Quadratic Forms

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INTRODUCTION

A quadratic form over a commutative ring R is just a homogeneous polynomial of degree two in a number of variables. For example

$$X^2 + Y^2 - Z^2$$

is a quadratic form over $\mathbb{Z}$ (or $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{Q}_p$, $\mathbb{F}_q$) in 3 variables. Quadratic forms play very impotent roles in many branches of mathematics, especially in number theory and arithmetic algebraic geometry.

The goal of this seminar is to study the quadratic forms over finite fields $\mathbb{F}_q$, local fields $\mathbb{Q}_p$, $\mathbb{R}$, global fields $\mathbb{Q}$, and the integer ring $\mathbb{Z}$. We are going to classify the quadratic forms over these rings. The general method is to find invariants from the quadratic forms and then classify these quadratic forms up to these invariants.

1 FINITE FIELDS (15/04/2015)

Aim of the talk: INTRODUCE FINITE FIELDS.

DETAILS: Introduce the following:
- the characteristic of a field [Ser1, §1.1] or [Lang, pp. 89 Last paragraph];
- classification of finite fields [Ser1, §1.1, Theorem 1] or [Ko, pp. 55, Chapter III, Theorem 9];
- the multiplicative group of a finite field [Ser1, §1.2, Theorem 2].

2 ABSOLUTE VALUES AND p-ADIC FIELDS (22/04/2015)

Aim of the talk: INTRODUCE THE p-ADIC RINGS AND THE p-ADIC FIELDS .

DETAILS: Introduce the following:
- the p-adic absolute value on the rational numbers [Ko, Chapter I, §1, §2, pp. 1-2] or [Neu, Chapter II, §2];
- the notion Archimedean and non-archimedean absolute values [Ko, Chapter I, §2, pp. 3];
- the buiding up of the p-adic field $\mathbb{Q}_p$ [Ko, §4] or [Neu, Chapter II, §2, pp. 109 -111, Prop 2.2];
- the notion of $\mathbb{Z}_p$ via valuations [Neu, Prop 2.3].

3 p-ADIC RINGS AND p-ADIC FIELDS (29/04/2015)

Aim of the talk: INTRODUCE THE p-ADIC RINGS AND THE p-ADIC FIELDS IN TERMS OF PROJECTIVE LIMITS.

DETAILS: Introduce the following:
- the projective limit definition of $\mathbb{Z}_p$ and $\mathbb{Q}_p$ [Ser1, §1, Def 1, Def 2] or [Lang, pp. 50-51];
- make the projective limit definition more concepture via [Neu, Chapter II, pp. 100-104, Def 1.1, Prop 1.2, Prop 1.3].
- the properties of $\mathbb{Z}_p$ and $\mathbb{Q}_p$ [Ser1, §1, Prop 1, Prop 2, Prop 3, Prop 4].

In the end, point out that $\mathbb{Z}_p$ is a complete discrete valution ring with maximal ideal $p\mathbb{Z}_p$.

4 p-ADIC EQUATIONS AND HENSEL'S LEMMA (06/05/2015)

Aim of the talk: STUDY EQUATIONS AND HENSEL'S LEMMA OVER p-ADIC FIELDS.

DETAILS: Explain everyting in [Ser1, Chapter II, §2]. Note that [Ser1, Chapter II, §2.2, Corollary 1] is usually called Hensel's Lemma. Compare also with [Neu, Chapter II, pp. 129, 4.6].

5 THE STURCTURE OF $\mathbb{Q}_p^*$ (13/05/2015)

Aim of the talk: STUDY THE GROUP STRUCTURE OF $\mathbb{Q}_p^*$ AND DESCRIBE SQUARES IN IT.

DETAILS:

1. Explain verything in [Ser1, Chapter II, §3.1-3.2], the goal is Theorem 2.

2. Compare with [Neu, Chapter II, §, Prop 3.9].

3. Play with squres in $\mathbb{Q}_p^*$ [Ser1, Chapter II, §3.3].

6 HILBERT SYMBOL FOR LOCAL FIELDS (20/05/2015)

Aim of the talk: STUDY HILBERT SYMBOL FOR LOCAL FIELDS.

DETAILS: Explain everyting in [Ser1, Chapter III, §1].

7 HILBERT SYMBOL FOR GLOBAL FIELDS (27/05/2015)

Aim of the talk: STUDY HILBERT SYMBOL FOR GLOBAL FIELDS.

DETAILS: Explain everyting in [Ser1, Chapter III, §2].

8 GENERAL QUADRATIC FORMS (03/06/2015)

Aim of the talk: INTRODUCE THE GENERAL NOTION OF QUADRATIC FORMS.

DETAILS: Explain following things from [Ser1, Chapter IV, §1].

1. The notion of quadraic forms over a commutative ring: §1.1.

2. The notion orthogonal elements, orthogonal decomposition, non-degenerate quadratic forms from §1.2.

3. The notion of hyperbolic plane from §1.3.

4. Prove that every quadratic module has an orthogonal basis: Theorem 1 in §1.4.

5. Explain everything in §1.6.

9 QUADRATIC FORMS OVER $\mathbb{Q}_p$ AND $\mathbb{R}$ (10/06/2015)

Aim of the talk: CLASSIFY QUADRATIC FORMS OVER $\mathbb{Q}_p$ AND $\mathbb{R}$.

DETAILS: Explain everything in [Ser1, Chapter IV, §2].

10 QUADRATIC FORMS OVER $\mathbb{Q}$ (17/06/2014)

Aim of the talk: CLASSIFY QUADRATIC FORMS OVER $\mathbb{Q}$.

DETAILS: Explain everything in [Ser1, Chapter IV, §3].

11 QUADRATIC FORMS OVER $\mathbb{F}_q$ AND SUMS OF THREE SQUARES (24/06/2015)

Aim of the talk: CLASSIFY QUADRATIC FORMS OVER $\mathbb{F}_q$ AND APPLY THE CLASSIFICATION OF QUADRATIC FORMS OVER $\mathbb{Q}$ TO THE STUDY OF SUMS OF THREE SQUARES.

(1) For quadratic forms over $\mathbb{F}_q$ see [Ser1, Chapter IV, §1.7].

(2) For the study of sums of three squares see [Ser1, Appendix of Chapter IV].

12 PRELIMINARIES ON INTEGRAL QUADRATIC FORMS (01/07/2015)

Aim of the talk: INTRODUCE ALL THE NECESSARY INVARIANTS NEEDED FOR THE STUDY OF INTEGRAL QUADRATIC FORMS.

DETAILS: Follow [Ser1, Chapter V, §1] closely.

13 STATE THE CLASSIFICATION RESULTS FOR INTEGRAL QUADRATIC FORMS (08/07/2015)

Aim of the talk: STATE THE CLASSIFICATION RESULTS FOR INTEGRAL QUADRATIC FORMS.

DETAILS: Follow [Ser1, Chapter V, §2] closely.

14 FINISH THE PROOFS (15/07/2014)

Aim of the talk: FINISH ALL THE PROOFS LEFT FROM THE LAST TALK.

DETAILS: Explain everything in [Ser1, Chapter V, §3].

REFERENCES

[Ko]   N. Koblitz, *p-adic Numbers, p-adic Analysis and Zeta-Functions*, second edition, GTM 58, Springer-Verlag, 1984.

[Lang]   S. Lang,*Algebra*, GTM 211, Springer-Verlag, 2002.

[Neu]   J. Neukirch, *Algebraic Number Theory*, Springer-Verlag, 1999.

[Ser1]   J. -P. Serre, *A Course in Arithmetic*, GTM 7, Springer-Verlag, 1973.

[Ser2]   J. -P. Serre, *Local Fields*, GTM 67, Springer-Verlag, 1979.