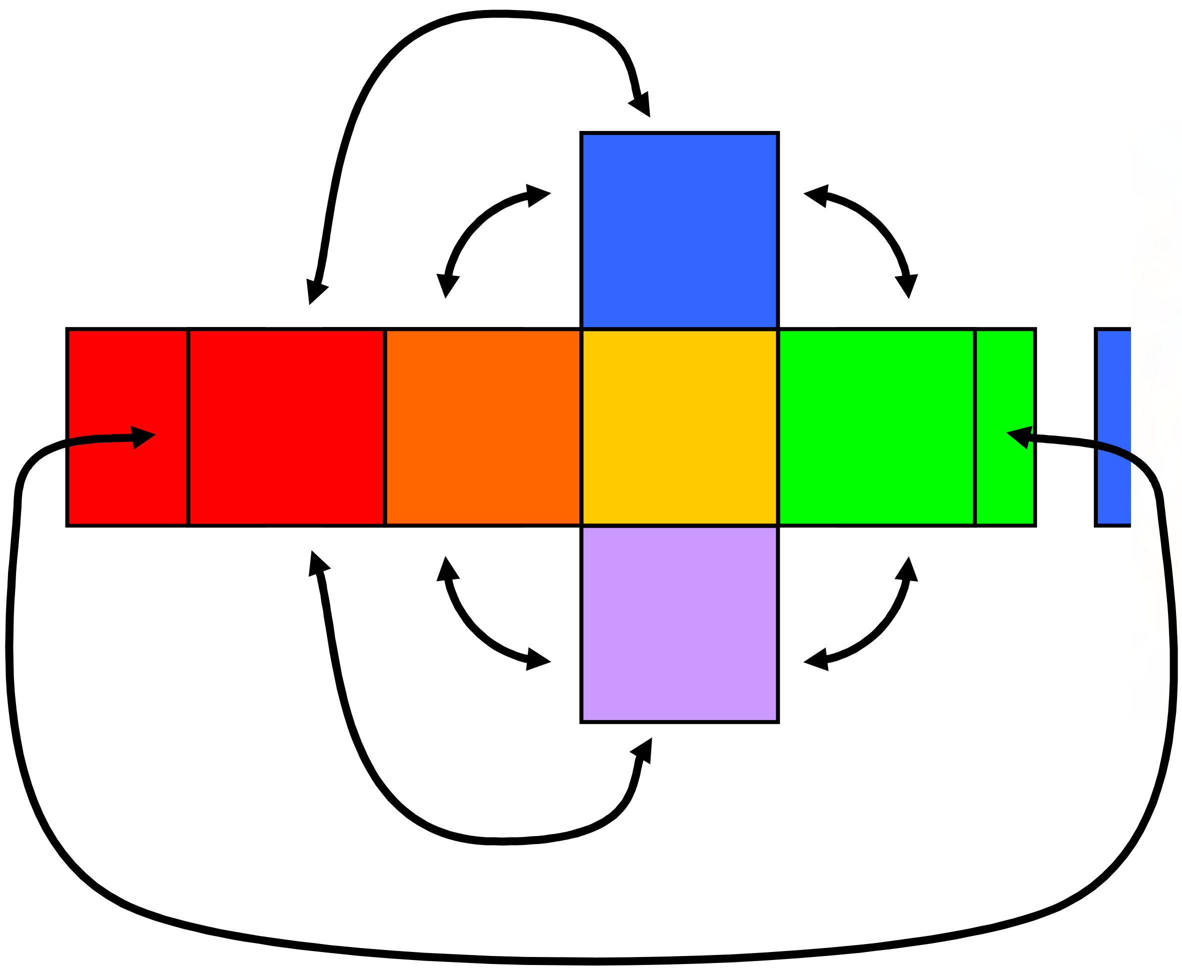
- Stitch them together, with maximal symmetry
- Stuff tightly with polyester fiber

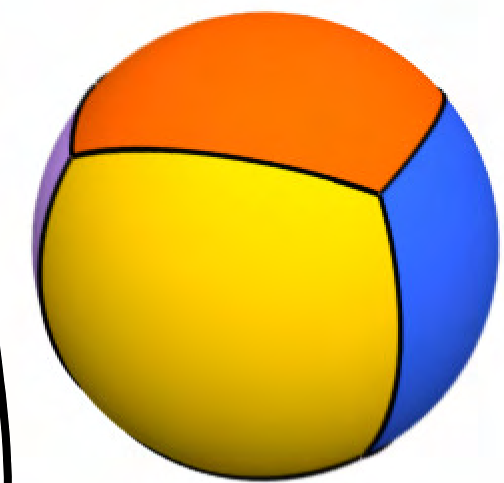
What shape could you get?

Puzzle: 6 squares



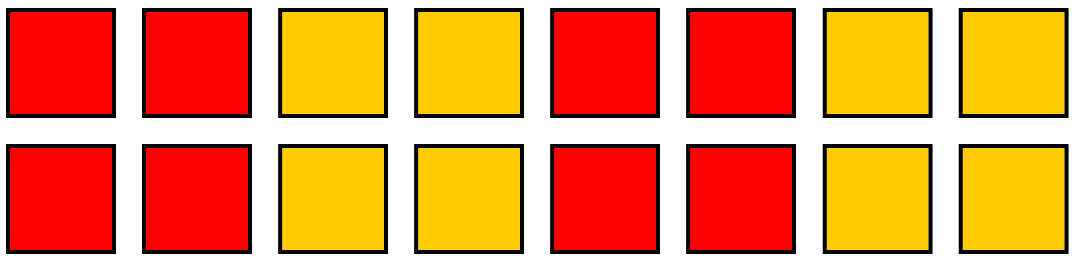
Puzzle: 6 squares





Puzzle: 16 squares

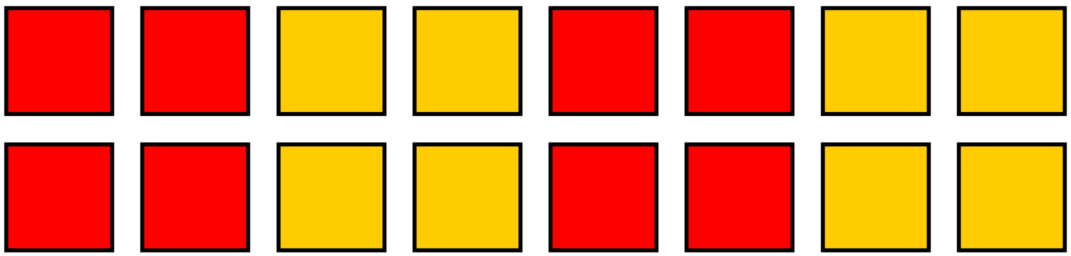
- Take a highly elastic, colorful fabric
- Cut out 16 squares



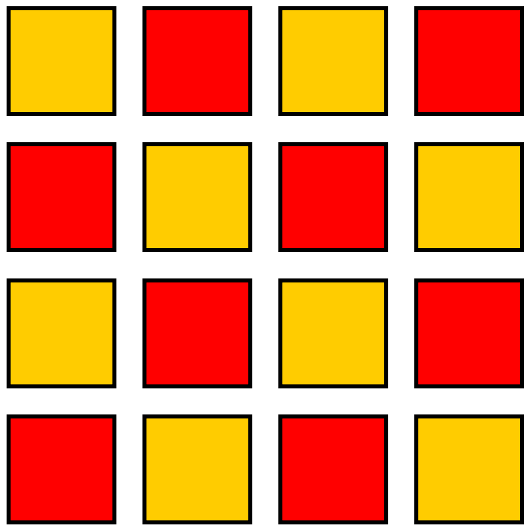
- Stitch them together, with maximal symmetry
- Stuff with polyester fiber

What shape could you get?

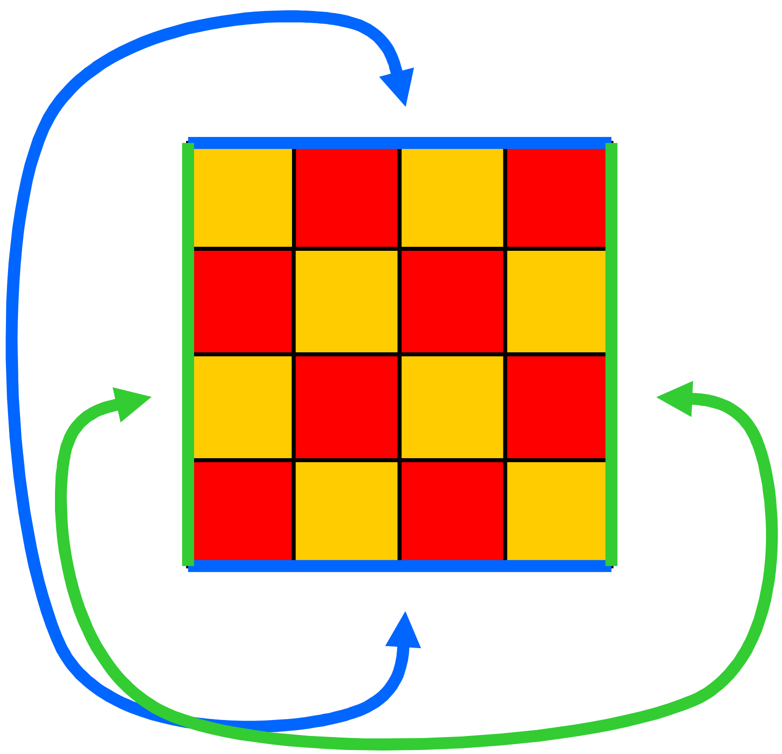
Puzzle: 16 squares

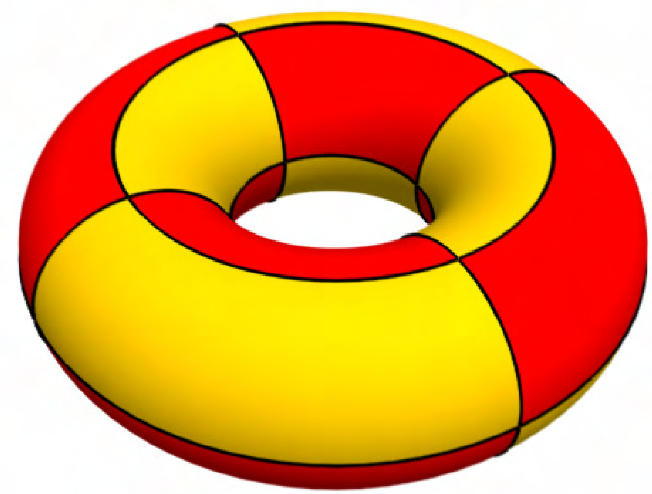


Puzzle: 16 squares



Puzzle: 16 squares

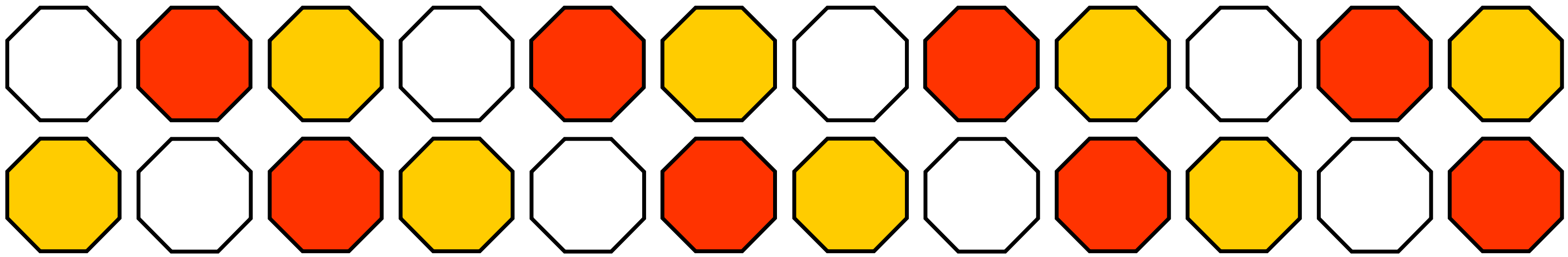




Torus (genus 1), with checkerboard pattern

Puzzle: 24 octagons

- Take a highly elastic, colorful fabric
- Cut out 24 octagons



- Sew them together, to get a closed surface
- Stuff with polyester fiber

What shape could you get?

More puzzles

- Take N polygons with p sides, stitch them together, such that at each corner q sides meet. Find shapes for (N, p, q) =

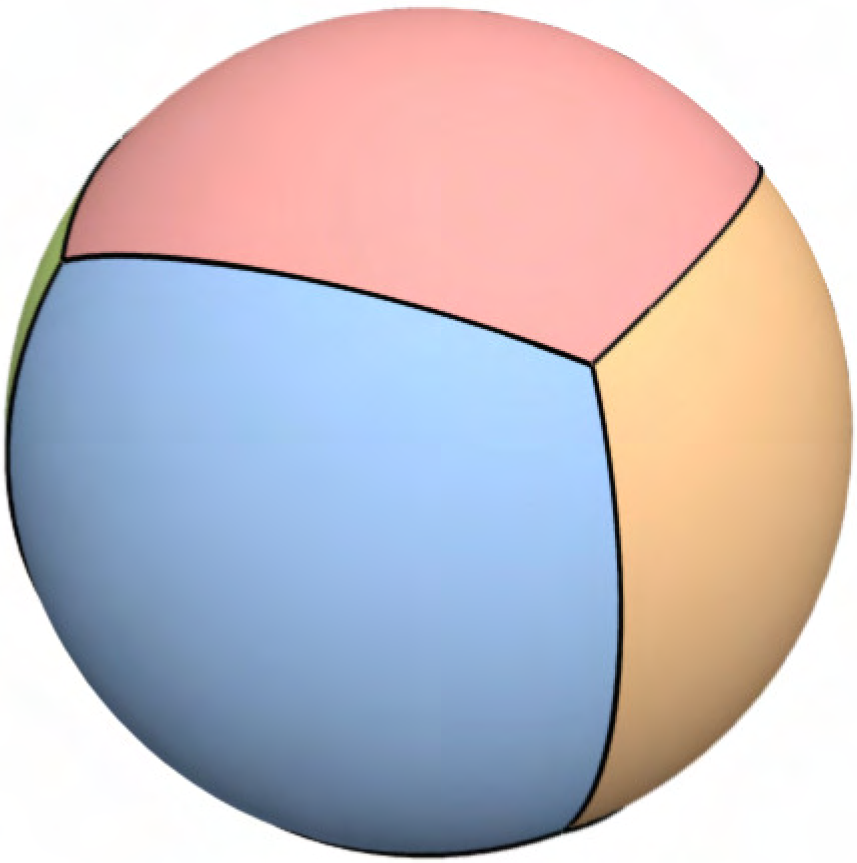
(16, 3, 8),	(6, 6, 4),	(2, 8, 5),	(2, 6, 6),	(1, 8, 8),
(12, 8, 3),	(12, 4, 6),	(6, 4, 12),	(4, 8, 4),	(2, 12, 4),
(18, 4, 6),	(10, 4, 10),	(64, 3, 8),	(40, 4, 5),	(16, 6, 4),
(16, 4, 8),	(12, 4, 12),	(4, 14, 4),	(48, 4, 6),	(32, 6, 4),
(8, 12, 4),	(20, 4, 20),	(54, 4, 6),	(60, 4, 6),	(96, 3, 12),
(96, 4, 6),	(16, 12, 4),	(56, 3, 7),	(168, 4, 6),	…

The general puzzle: regular maps

Construct space models of *regular maps*

- Surface topology, combinatorial group theory, graph theory, algebraic geometry, hyperbolic geometry, physics, chemistry, …

Regular maps

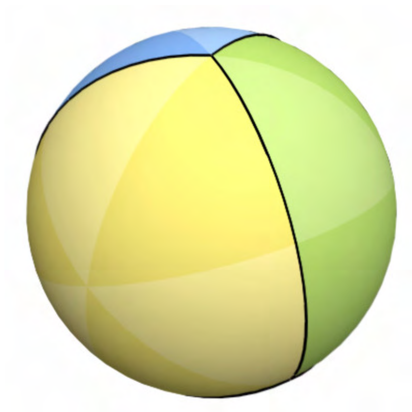


Regular map:

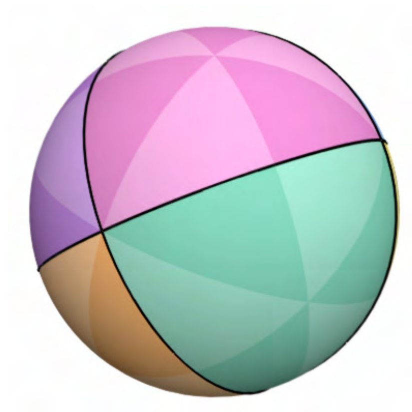
Embedding of a graph in a closed surface, such that topologically

- faces are identical
- vertices are identical
- edges are identical

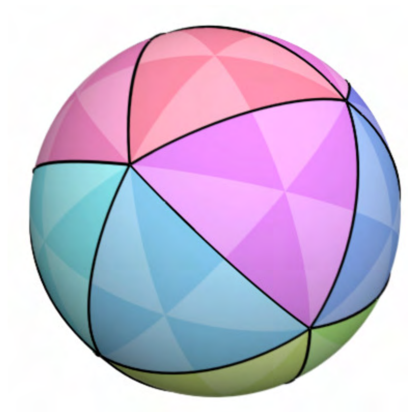
Genus 0: Platonic solids



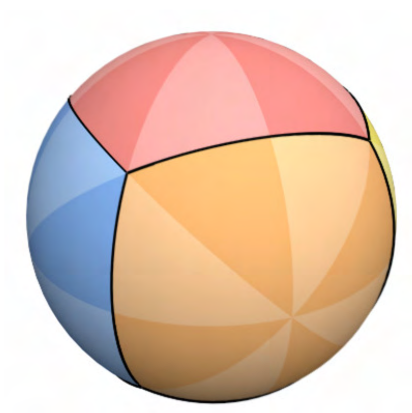
{3, 3}



{3, 4}



{3, 5}

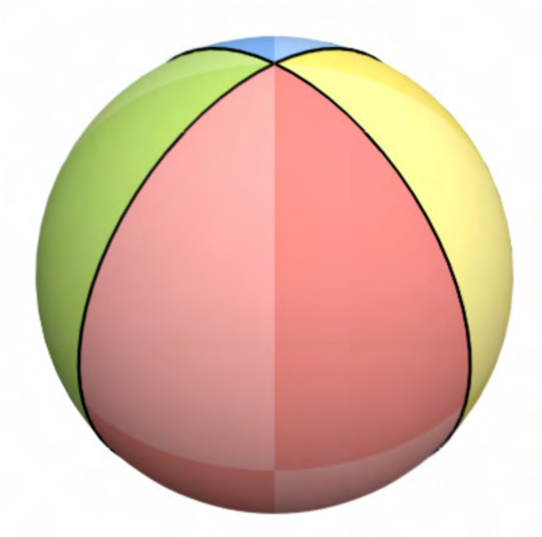


{4, 3}

{5, 3}

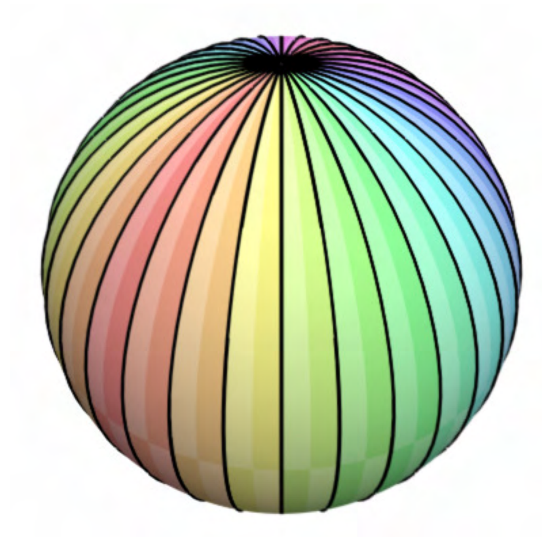
Genus 0: hosohedra

- hosohedron: faces with two edges



{2, 4}

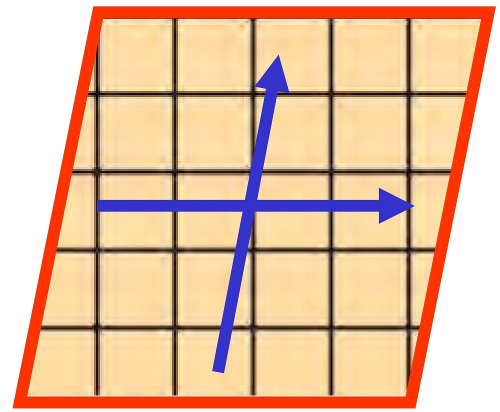
{2, 9}

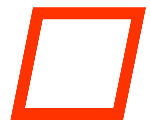


{2, 32}

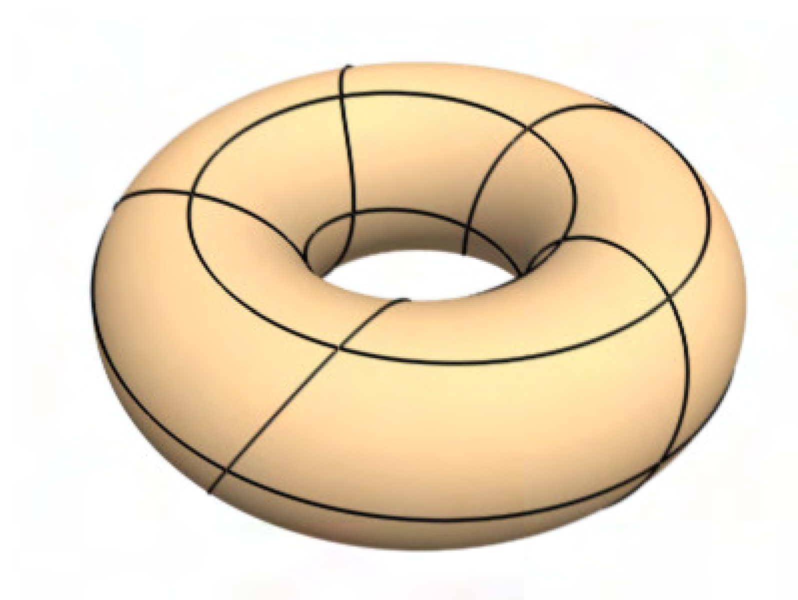
Tori (genus 1)

- Tile the plane



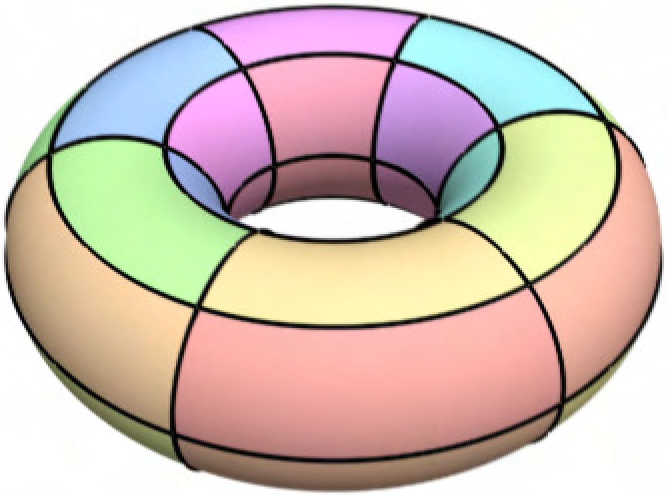
- Define a rhombus  (all sides same length)

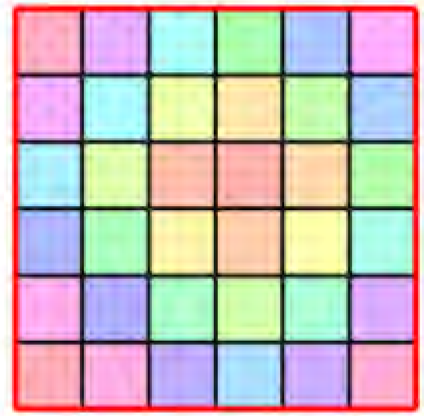
- Project tiling

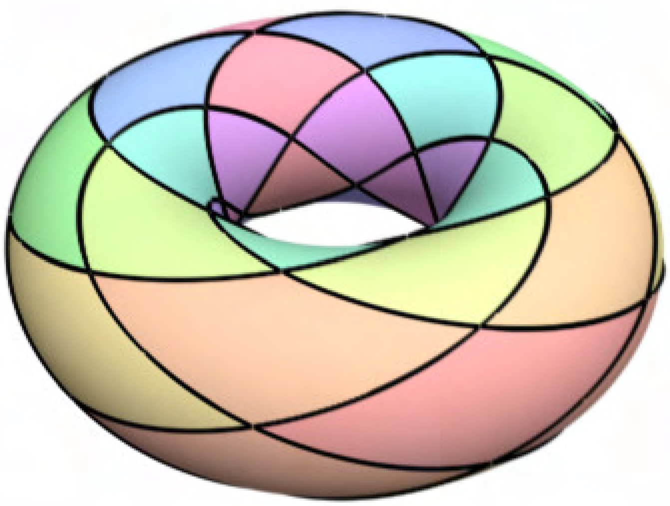


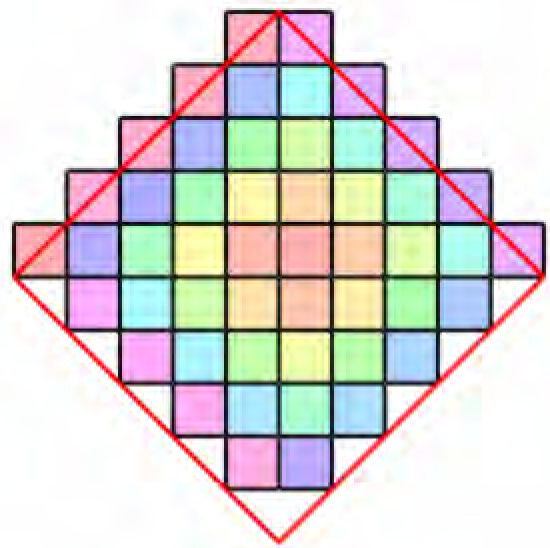
- Fold rhombus to torus

{4, 4}: torii

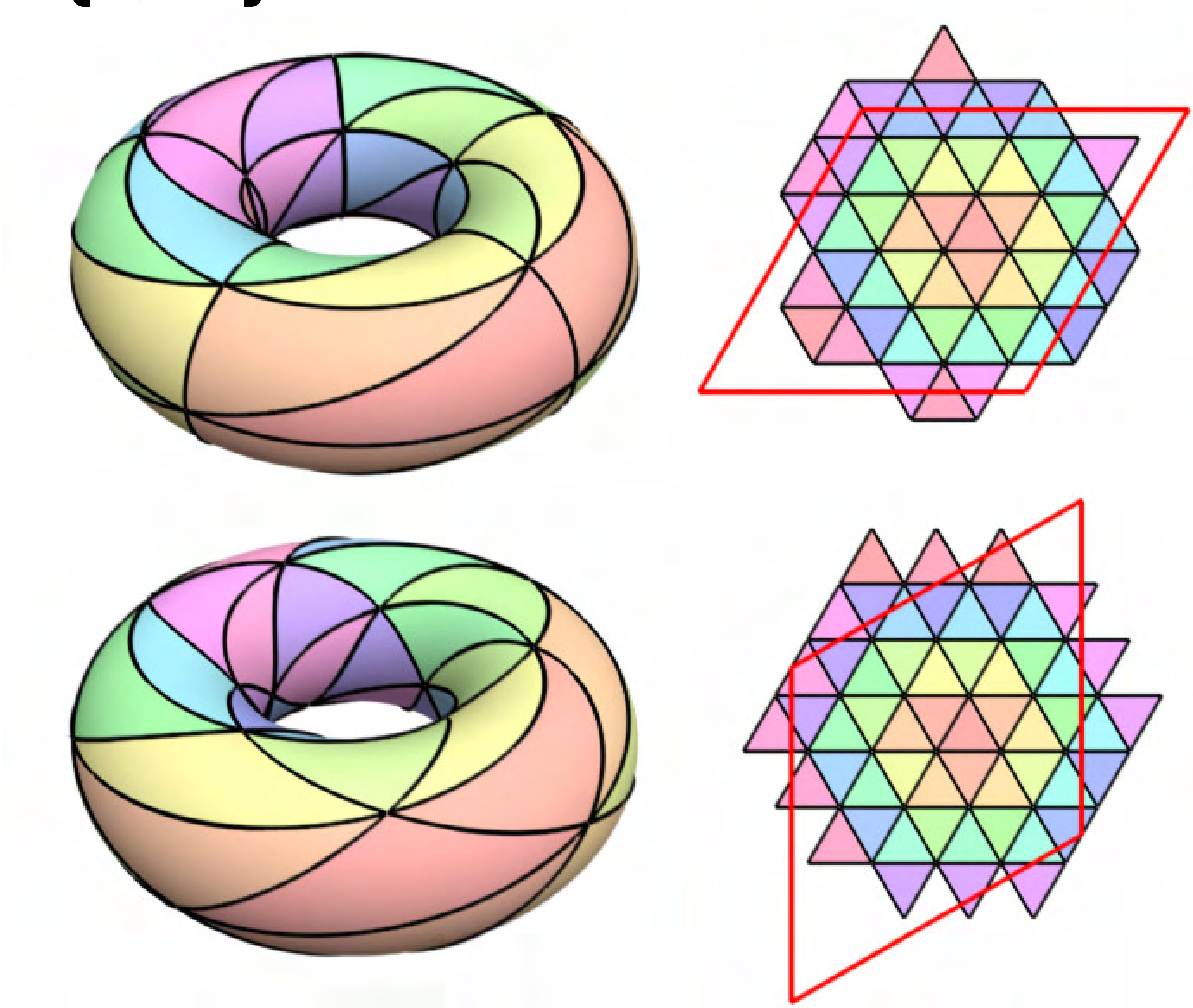




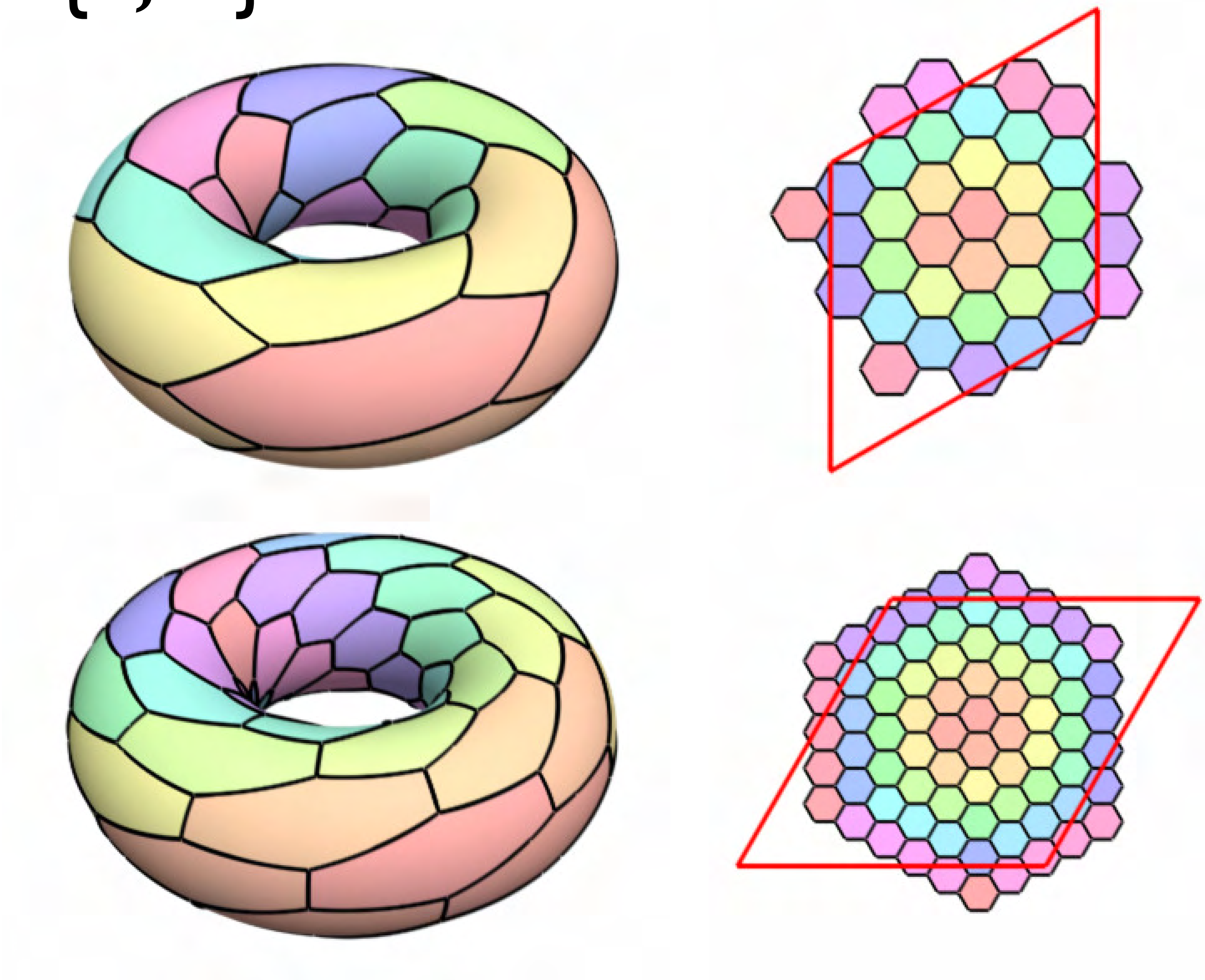




{3, 6}: torii



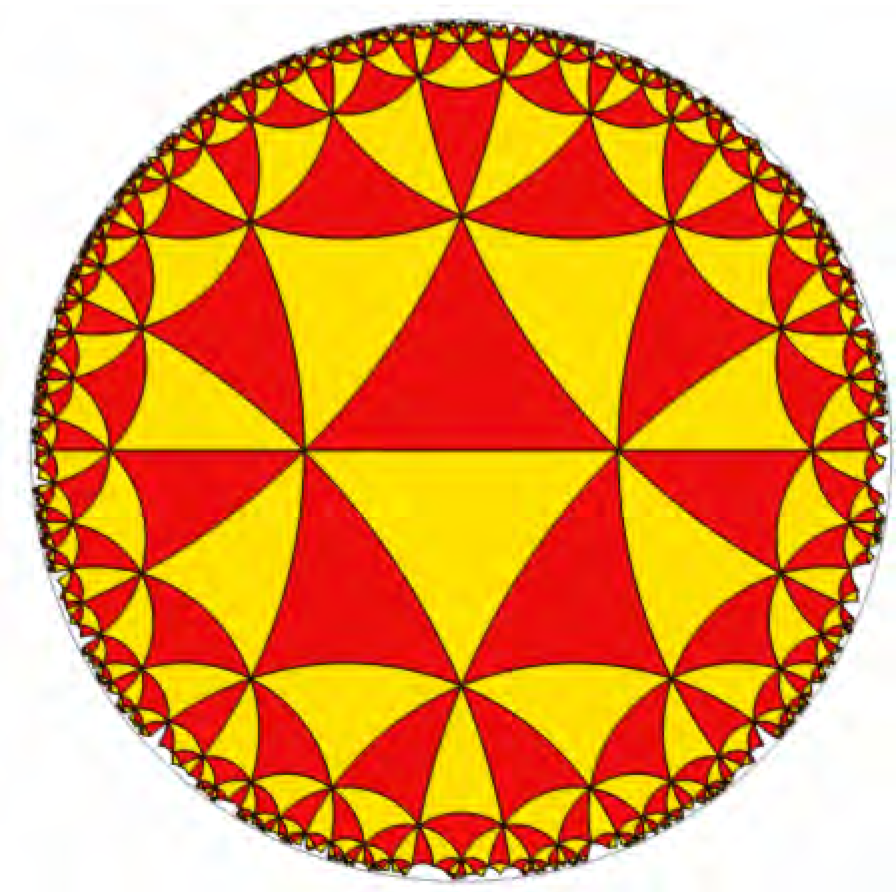
{6, 3}: torii



Genus $g \geq 2$

g	*shape*	*geometry*	*transf.*	*tilings*
0	sphere	spherical	3D rotation	{3,3}, {3,4}, {4,3}, {3,6}, {6,3}, {2,*n*}
1	torus	planar	2D Euclidean	{4,4}, {3,6}, {6,3}
≥ 2	?	hyperbolic	Möbius	{3,7}, {4,5}, {5,4}, {4,6}, {6,4}, {5,5}, …

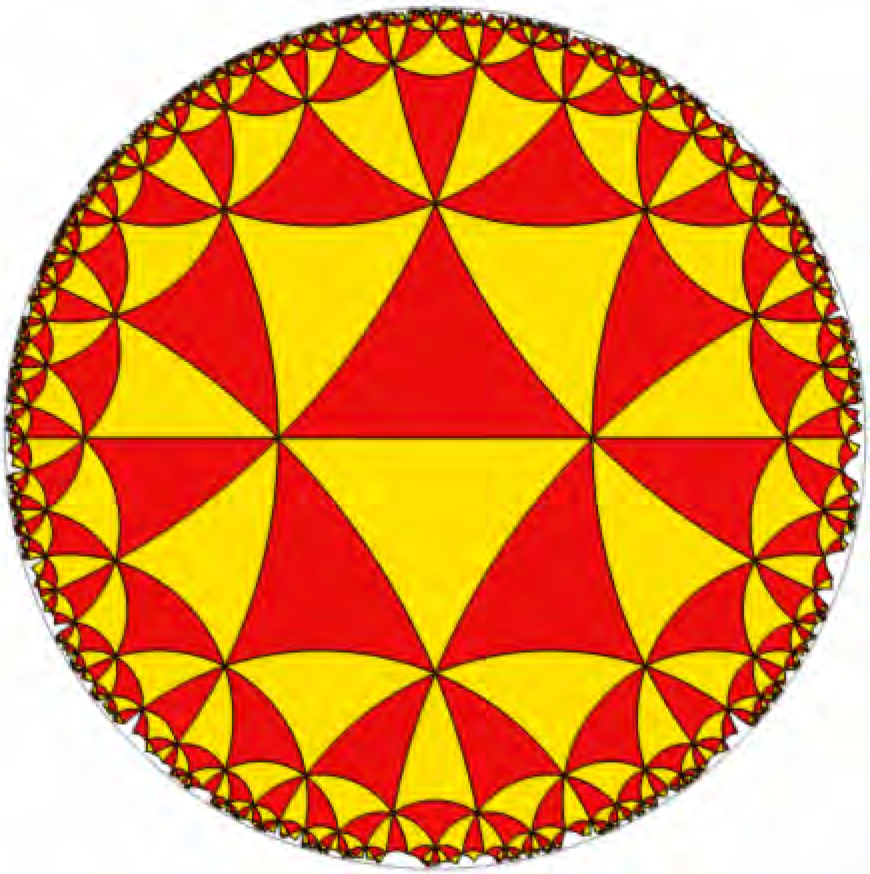
Hyperbolic geometry



Poincaré model of hyperbolic plane:

- conformal
- area distorted: all triangles equal
- hyperbolic line: circle

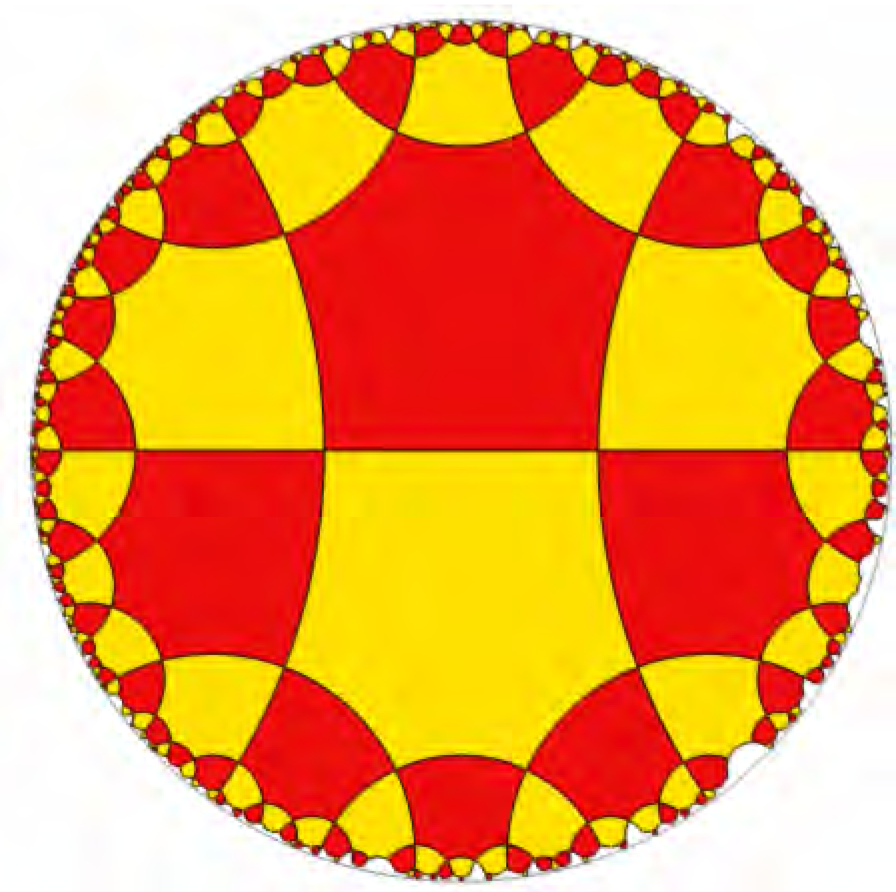
Tilings



{3, 8} tiling

hyperbolic plane

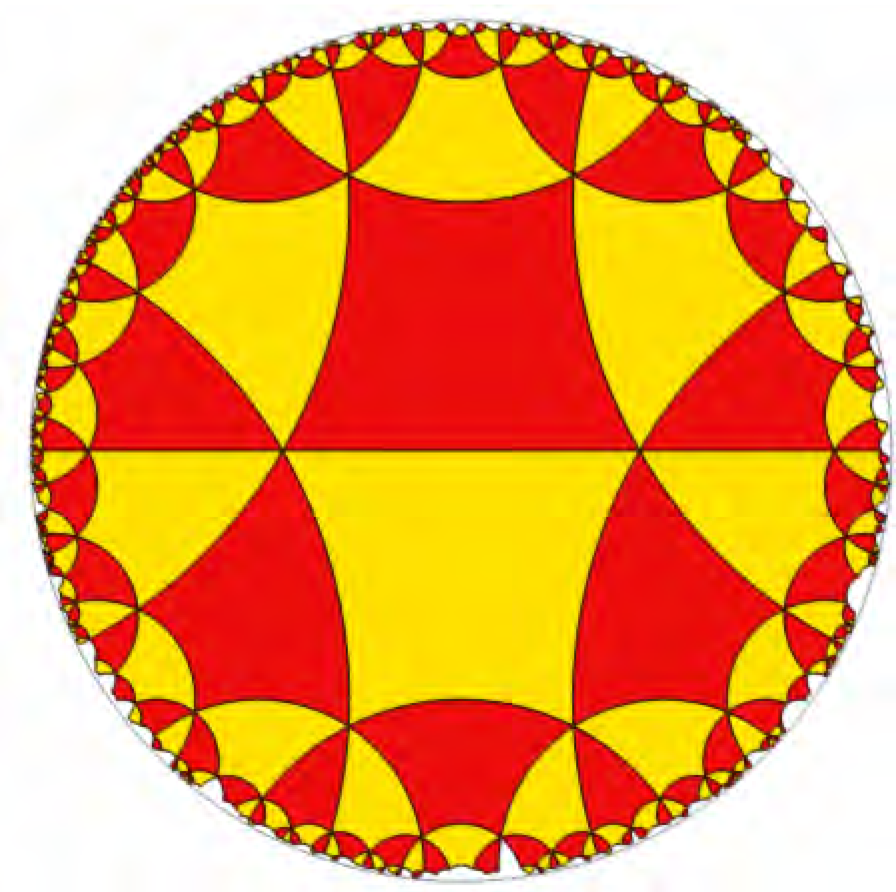
Tiling



{6, 4} tiling

hyperbolic plane

Tilings



{4, 6} tiling

hyperbolic plane

Regular maps

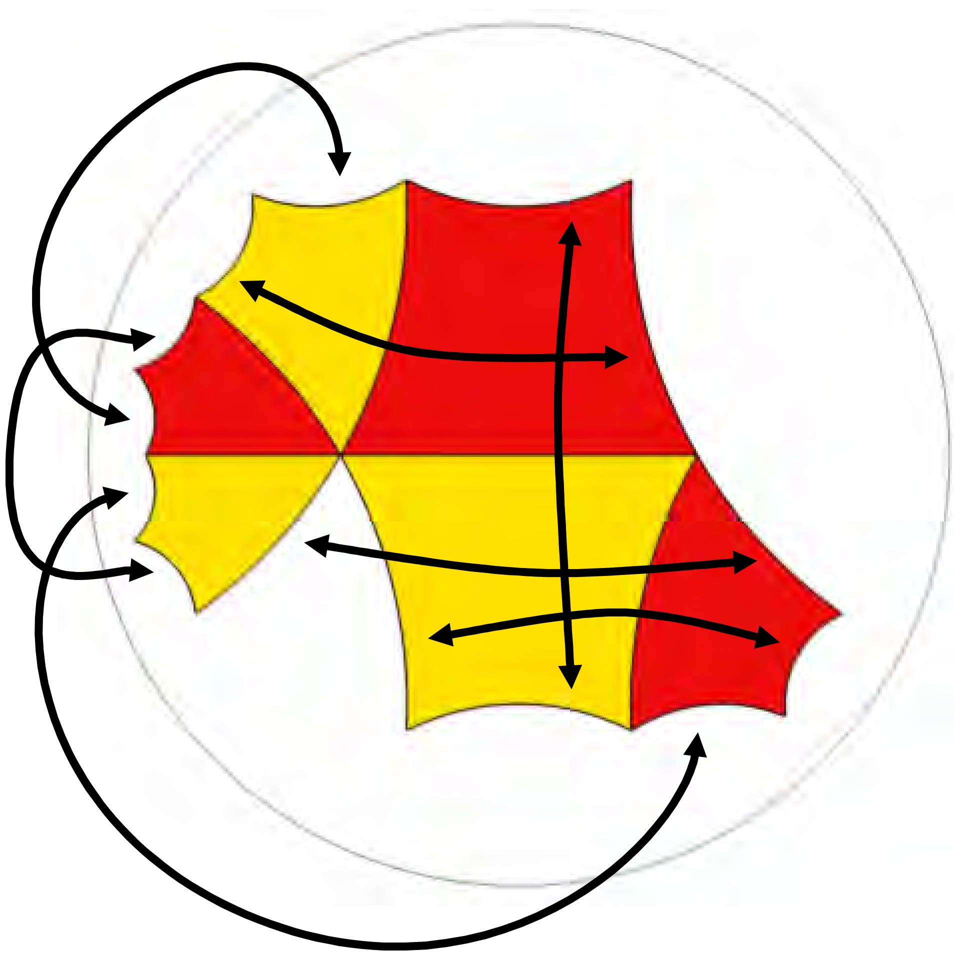
Regular map:

Cut out part of tiling hyperbolic plane

For instance:

6 quads

Regular maps



Regular map:

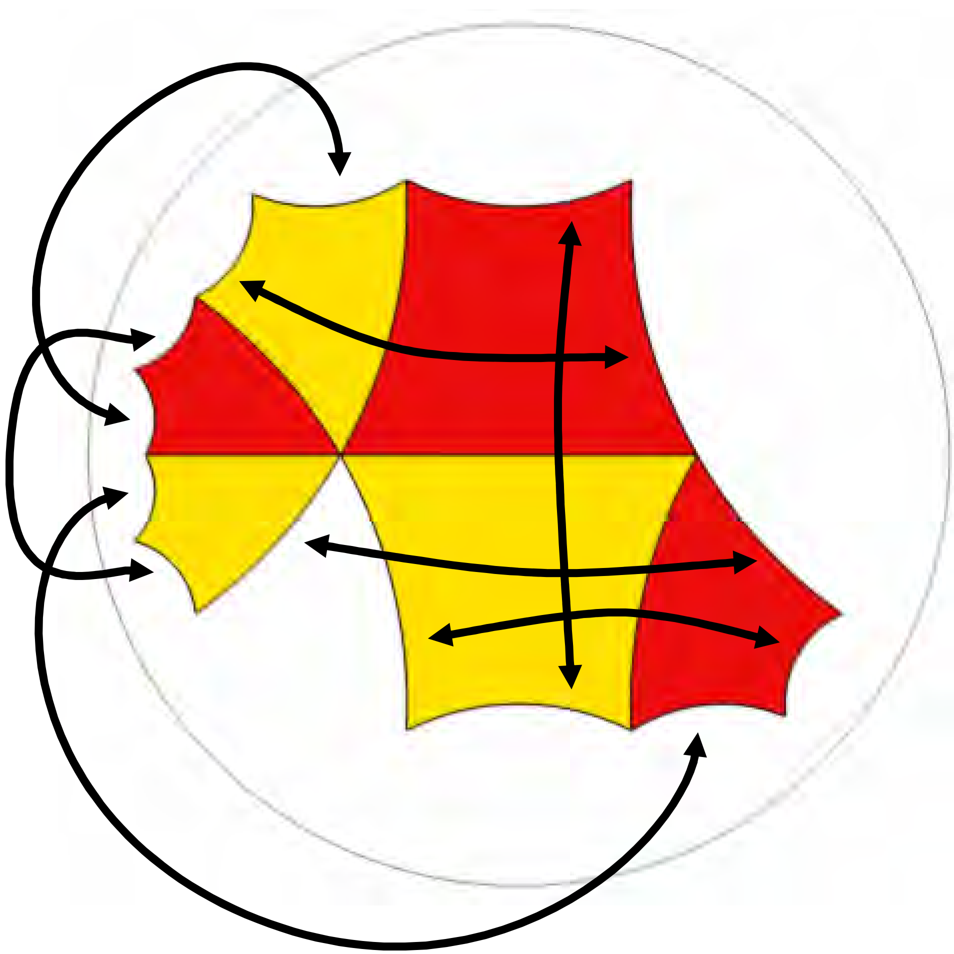
Cut out part of tiling hyperbolic plane

For instance:

6 quads,

and match edges

Regular maps



Regular map:

Cut out part of tiling hyperbolic plane

M. Conder (2006): enumerated all regular maps for $g \leq 101$

Conder's list

R2.1 : Type {3,8}_12 Order 96 mV = 2 mF = 1
Defining relations for automorphism group:
[T^2, R^-3, (R * S)^2, (R * T)^2, (S * T)^2, (R * S^-3)^2]

R2.2 : Type {4,6}_12 Order 48 mV = 3 mF = 2
Defining relations for automorphism group:
[T^2, R^4, (R * S)^2, (R * S^-1)^2, (R * T)^2, (S * T)^2, S^6]

R2.3 : Type {4,8}_8 Order 32 mV = 8 mF = 2
Defining relations for automorphism group:
[T^2, R^4, (R * S)^2, (R * S^-1)^2, (R * T)^2, (S * T)^2, S^-2 * R^2 * S^-2]

R2.4 : Type {5,10}_2 Order 20 mV = 10 mF = 5
Defining relations for automorphism group:
[T^2, S * R^2 * S, (R, S), (R * T)^2, (S * T)^2, R^-5]

……

R101.55 : Type {204,204}_2 Order 816 mV = 204 mF = 204 Self-dual
Defining relations for automorphism group:
[T^2, S * R^2 * S, (R, S), (R * T)^2, (S * T)^2, R^92 * S^-1 * R^3 * T * S^2 * T * R^16 * S^-89 * R]

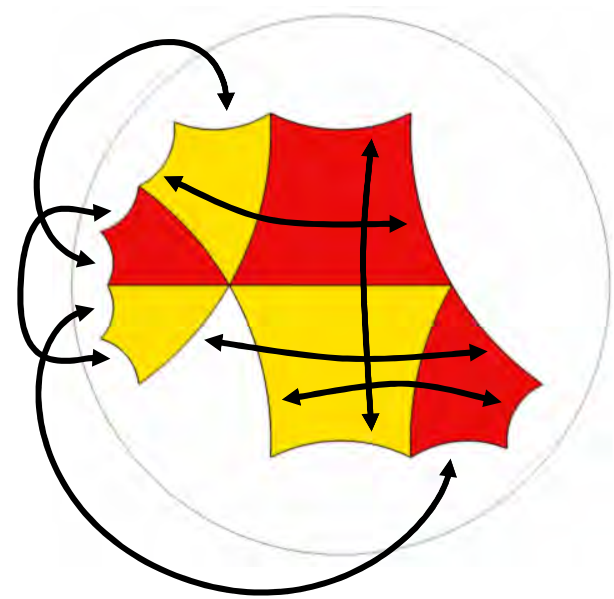
R101.56 : Type {404,404}_2 Order 808 mV = 404 mF = 404 Self-dual
Defining relations for automorphism group:
[T^2, S * R^2 * S, (R, S), (R * T)^2, (S * T)^2, R^98 * T * S^2 * T * R^10 * T * R^-3 * T * R^4 * S^-85]

Total number of maps in list above: 3378

Conder's list

R2.2 : Type {4,6}_12 Order 48 mV = 3 mF = 2
Defining relations for automorphism group:
[T^2, R^4, (R * S)^2, (R * S^-1)^2, (R * T)^2, (S * T)^2, S^6]

- Rg.i: genus g, member i
- complete definition topology (connectivity)
- combinatorial group theory



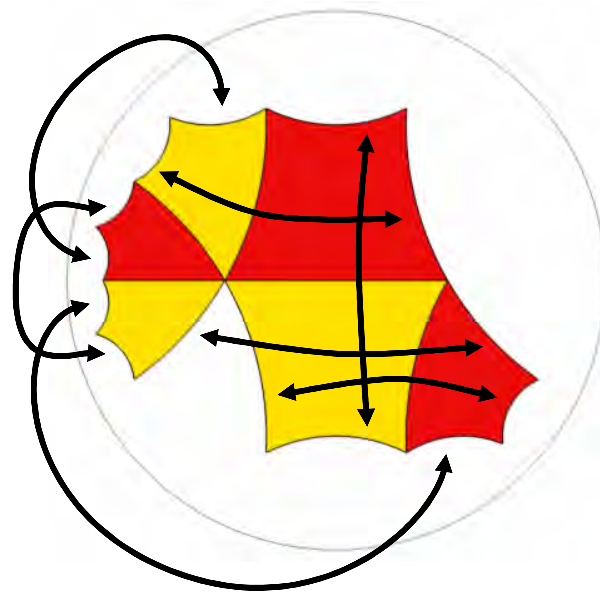
- No cue on possible geometry

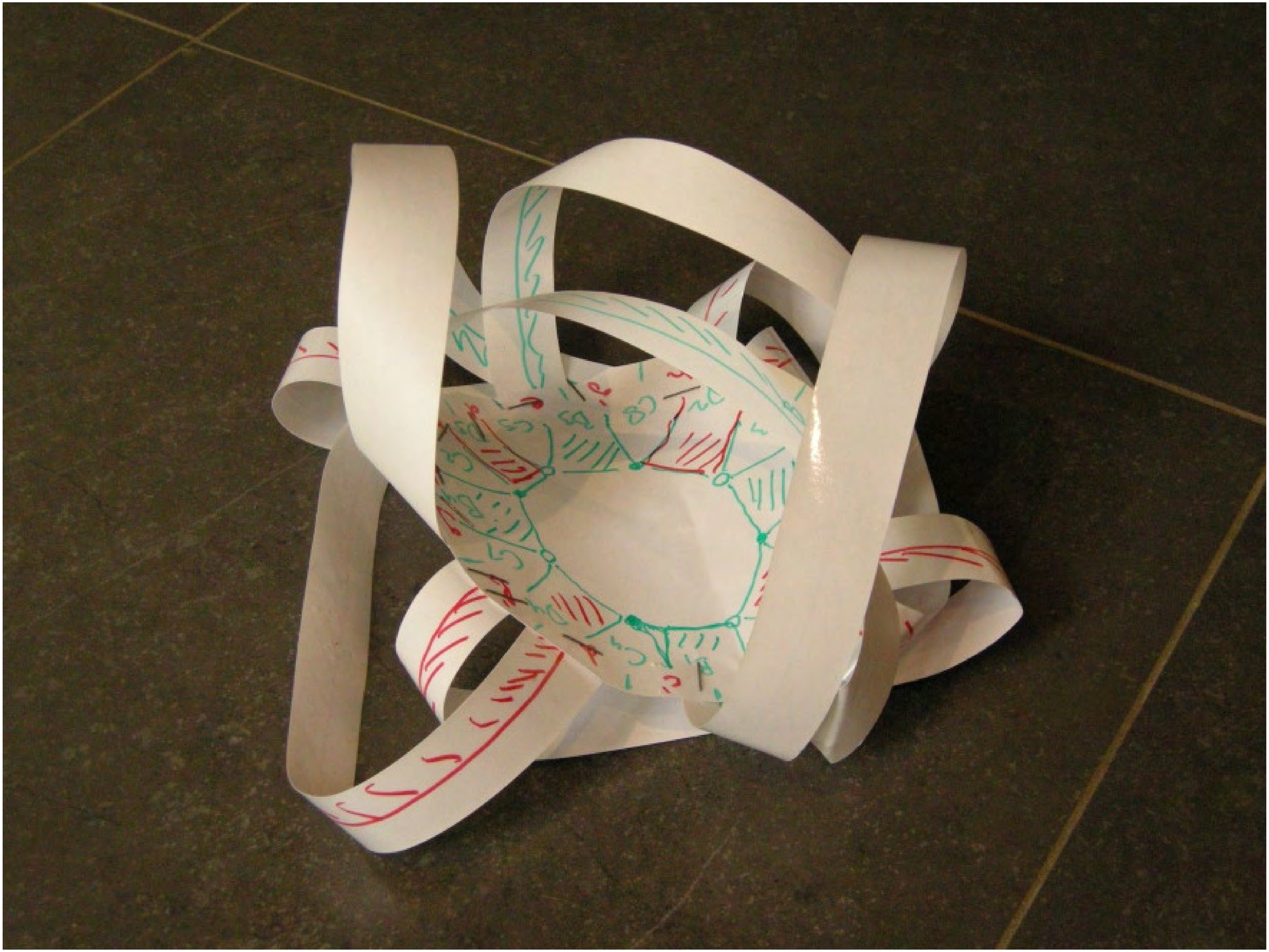
Conder's list

R2.2 : Type {4,6}_12 Order 48 mV = 3 mF = 2
Defining relations for automorphism group:
[T^2, R^4, (R * S)^2, (R * S^-1)^2, (R * T)^2, (S * T)^2, S^6]

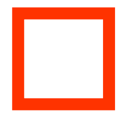
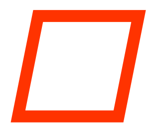
The challenge:

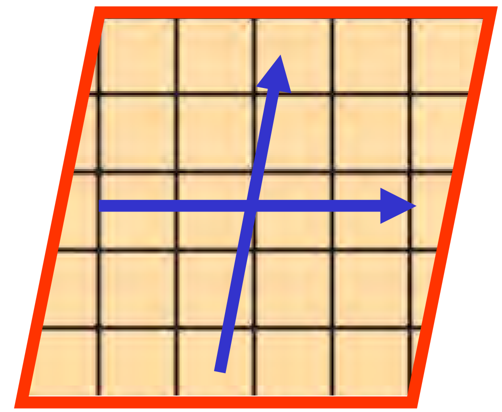
Given the complete topology, find a *space model*: an embedding of faces, edges and vertices in 3D space

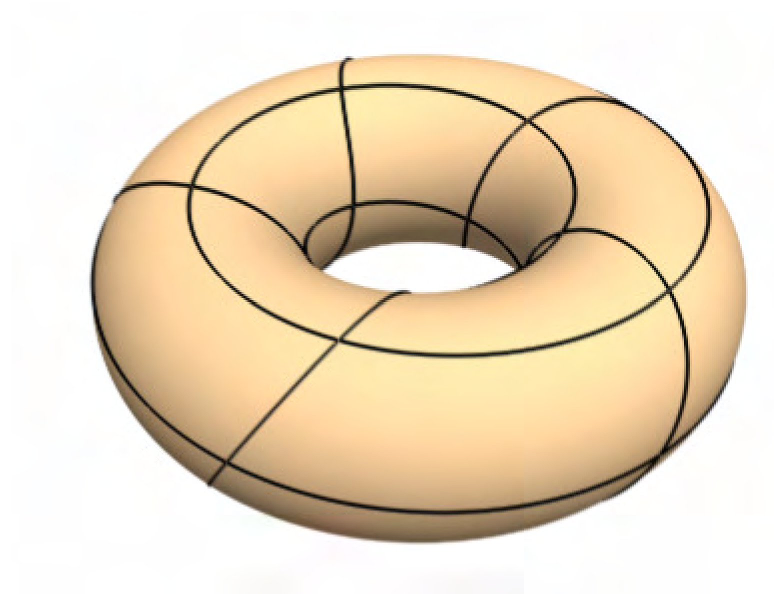




Tori (genus 1) (*reprise*)

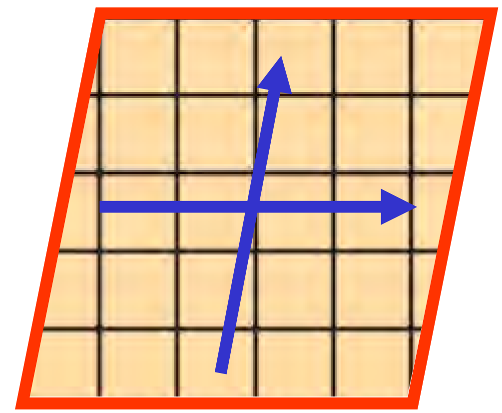
- Tile the plane
- Take a torus
- Unfold to square 
- Warp to a rhombus 
- Project tiling
- Map rhombus to torus

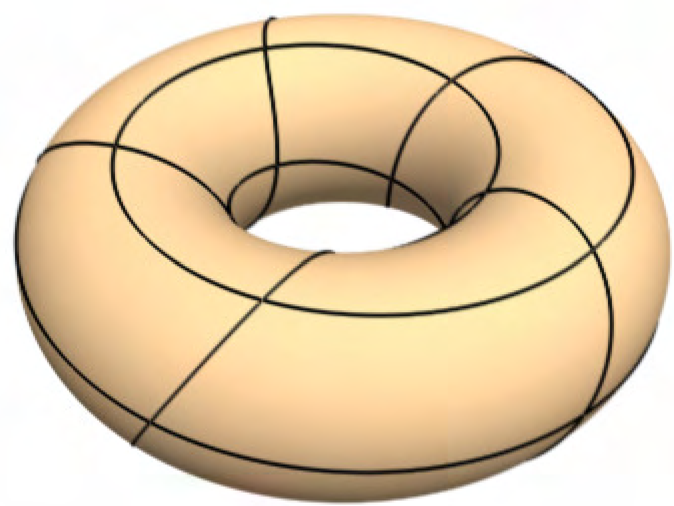




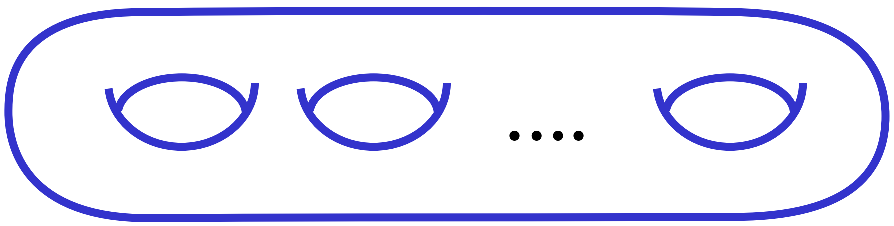
Approach for $g \geq 2$

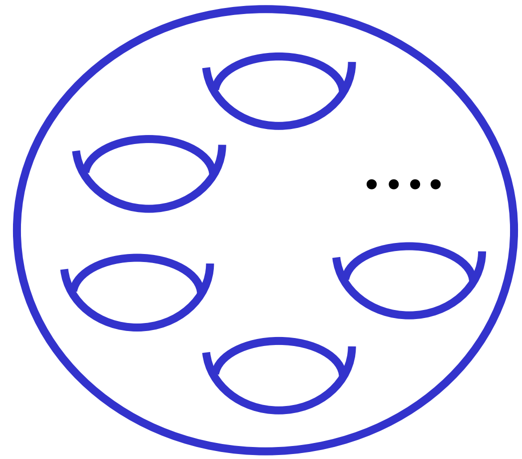
- Tile the hyperbolic plane
- Take a nice genus g shape
- Unfold to cut out
- Warp to match shape
- Project tiling
- Map cut out to nice shape



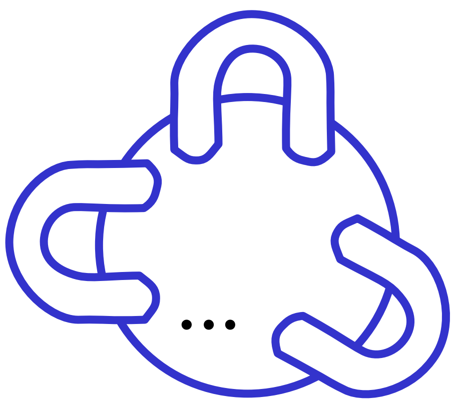


Nice genus g shape?





Solid shape with holes?



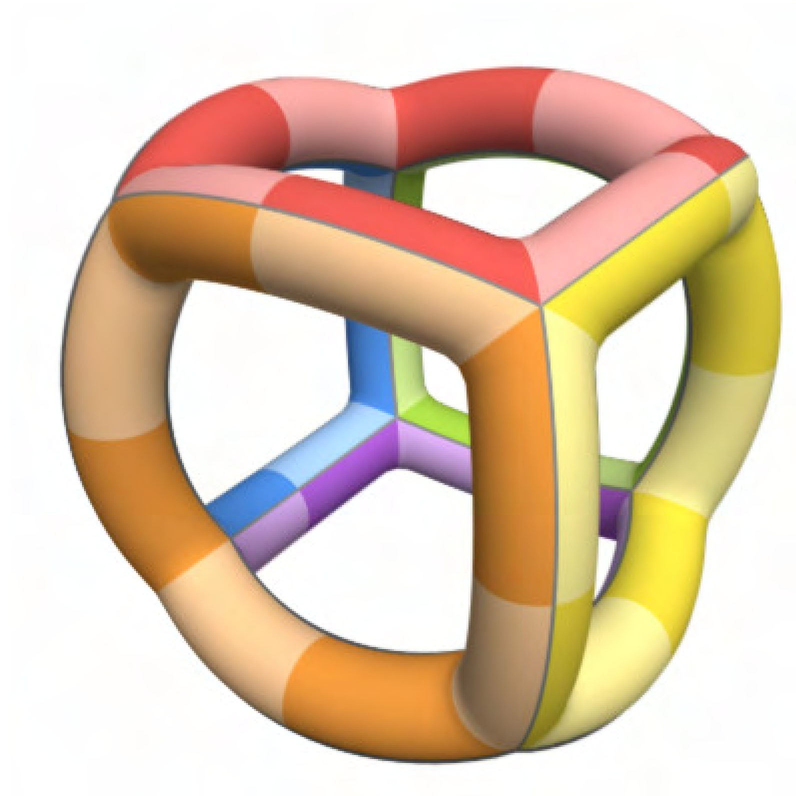
Sphere with handles?

Where to place holes or handles to get maximal symmetry?

For g = 6, 13, 17, …?

Tubified regular maps

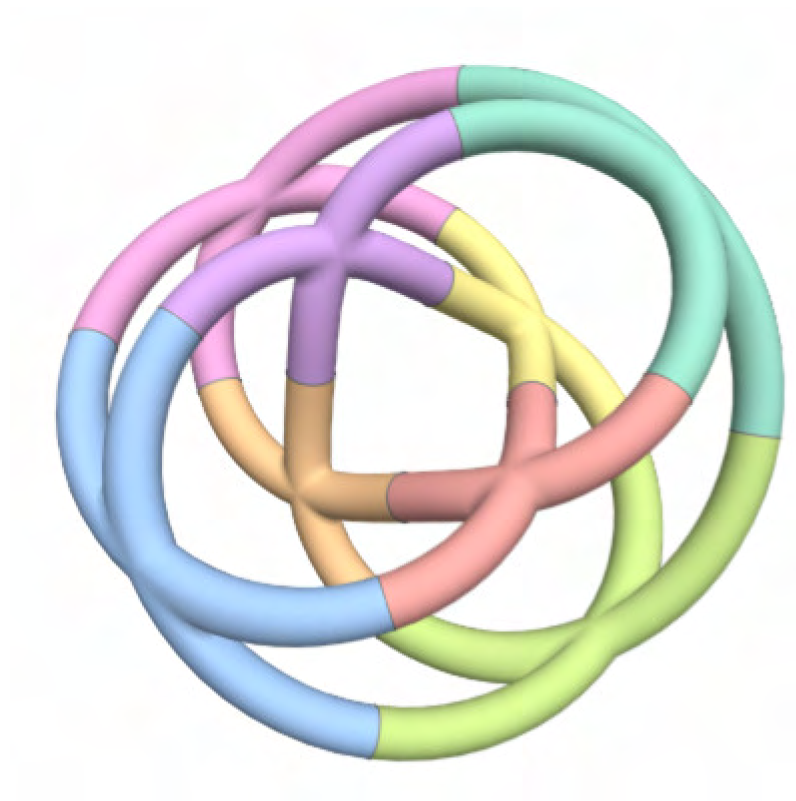
- Take a regular map
- Turn edges into tubes
- Remove faces



- edges → tubes
- vertices → junctions
- faces → holes
- triangles → ¼ tubes

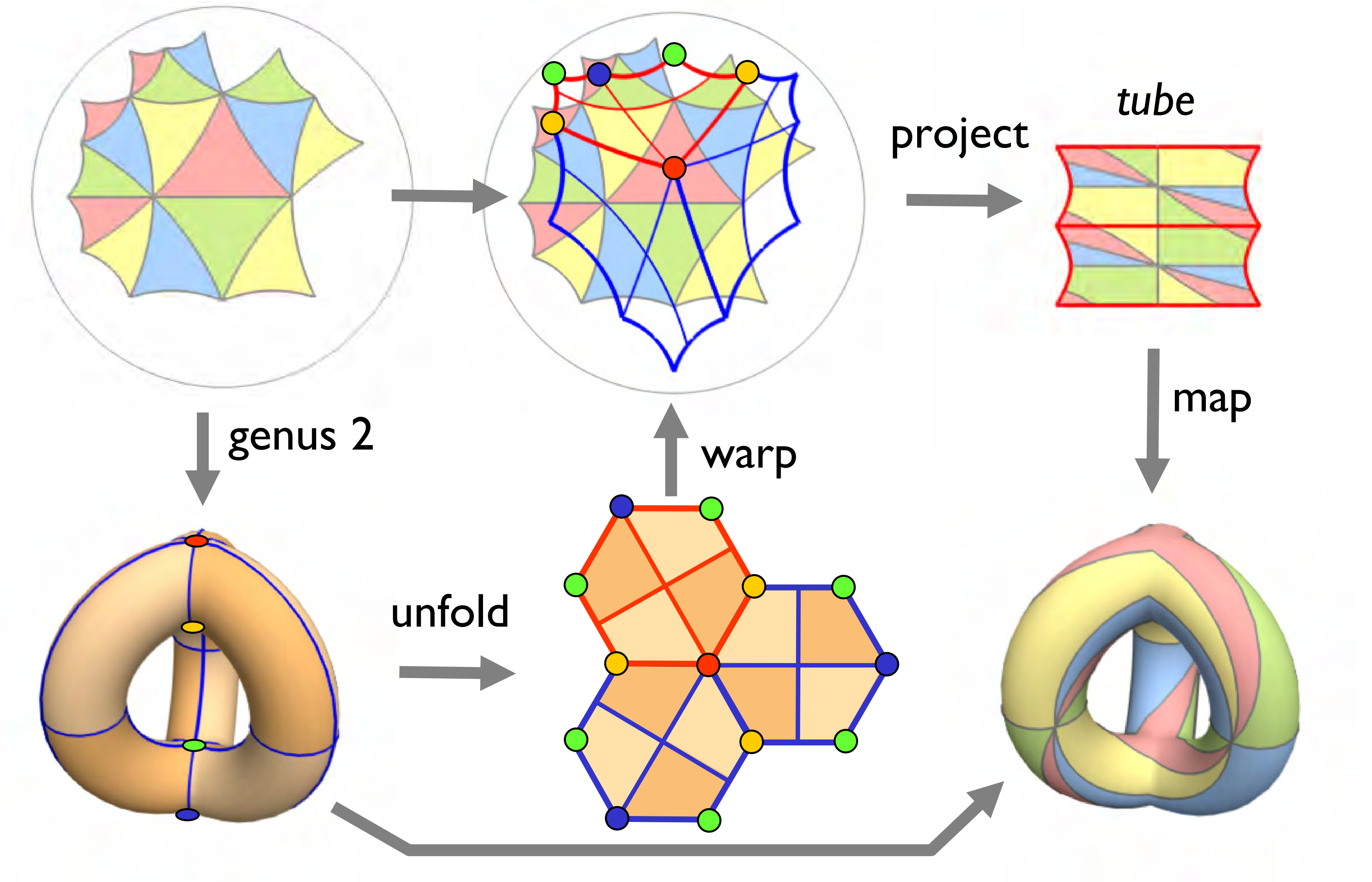
Tubified regular maps

- Take a regular map
- Turn edges into tubes
- Remove faces

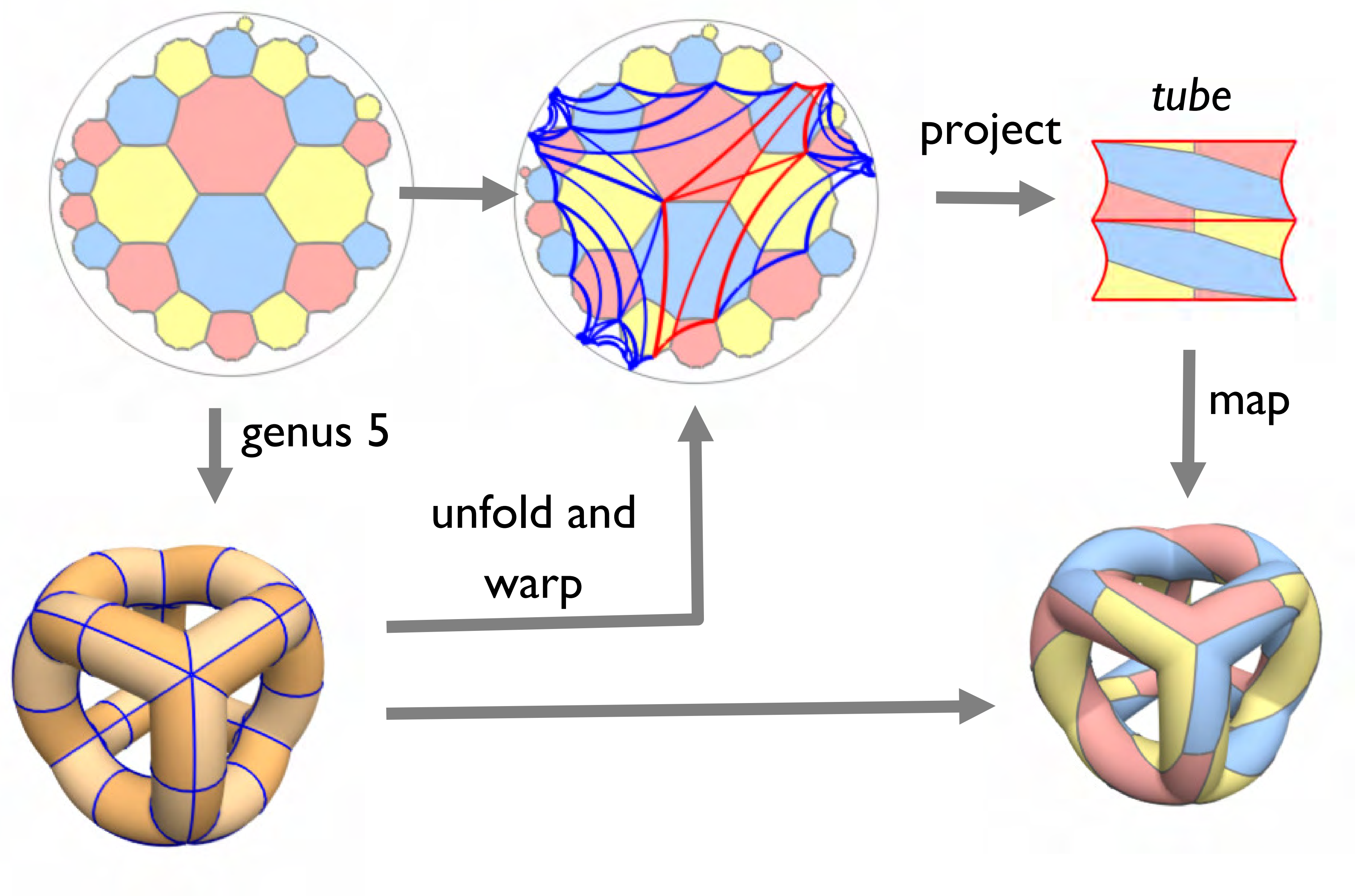


- edges $\rightarrow$ tubes
- vertices $\rightarrow$ junctions
- faces $\rightarrow$ holes
- triangles $\rightarrow$ ¼ tubes

Solving R2.1{3, 8}, 16 triangles



Solving R5.1{3, 8}, 24 octagons



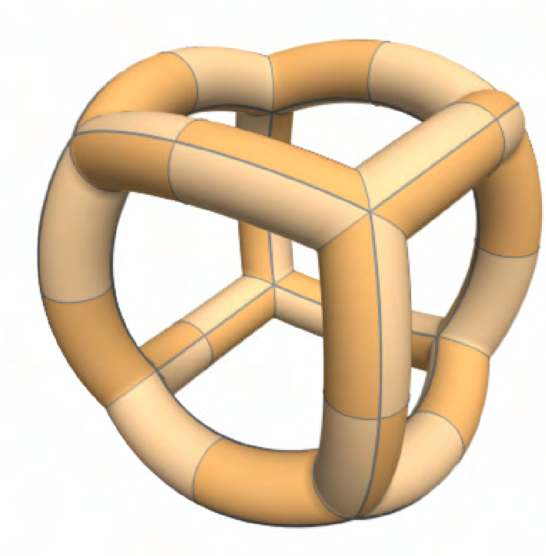
Symmetric Tiling of Closed Surfaces:

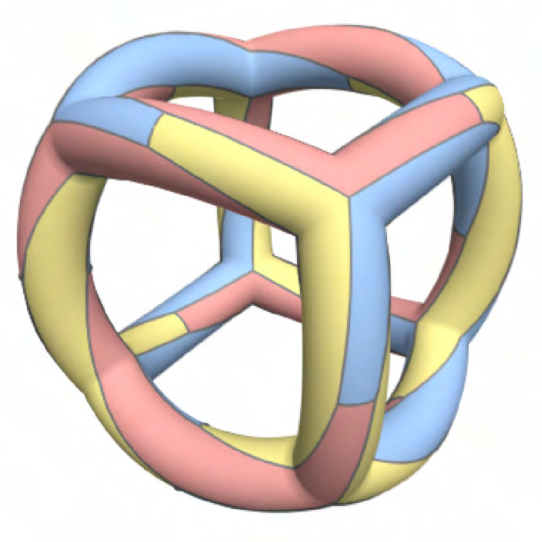
Visualization of Regular Maps

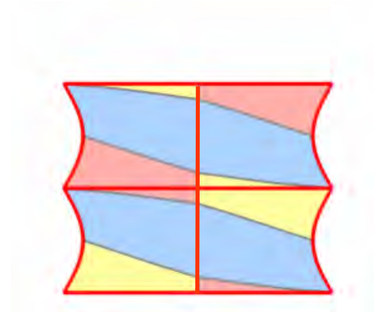
ACM SIGGRAPH 2009

Status

- About 50 different space models for regular maps found automatically
- Future work: solve more cases, by using less symmetric target shapes







4 fold symmetry

Finally

- Three puzzles: Knots, maps, and tiles
- Much more detail in the papers
- Thanks!