Polytopes, tilings, and compact moduli of algebraic varieties

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Overview

1. A well known fact: correspondence
   lattice polytopes $\longleftrightarrow$ toric varieties in algebraic geometry.
2. A less known fact: correspondece
   lattice polytopes, tilings (finite or ∞ periodic) $\longleftrightarrow$
   algebraic varieties which are seemingly very far from toric:
   curves, abelian varieties, K3 surfaces, surfaces of general type, etc.

The polytopes and tilings appear naturally when one investigates the degenerations of varieties and compactifications of their moduli spaces.

Goal: to explain the correspondence (2), and what algebraic geometers could learn from experts on polytopes and tilings.

TORIC VARIETIES AND LATTICE POLYTOPES

Toric variety = algebraic variety X with group action by an algebraic torus $T = (\mathbb{C}^*)^n$ such that:

- X is normal,
- $T \subset X$ as the largest, dense orbit.

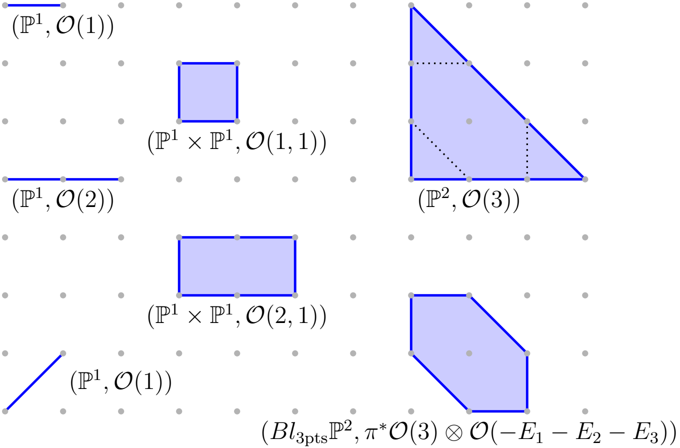
Two dual lattices:

- $M = \mathrm{Hom}(T, \mathbb{C}^*) \simeq \mathbb{Z}^n$ (characters, “monomials”)
- $N = \mathrm{Hom}(\mathbb{C}^*, T) \simeq \mathbb{Z}^n$ (1-parameter subgroups)

Toric geometry $\longleftrightarrow$ Polytopes in two mirror-symmetric ways:

- Projective toric variety (X, L) with an ample line bundle $\longleftrightarrow$ polytope Q with vertices in M (direct picture)
- Arbitrary toric variety $X \longleftrightarrow$ fan in $N \otimes \mathbb{Q}$ (inverted pic) E.g, the fan could be the cone over faces of a polytope.

POLYTOPES IN LATTICE M (DIRECT PICTURE)



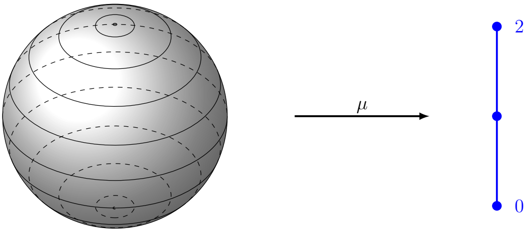
Polytope $Q = \sqcup$ (open faces). Variety $X = \sqcup$ $(T -$ orbits$)$.
Faces of $Q \longleftrightarrow$ T-orbits of X of the same dimension.

Polytope as the moment polytope

Polytope Q = image of the moment map $\mu_L : X \to \mathbb{R}^n$.

Example

$\mu_L : X = \mathbb{CP}^1 \to Q = [0, 2]$ for $L = \mathcal{O}(2)$.



Here, $X = \{x_0 x_2 = t x_1^2\}$, $t \neq 0$, and

$$\mu_L(x_0, x_1, x_2) = \frac{0 \cdot |x_0| + 1 \cdot |x_1| + 2 \cdot |x_2|}{\sqrt{|x_0|^2 + |x_1|^2 + |x_2|^2}}$$

From Combinatorics to Algebraic geometry

(Almost) *everything* about a projective toric variety is encoded in its polytope:

- Singularities (e.g. simplicial polytopes $\longleftrightarrow$ varieties with abelian quotient singularities)
- Divisors and line bundles
- Sheaves of differential forms, canonical class
- Kähler-Einsten metrics, ...

So, (almost) any algebro-geometric question about X becomes a purely combinatorial question about the polytope Q.

From Algebraic geometry to Combinatorics

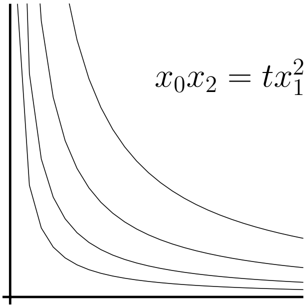
- Proof of Upper Bound Conjecture for the number of faces of different dimension of a simplicial polytope by McMullen (1970) and Stanley (1975).
  Stanley's proof uses Hard Lefshetz Theorem for cohomology of algebraic varieties.
- Using Riemann-Roch Theorem to compute integrals of polynomial functions over polytopes by Khovanskii-Pukhlikov (1993).

DEGENERATIONS OF TORIC VARIETIES AND TILINGS

Example

A family $X_t = \{x_0 x_2 = t x_1^2\}$ in $\mathbb{P}^2 \times \mathbb{A}^1_t$, $t \to 0$. For $t \neq 0$,
$X_t = \mathbb{P}^1 = (z_0 : z_1) \mapsto (x_0 : x_1 : x_2)$, $x_0 = z_0^2$, $x_1 = z_0 z_1$, $x_2 = t \cdot z_1^2$.

For $t \neq 0$, $(X_t, L_t) = (\mathbb{P}^1, \mathcal{O}(2))$. $\quad X_0 = \mathbb{P}^1 \cup \mathbb{P}^1$.



DEGENERATION OF THE MOMENT MAP

Example

A family $X_t = \{x_0x_2 = tx_1^2\}$ in $\mathbb{P}^2 \times \mathbb{A}^1_t$, $t \to 0$. For $t \neq 0$,
$X_t = \mathbb{P}^1 = (z_0 : z_1) \mapsto (x_0 : x_1 : x_2)$, $x_0 = z_0^2$, $x_1 = z_0z_1$, $x_2 = t \cdot z_1^2$.

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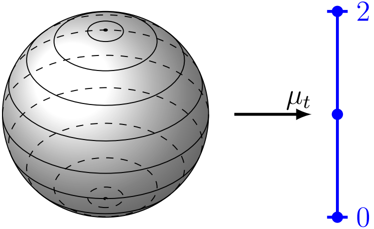

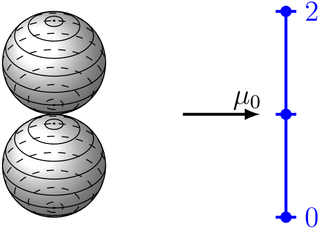

Degenerations and height functions

Example

A family $X_t = \{x_0 x_2 = t x_1^2\}$ in $\mathbb{P}^2 \times \mathbb{A}^1_t$, $t \to 0$. For $t \neq 0$,
$X_t = \mathbb{P}^1 = (z_0 : z_1) \mapsto (x_0 : x_1 : x_2)$, $x_0 = z_0^2$, $x_1 = z_0 z_1$, $x_2 = t \cdot z_1^2$.

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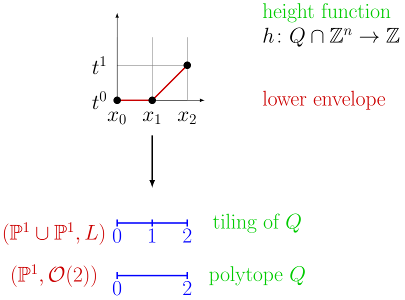


Height functions and secondary fan

- Let: X toric variety, $f\colon X \to \mathbb{P}^N$ finite T-equivariant map, $L = f^*\mathcal{O}_{\mathbb{P}^N}(1)$.

$$H^0(X, L) = \oplus_{m \in Q \cap \mathbb{Z}^n} \mathbb{C} z^m, \quad x_m \mapsto c_m z^m$$

- Family $f_t\colon X_t \to \mathbb{P}^N$, $t \to 0$, $X_t \simeq X$ for $t \neq 0$ gives

$$c_m(t) = t^{h(m)} c'_m(t), \quad c'_m(0) \neq 0$$

- $\rightsquigarrow$ height function $h\colon Q \cap \mathbb{Z}^n \to \mathbb{Z}$. The lower convex envelope of the points $(m, h(m))$ determines a convex tiling of Q by lattice polytopes, also the degeneration X_0.
- Define $h \sim h'$ if $\mathrm{Tiling}(h) = \mathrm{Tiling}(h')$. This divides all height functions into equivalence classes and gives a fan on $\mathbb{R}^N / \mathbb{R}^{1+\dim Q}$, where $N = \# Q \cap \mathbb{Z}^n$.
- $=$ secondary fan of Gelfand-Kapranov-Zelevinsky, normal fan of the secondary polytope $\Sigma(Q)$.

Corollary

The poset of convex tilings = the poset of faces of the polytope $\Sigma(Q)$, and so is homeomorphic to a sphere.

Definition

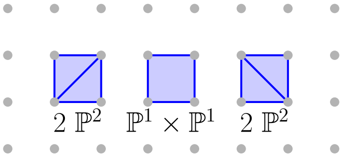
Stable toric variety = seminormal union $T \curvearrowright X = \cup X_j$ of toric varieties. STV over $\mathbb{P}^N$: finite T-map $f: X \to \mathbb{P}^N$.

Theorem (VA'02, VA-Brion'06)

For any polytope Q, there exists a projective moduli space $\overline{\mathrm{M}}_Q = \{f: X \to \mathbb{P}^N\}$ of STVs over $\mathbb{P}^N$ "of numerical type Q".
Strata of $\overline{\mathrm{M}}_Z \longleftrightarrow$ tilings of Q.
Strata of the main irr component of $\overline{\mathrm{M}}_Z \longleftrightarrow$ convex tilings of Q.

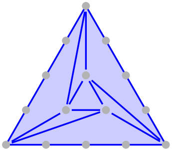
Example

The quadric $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ and its degenerations.

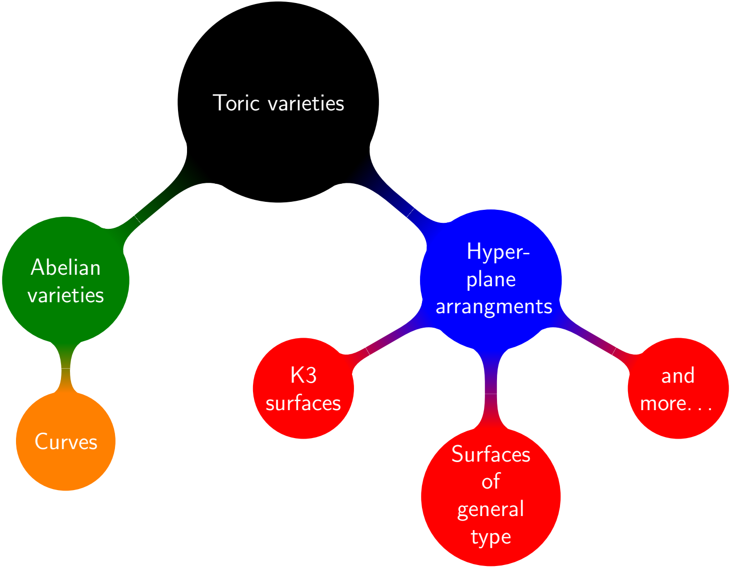


Example

Noncovex tiling $\implies$ extra irreducible components in the moduli space $\overline{\mathrm{M}}_Q$.



Beyond toric varieties



Abelian varieties

- Abelian variety A = smooth connected projective algebraic variety with an algebraic group structure. Comes with a point $0 \in A$.
- Over $\mathbb{C}$, $A = \mathbb{C}^g/\mathbb{Z}^{2g} = \mathbb{C}^g/(I_{g\times g}, \Omega) = (\mathbb{C}^*)^g/\exp(2\pi i\,\Omega)$. Here, $\Omega \in \mathrm{Mat}_{g\times g}(\mathbb{C})$ is the matrix of periods.
- Polarization on X is an ample line bundle L (up to $\underset{\mathrm{alg}}{\sim}$)

  Degree of polarization $d := h^0(A, L) = L^g/g! \in \mathbb{N}$.
  Principal polarization: $d = 1$.
- For principally polarized abelian variety (PPAV), there is a choice of period vectors in $\mathbb{C}^g$ such that $\Omega^t = \Omega$ and $\mathrm{Im}\,\Omega > 0$.
- Abelian torsor = projective variety X with action $A \curvearrowright X$, a principal homogeneous space over A. No special point $x \in X$.
- Abelian variety A with principal polarization $\overset{1:1}{\longleftrightarrow}$ abelian torsor (X, Θ) with a divisor.

DEGENERATIONS OF AV'S AND PERIODIC TILINGS

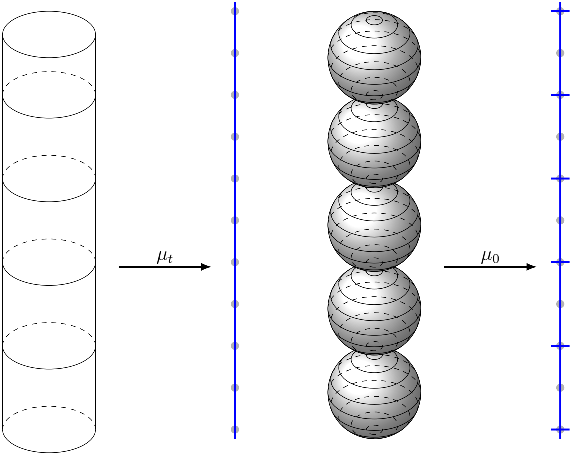
Consider a family $A_t \curvearrowright X_t \supset \Theta_t$, $t \in \mathbb{A}^1$. Suppose:

- For $t \neq 0$, A_t is an abelian variety, X_t abelian torsor.
- As $t \to 0$, $\quad \Omega_t \to 0_{g\times g}$.

Then

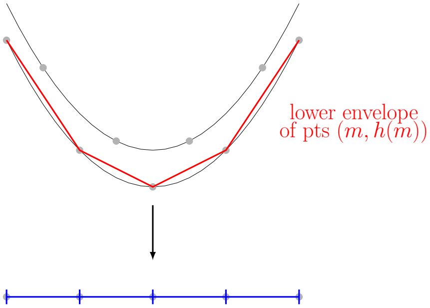
- $A_t = (\mathbb{C}^*)^g / \exp(2\pi i\, \Omega_t) \to A_0 = (\mathbb{C}^*)^g$, a torus T.
- $\big(A_t \curvearrowright X_t \supset \Theta_t\big) \to \big(A_0 \curvearrowright X_0 \supset \Theta_0\big)$,
  a projective variety with T-action.
  X_0 is a kind of "toric" variety.

EXAMPLE: DEGENERATION OF ELLIPTIC CURVES



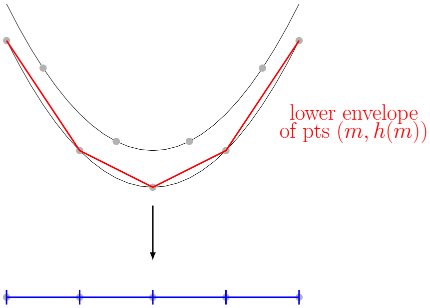
For $t \neq 0$, $X_t = \mathbb{C}^*/\mathbb{Z}$ = elliptic curve.
For $t = 0$, X_0 = $\mathbb{P}^1$ with poles identified, a rational nodal curve.
$\longleftrightarrow$ to the $\mathbb{Z}$-periodic tiling of $\mathbb{R}$ into intervals $[n, n+1]$.

QUADRATIC HEIGHT FUNCTIONS AND ∞-ANALOGUE OF SECONDARY FAN



Consider

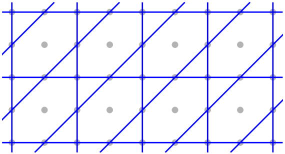
- two lattices $L' \subset L = \mathbb{Z}^g$ with finite L/L'
- semi positive definite quadratic form $q\colon L \to \mathbb{R}$
- function $h\colon L \to \mathbb{R}$ such that $h(m) = q(m) + \bar{r}(m)$, where $\bar{r}(m)$ only depends on $r \mod L' \in L/L'$.



- The convex hull (lower envelope) of the points $(m, f(m))$, projected to L, defines a convex tiling of $\mathbb{R}^g$ periodic w.r.t. L' into polyhedra with vertices in L'.
- All such height functions mod constants are devided into equivalence classes $h \sim h'$ if they give the same tiling of $\mathbb{R}^g$
- $\rightsquigarrow$ get the fan $\mathrm{Fan}(g, L/L')$ of dimension $\frac{g(g+1)}{2} + |L/L'| - 1$.

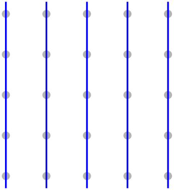
Example

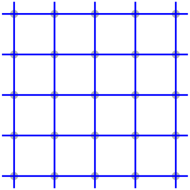
A periodic subdivision for $g = 2$ and $L' = 2L$.

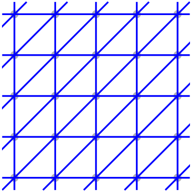


Example

All periodic tilings for $g = 2$ with $L' = L$.







COMPACTIFIED MODULI OF ABELIAN VARIETIES

Theorem (VA'02)

$\exists$ *moduli* $\overline{\mathrm{AP}}_{g,d}$ *of stable semiabelic pair, compactifying the moduli space of polarized abelian torsors* (A, Θ) *of degree* d.

$\dim \overline{\mathrm{AP}}_{g,d} = \frac{g(g+1)}{2} + d - 1$.

strata of $\overline{\mathrm{AP}}_{g,d} \longleftrightarrow$ *periodic tilings of* $\mathbb{R}^g$ *with* $|L/L'| = d$

strata of the main irr comp of $\overline{\mathrm{AP}}_{g,d} \longleftrightarrow$ *convex periodic tilings*

In particular, when $d = |L/L'| = 1$, $\overline{\mathrm{AP}}_{g,1}$ compactifies the moduli space A_g of principally polarized abelian varieties.

Convex tilings and the fan

When $L' = L$:

- convex tilings = Delaunay tilings,
- $\mathrm{Fan}(g, \{1\})$ = 2nd Voronoi fan = L-type decomposition = Delaunay-Voronoi fan (cf. Voronoi '1908)

Question

Describe convex periodic tilings and the the fan $\mathrm{Fan}(g, L/L')$ when $d = |L/L'| \neq 1$, at least for low g.

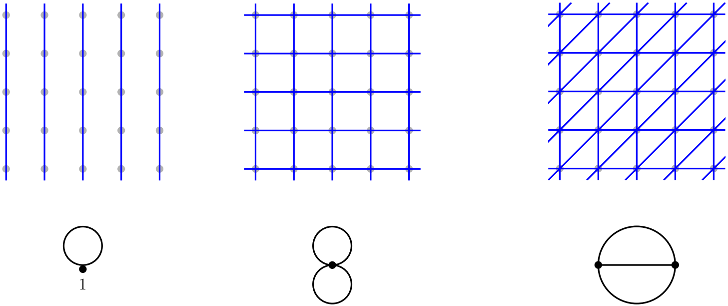
$\mathrm{Fan}(g, \mathbb{Z}_2^g)$, i.e. with $L' = 2L$, is especially important for applications), corresponds to degenerations of abelian varieties with twice the principal polarization.

CURVES AND TORELLI MAP

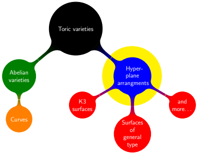
- Degenerations of curves are described by graphs
- Torelli map $\mathrm{M}_g \to \mathrm{A}_g$, curve $C \mapsto$ Jacobian JC, a principally polarized abelian variety
- extends to compactifications $\overline{\mathrm{M}}_g \to \overline{\mathrm{A}}_g^{\mathrm{2ndVor}}$
- Torelli map near the boundary of moduli space is described by:
  graph $\Gamma \mapsto$ cographic regular matroid $\{f_i \in (\mathbb{Z}^g)^* \mapsto$ dicing of $\mathbb{R}^g$ by systems of hyperplanes $\{f_i(x) = n_i \in \mathbb{Z}\}$.
- regular matroid = matroid which can be defined over field of arbitrary characteristic. By Seymour, all regular matroids are: graphic, cographic, R_5, and their "amalgamations".

Example

Cographic dicings for $g = 2$ corresponding to graphs.



HYPERPLANE ARRANGEMENTS

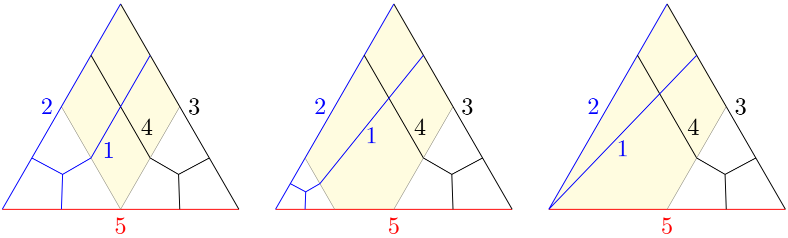


- h.a.: $(\mathbb{P}^{r-1}, B_1, \dots, B_n)$, B_i are hyperplanes
- dually: n vectors in $\mathbb{C}^n$, a realizable matroid
- Up to isomorphism, so mod $\mathrm{PGL}(r)$. First $r+1$ hyperplanes can be fixed, $(n-r-1)(r-1)$ parameters remaining.
- Will consider with weights $\beta = (b_1, \dots, b_n)$, $0 < b_i \le 1$. Then $(\mathbb{P}^{r-1}, \sum b_i B_i)$ is log canonical if for any $I \subset \{1, \dots, n\}$ one has
$$\sum_{i \in I} b_i \le \operatorname{codim} \cap_{i \in I} B_i.$$

Degenerations and moduli

What happens if you have a family $(\mathbb{P}^{r-1}, \sum b_i B_i)_t$ that degenerates as $t \to 0$? In the limit $\mathbb{P}^{r-1}$ splits up into several irr components $X = \cup X_j$, and $(X, \sum b_i B_i)$ is a stable pair.

- For curves, i.e. $(\mathbb{P}^1, \sum b_i B_i)$ one gets $\overline{\mathrm{M}}_{0,n}$ or $\overline{\mathrm{M}}_{0,\beta}$, the moduli space of stable n-pointed curves of genus 0.
- In higher dimension $r - 1 \geq 2$, one gets an analogous compact moduli space $\overline{\mathrm{M}}_\beta(r, n)$ of weighted stable hyperplane arrangements.



WHERE ARE TORIC VARIETIES?

- related to toric varieties in grassmannian

$$T = (\mathbb{C}^*)^n / \operatorname{diag} \mathbb{C}^* \curvearrowright G(r,n) = \{V^r \subset \mathbb{C}^n\}$$

- toric variety $Y \subset G(r,n)$ is $Y = \overline{T.[V]}$.
  $\rightsquigarrow$ h.a. $(\mathbb{P}^{r-1} = \mathbb{P}V, B_i = \mathbb{P}V \cap \{z_i = 0\}$
  (Gelfand-McPherson correspondence)
- For a generic h.a., the moment polytope is hypersimplex

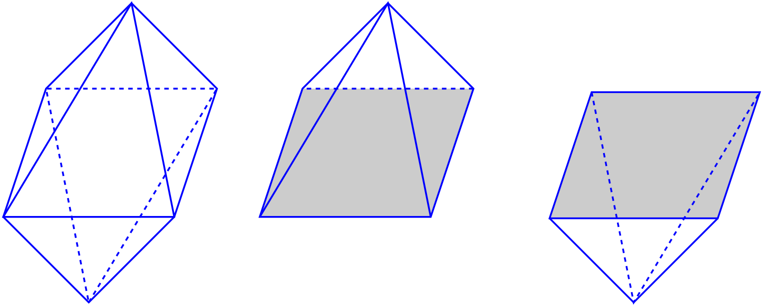
$$\begin{aligned}\Delta(r,n) &= \{(x_i) \in \mathbb{R}^n \mid 0 \le x_i \le 1, \sum x_i = r\} \\ &= \operatorname{Conv}\{(1^r, 0^{n-r})\}\end{aligned}$$

- For arbitrary h.a., get matroid polytopes

$$\begin{aligned}Q_V &= \{(x_i) \in \mathbb{R}^n \mid \forall I \subset \bar{n}, \sum_{i \in I} \le \operatorname{codim} \cap_{i \in I} B_i, \sum x_i = r\} \\ &= \operatorname{Conv}\{(1_I, 0_{I^c}) \mid \cap_{i \in I} B_i = \emptyset\}\end{aligned}$$

Example

Matroid polytopes in $\Delta(2,4)$.



Combinatorial structure of $\overline{\mathrm{M}}_\beta(r,n)$

Theorem (HKT'05, VA'08)

For all $r, n, \beta = (b_1, \dots, b_n)$, there exists a projective moduli space $\overline{\mathrm{M}}_\beta(r,n)$ of stable weighted hyperplane arrangements $(X, \sum b_i B_i)$

strata of $\overline{\mathrm{M}}_\beta(r,n) \longleftrightarrow$ tilings of cut hypersimplex $\Delta_\beta(r,n)$ by matroid polytopes

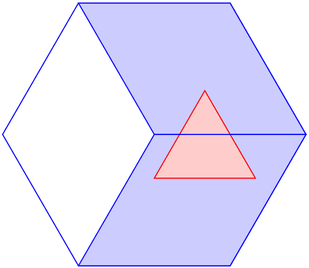
strata of the main irr comp of $\overline{\mathrm{M}}_\beta(r,n) \longleftrightarrow$ convex tilings of $\Delta_\beta(r,n)$ by matroid polytopes

Here, cut hypersimplex is defined as

$$\Delta_\beta(r,n) = \{(x_i) \in \mathbb{R}^n \mid 0 \le x_i \le b_i, \sum x_i = r\}$$

Example

A cover of the cut hypersimplex $\Delta_\beta(r,n)$ by matroid polytopes in $\Delta(r,n)$.



Question: What is the structure of the poset of convex tilings of cut hypersimplex $\Delta_\beta(r,n)$ by matroid polytopes?
(For $r = 2$ and $\beta = (1,\dots,1)$, this is the same as tropical $\overline{\mathrm{M}}_{0,n}$ = the space of "phylogenetic trees", and the answer is known for low n.)
Question: How are Poset(β_1) and Poset(β_2) related for $\beta_1 > \beta_2$?

GENERALIZED GROMOV-WITTEN INVARIANTS

- GW invariants are defined using moduli spaces of stable curves $\overline{\mathrm{M}}_{g,n}$ and of stable maps from curves to other varieties $\overline{\mathrm{M}}_{g,n}(V, \gamma)$.
- Speculatively, "higher" GW invariants could be defined by using moduli $\overline{\mathrm{M}}$ of higher-dimensional pairs (X, B) and of maps $f\colon (X, B) \to V$.
- The first really large and computable collection of such higher-dimensional moduli spaces is $\overline{\mathrm{M}}_{\beta}(r, n)$.
- Recall: $\overline{\mathrm{M}}_{0,n} = \overline{\mathrm{M}}_{1,\dots,1}(2, n)$
- Work is being done in this direction…

Applications of hyperplane arrangements to other varieties

- Through Galois covers $X \to \mathbb{P}^{r-1}$ ramified in a collection of hyperplanes $B_1, \dots, B_n$.
- For this, need to work with weights b_i such as $\frac{1}{2}$ or $\frac{2}{3}$.

SURFACES AND TILINGS

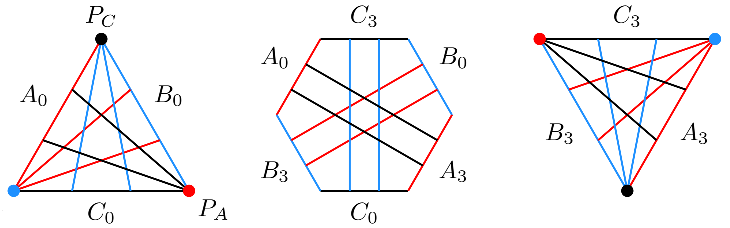
Algebraic surfaces are classified by their Kodaira dimension:

- $\kappa = -\infty$: rational, ruled surfaces.
- $\kappa = 0$: K3, Enriques, abelian, bielliptic surfaces.
- $\kappa = 1$: elliptic surfaces.
- $\kappa = 2$: surfaces of general type.

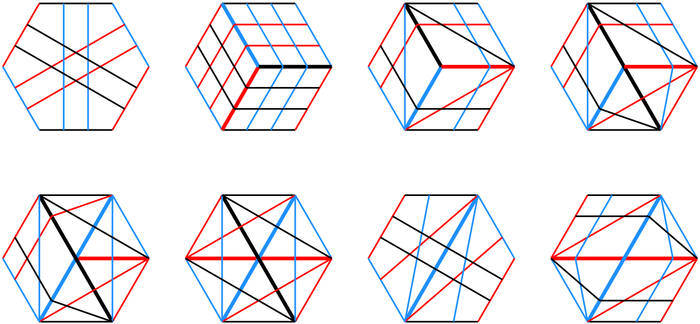
Surfaces of general type are the hardest, and among them surfaces with geometric genus $p_g = 0$ and regularity $q = 0$ are the rarest and most prized.

Among these prized surfaces, there are two classes closely related to line arrangements in $\mathbb{P}^2$:

- Campedelli surfaces $X \xrightarrow{\mathbb{Z}_2^3} \mathbb{P}^2$ ramified in 7 lines D_g, $g \in \mathbb{Z}_2^3 \setminus 0$.
  Computing degenerations $\mapsto$ computing matroid covers of cut hypersimplex $\Delta_{(\frac{1}{2},\dots,\frac{1}{2})}(3,7)$ (turns out to be very easy).
- Burniat surfaces $X \xrightarrow{\mathbb{Z}_2^2} \mathrm{Bl}_{3\mathrm{pts}}\,\mathbb{P}^2$ ramified in 9 curves labeled by $g \in \mathbb{Z}_2^2 \setminus 0 \;= \{$ black, red, blue $\}$
  Computing degenerations $\mapsto$ computing matroid covers of cut hypersimplex $\Delta_{(\frac{1}{2},\dots,\frac{1}{2})}(3,9)$ (I used *polymake*).

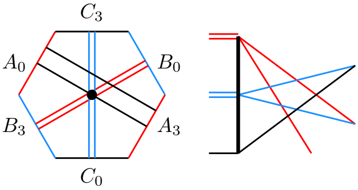


DEGENERATIONS OF BURNIAT SURFACES (8 /10)

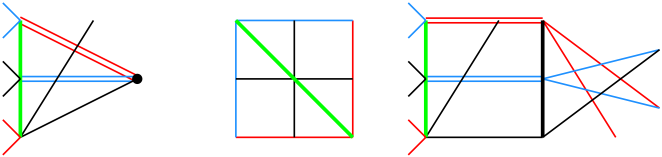


hexagon = $\mathrm{Bl}_{3\mathrm{pts}}\, \mathbb{P}^2$, rhombus = $\mathbb{P}^1 \times \mathbb{P}^1$,
triangle=$\mathbb{P}^2$, trapezoid = $\mathrm{Bl}_{pt}\, \mathbb{P}^2 = \mathbb{F}_1$.

Degenerations 9 and 10 (non-toric)



$\mathrm{Bl}_{4\mathrm{pts}}\,\mathbb{P}^2 \cup \mathbb{P}^2$



$(\mathbb{P}^1 \times \mathbb{P}^1) \cup \mathbb{F}_1 \cup \mathbb{P}^2$

K3 SURFACES

- K3 surface: $K_X \sim 0, h^1(\mathcal{O}_X) = 0$
- 19-dim moduli spaces $F_{2d} = \{(X, L) \mid L^2 = 2d\}, d \in \mathbb{N}$.
- The Big Question: find an analogue of 2nd Voronoi fan for F_{2d}. Instead of tilings of $L \otimes \mathbb{R}/L' = \mathbb{R}^g/\mathbb{Z}^g$, what are tilings of the sphere S^2 with 24 singular points?
- Special case: covers of $\mathbb{P}^2$ ramified in 6 lines
- 6 lines on $\mathbb{P}^2$, grassmannian $G(3,6)$ and Aomoto-Gelfand hypergeometric function

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