Profiles of random trees

featuring infinitely-many-colour Pólya urns and “monkey” walks

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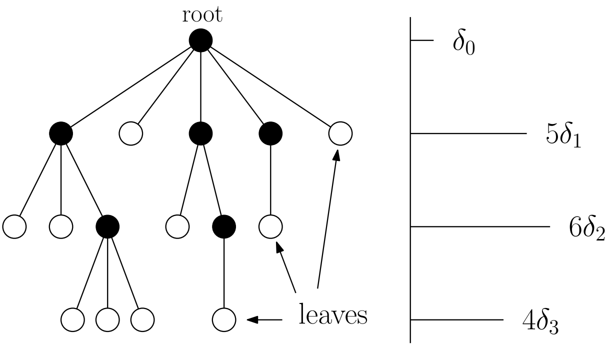
joint work with Jean-François Marckert (Bordeaux), Tim Rogers (Bath), and Gerónimo Uribe-Bravo (UNAM)

August 21st, 2017

Tree? Profile?



The profile of a tree τ is the probability measure

$$\Pi_\tau \propto \sum_{\nu \in \tau} \delta_{|\nu|},$$

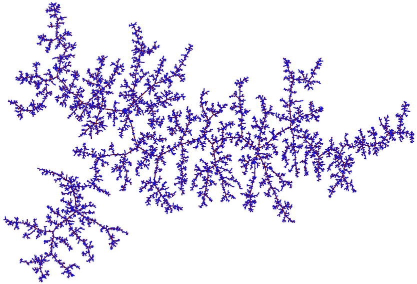
where the sum's indices are the nodes ν of τ, and $|\nu|$ is the (graph) distance from ν to the root.

Describing the "shape" of random trees

One famous example: the Catalan tree (or Galton-Watson tree)

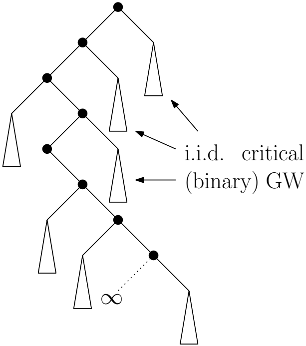
Pick a tree T_n uniformly among all n-leaf binary trees.
What is its "typical shape" when $n \to \infty$?

Scaling limit: the CRT



[I. Kortchemski]

Local limit:



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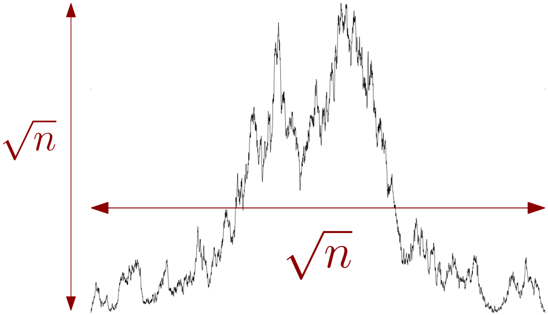
Profile: "local time of a Brownian excursion"



[Drmota & Gittenberger '96]

Cv. of the profile in prob.

[J.-F. Marckert]

Another example: the BST

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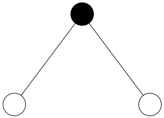

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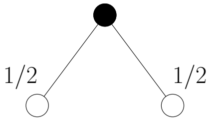

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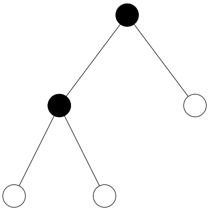

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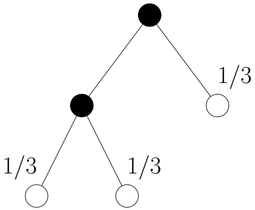

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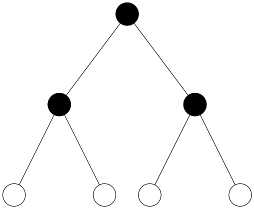

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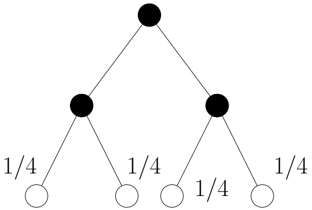

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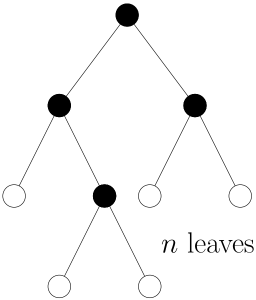

n leaves $\to \mathtt{BST}_n$

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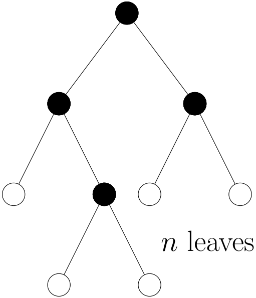

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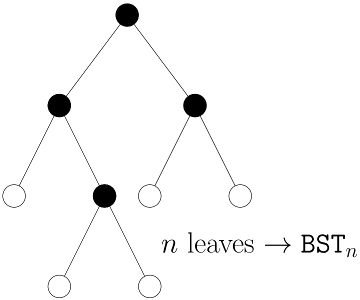

- no scaling limit
- local limit: full infinite binary tree [Devroye '86]

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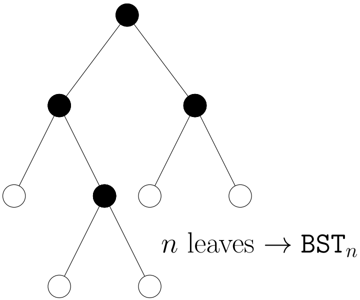
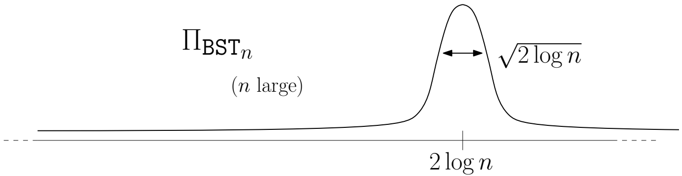


- no scaling limit
- local limit: full infinite binary tree [Devroye '86]
- profile: almost surely

$$\Pi_{\mathtt{BST}_n}(\sqrt{2\log n}\cdot + 2\log n) \to \mathcal{N}(0,1),$$

in the weak topology.
[Chauvin, Drmota, Jabbour-Hattab '01]



Trailer of the talk

1. The random recursive tree and its profile seen as an infinitely-many-colour Pólya urn. [with J.-F. Marckert]
2. The weighted random recursive tree and its profile as a tool to study a "reinforced" random walk. [with G. Uribe-Bravo]
3. Preferential attachment trees and their profile: an exciting open problem. [with T. Rogers]

The random recursive tree

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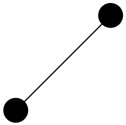

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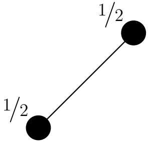

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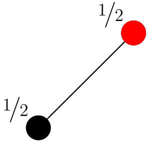

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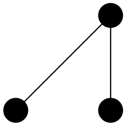

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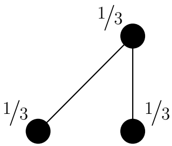

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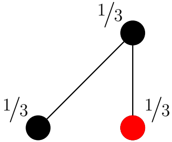

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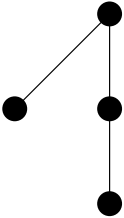

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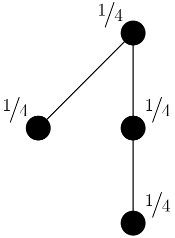

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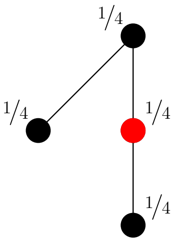

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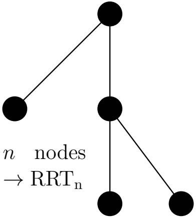

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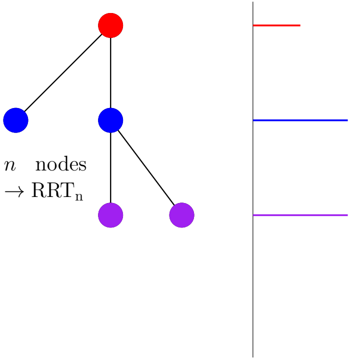

The profile is a Pólya urn: at each time-step, we

- pick a ball (node) uniformly at random, say, of colour (height) k
- add a new ball (node) of colour (height) $k+1$.

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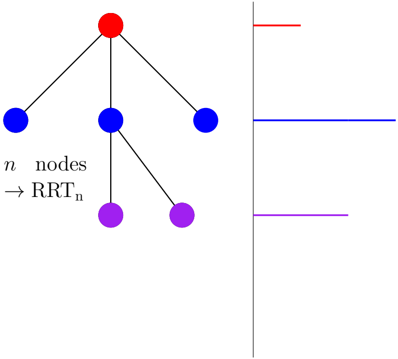

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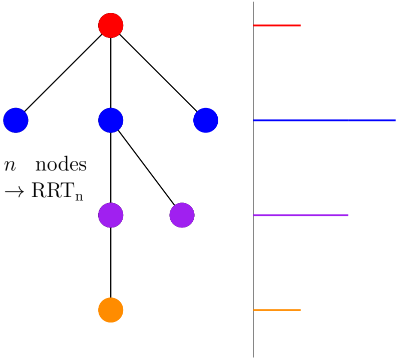

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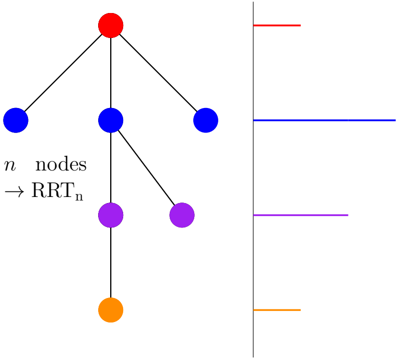

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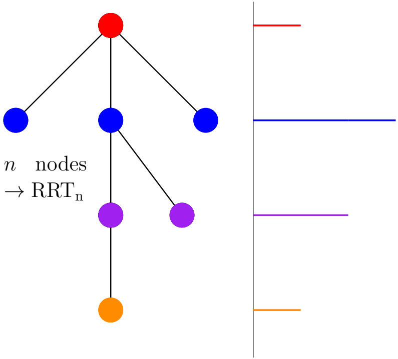

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Don't panick!

Proving convergence in probability of the profile of $\mathtt{RRT}_n$ is actually the key to a general theory of infinitely-many-colour Pólya urns.

Convergence in probability of the profile of $\mathtt{RRT}_n$

Theorem: [MM17]

When n goes to infinity, in probability

$$\Pi_{\mathtt{RRT}_n}(\sqrt{\log n}\,\cdot\,+\log n) \to \mathcal{N}(0,1),$$

for the weak topology on the space of probability distributions.

Ideas of the proof:

- Take U_n a node taken uniformly at random in $\mathtt{RRT}_n$ (cond. on $\mathtt{RRT}_n$). It is easy to show that

$$\frac{|U_n| - \log n}{\sqrt{\log n}} \Rightarrow \mathcal{N}(0,1), \text{in distribution.}$$

This only implies that $\mathbb{E}\big[\Pi_{\mathtt{RRT}_n}(\sqrt{\log n}\,\cdot\,+\log n)\big] \to \mathcal{N}(0,1)$.

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- To prove the result, we need to take U_n and V_n two independent uniform nodes in $\mathtt{RRT}_n$ (cond. on $\mathtt{RRT}_n$), and show that

$$\left(\frac{|U_n| - \log n}{\sqrt{\log n}}, \frac{|V_n| - \log n}{\sqrt{\log n}}\right) \Rightarrow (\Lambda_1, \Lambda_2),$$

where Λ_1 and Λ_2 are two independent standard Gaussians.

Convergence in probability of the profile of $\mathtt{RRT}_n$

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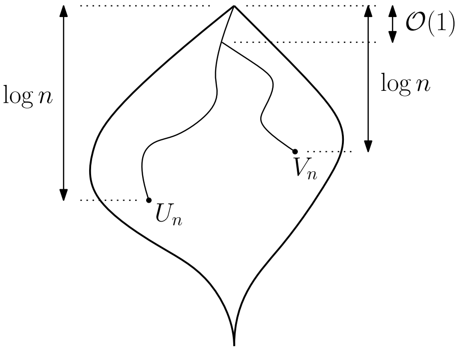

The key arguments are

1. $|U_n \wedge V_n| \to G \sim \mathtt{Geom}(1/3)$
2. given G, the subtrees rooted at the two children of G
   - are i.i.d. random recursive trees and
   - have linear size in n.

Convergence of the profile of $\mathtt{RRT}_n$

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Advertisement for [MM17]:

- Actually, using different techniques, we can prove convergence almost sure of the profile of the random recursive tree.
- One can define infinitely-many-colour urns as branching Markov chains on the random recursive tree.

A model of “reinforced random walk”

Take: [Boyer et al. ’14-’17]

- a “step” distribution on $\mathbb{Z}$, e.g. $\mathbb{P}(\xi = 1) = \mathbb{P}(\xi = -1) = 1/2$;
- a “run-length” distribution on $\mathbb{N}$, e.g. $\mathbb{P}(\omega = k) = (1/2)^k$.

Let $(\xi_n)_{n\geq 1}$, and $(\omega_i)_{i\geq 1}$ be i.i.d. copies of ξ and ω.

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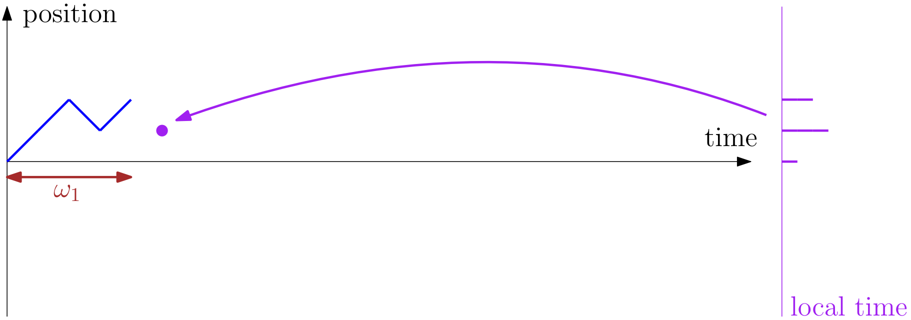

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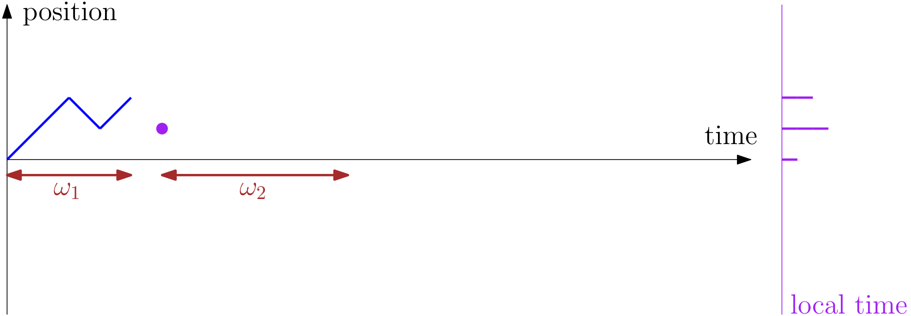

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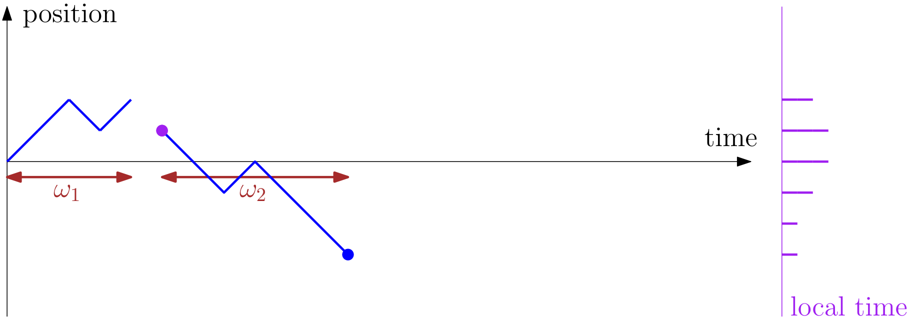

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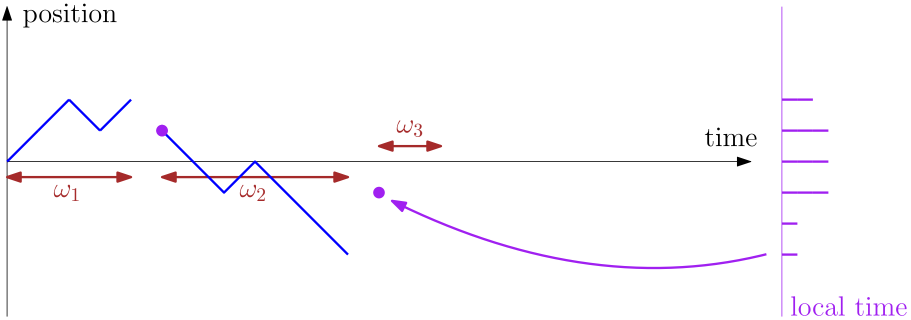

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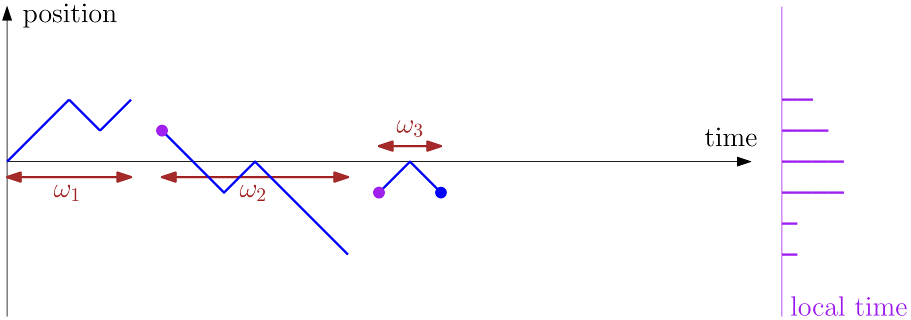

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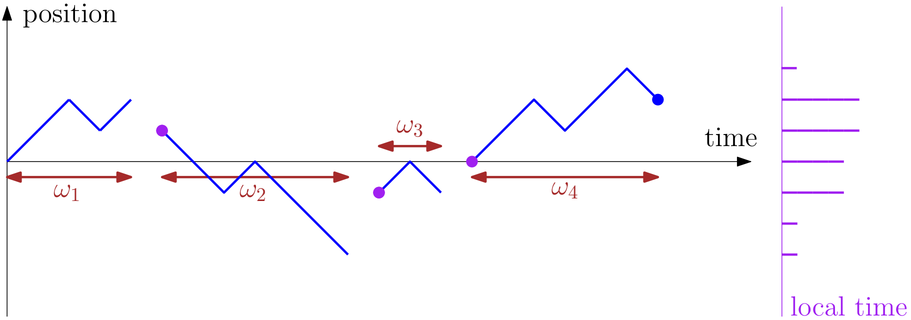

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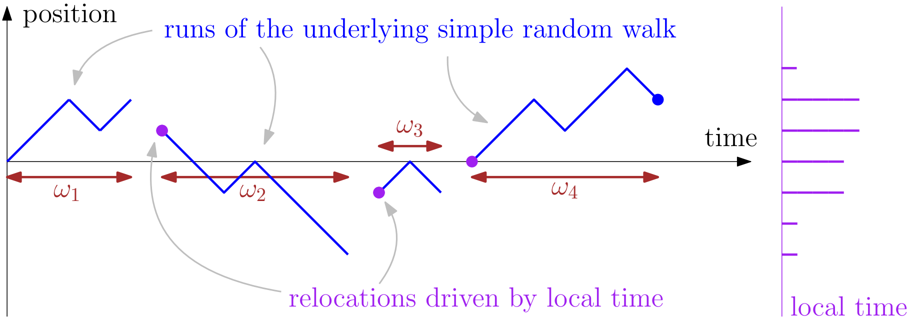
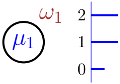


The weighted random recursive tree (WRRT)



Main idea: drawing a position according to $\sum_{i=1}^{k} \mu_k$ is the same as

1. drawing an integer I_k with probability $\mathbb{P}(I_k = i) \propto \omega_i$;
2. and then draw a position according to μ_{I_k}.

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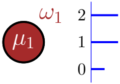

We couple the monkey walk with a (randomly) labelled WRRT: at every time step, we

- draw a node (say node number I_k) of the WRRT at random with probability proportional to the weights $(\omega_i)_{i\geq 1}$;
- we add a child to this node, we draw a position X_k according to μ_{I_k}, and label the new node by the local time measure of a random walk starting at X_k and of length ω_k.

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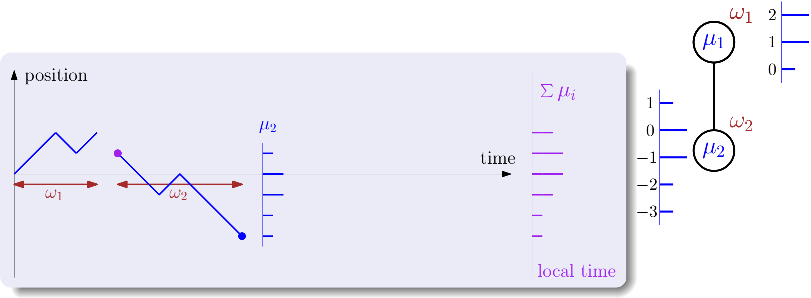

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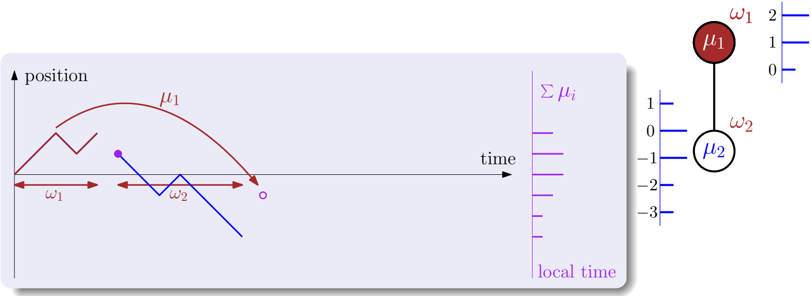
- draw a node (say node number I_k) of the WRRT at random with probability proportional to the weights $(\omega_i)_{i\geq 1}$;
- we add a child to this node, we draw a position X_k according to μ_{I_k}, and label the new node by the local time measure of a random walk starting at X_k and of length ω_k.

The weighted random recursive tree (WRRT)



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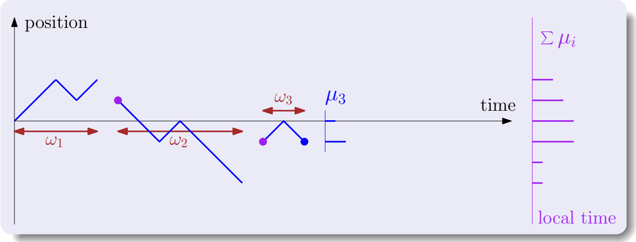

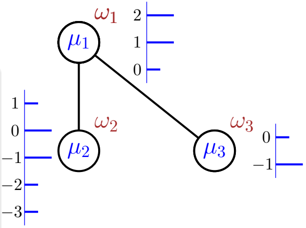

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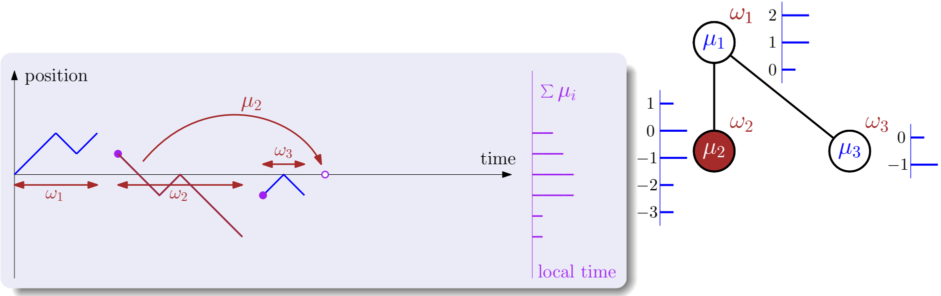

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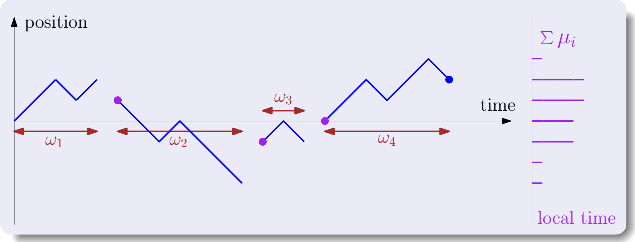

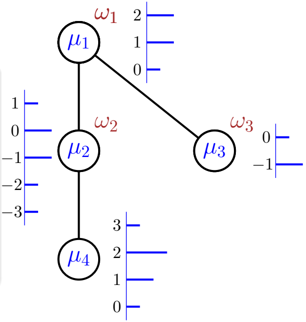

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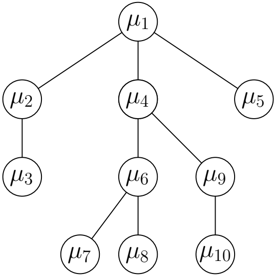
A branching Markov chain



- underlying tree = WRRT;

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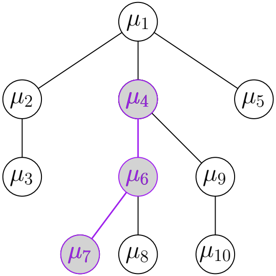
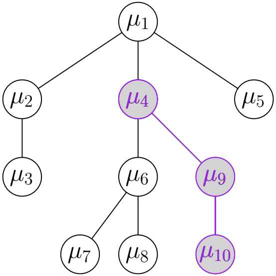


- underlying tree = WRRT;
- the labels are a branching Markov chain (on the space of measures), i.e.
  - the sequence of labels along each branch is a Markov chain (all of the same Kernel);

A branching Markov chain



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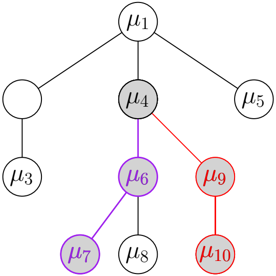

- underlying tree = WRRT;
- the labels are a branching Markov chain (on the space of measures), i.e.
  - the sequence of labels along each branch is a Markov chain (all of the same Kernel);
  - two distinct branches are independent after they branch.

We are interested in $\sum_{i=1}^{k} \mu_i / \sum_{i=1}^{k} \omega_i$ which is

- the local time of the monkey walk just before the $k+1$-th relocation
- the distribution of the monkey walk at the $k+1$-th relocation.

Understanding the Markov chain + the profile of the WRRT is enough to understand $\sum_{i=1}^{k} \mu_i$.

Result and conjecture

Theorem [MUB++]

Let $(\omega_i)_{i\geq 1}$ be an i.i.d. sequence of weights of mean m and finite variance. Let $\mathtt{WRRT}_k$ be the k-node weighted random recursive tree with weights $(\omega_i)_{i\geq 1}$. Then, $(\omega_i)_{i\geq 1}$-a.s., in probability when $n \to \infty$,

$$\Pi_{\mathtt{WRRT}_k}(\sqrt{\log n}\ \cdot\ + \log n) \to \mathcal{N}(0,1).$$

Conjecture:

Let $(M_n)_{n\geq 0}$ be the monkey walk. In distribution when $n \to \infty$,

$$\frac{M_n - c_1 \log n}{\sqrt{c_2 \log n}} \Rightarrow \mathcal{N}(0,1).$$

Further: can we prove convergence as a process to, say, a Brownian motion with random reinforced relocations?

The preferential attachment tree - Barabási and Albert

- At time 1, one node (the root).
- At time n, add the n^{th} node in the tree: link it to a random node chosen with probability proportional to the degrees.

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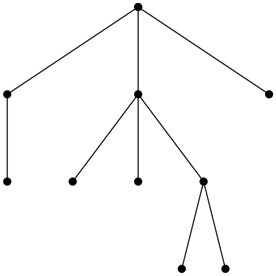

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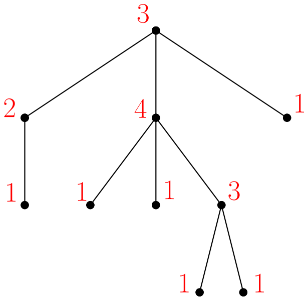

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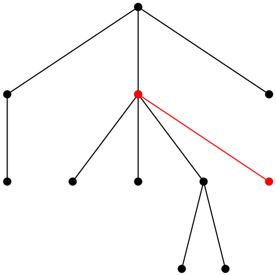

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Theorem [Katona '05]

Almost surely when $n \to \infty$ (for the weak topology),

$$\Pi_{\mathrm{BA}_n}\left(\sqrt{\frac{\log n}{2}} \cdot + \frac{\log n}{2}\right) \to \mathcal{N}(0,1).$$

The preferential attachment tree with fitnesses

- Fix $(X_n)_{n\geq 1}$ i.i.d. fitnesses.
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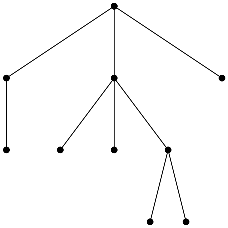

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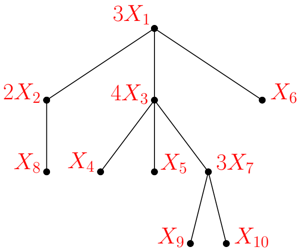

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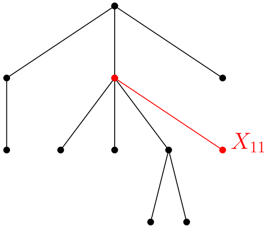

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Two competing dynamics: rich-gets-richer and fit-gets-richer.

Conjecture – the winner takes it all [Bianconi & Barabási]

$$\liminf_{n\to\infty} \frac{\text{max degree at time } n}{n} > 0$$ [see P. Mörters' course]

Condensation

μ = fitness distribution on $[0,1]$ X_i = fitness of node ν_i (i.i.d.)

Empirical fitness distribution: $\Xi_n = \frac{1}{n}\sum_{i=1}^{n} \mathrm{deg}(\nu_i)\delta_{X_i}$.

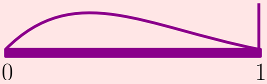
Ξ_n converges (weakly) almost surely to

If (cond) fails:

(cond)

$$\boxed{\int_0^1 \frac{\mathrm{d}\mu(x)}{1-x} < 2}$$

If (cond) holds:



Conjecture:

In probability when $n \to \infty$ (for the weak topology):
if (cond) fails, then $\Pi_{\mathrm{BB}_n}(\sqrt{c\log n}\ \cdot\ + c\log n)$;
if (cond) holds, then $\Pi_{\mathrm{BB}_n}(\alpha_n\ \cdot\ +\beta_n) \to \mathcal{N}(0,1)$ with $\alpha_n, \beta_n = o(\log n)$.

Conclusion

This talk in a nutshell:

- The profile of a random tree is a way to describe its shape - particularly useful when there is no scaling/local limit, typically for "short fat" trees.
- Proving convergence in probability is done by taking two nodes at random in the tree and look at the joint distribution of their respective heights. (Almost sure convergence is much harder.)
- Convergence of the profile can be the first step towards studying more intricate objects such as the infinitely-many-colour urns and the monkey walk.
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Thanks!!