Weierstrass-Institute for
Applied Analysis and Stochastics

Leibniz-Gemeinschaft

Stochastic geometry in telecommunications

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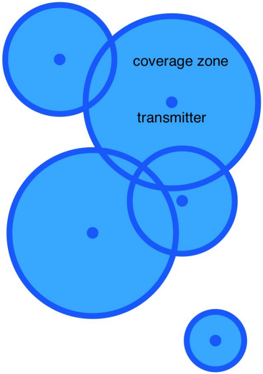
Challenges

- High complexity in space and time
- Large number of network components
- Random positioning and mobility of components
- Common communication technology

Let us consider networks with the following properties:

- random spatial distribution of network components
- static networks without time
- no additional infrastructure



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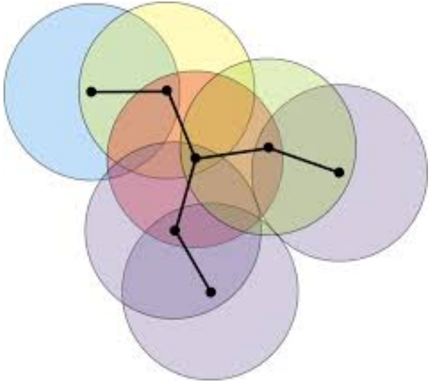
Idea since the 1960's (Gilbert): Use stochastic geometry to model telecommunication networks.



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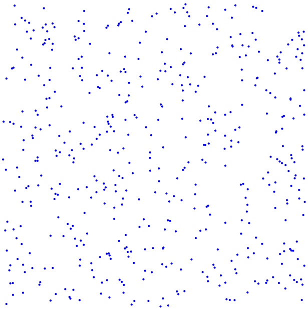
The Poisson point process

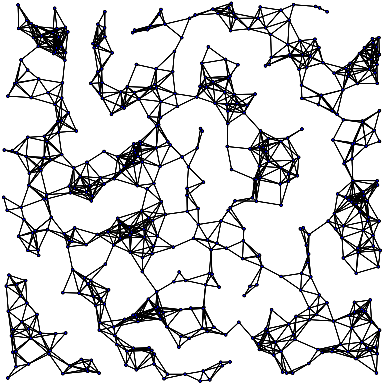
- A Poisson point process X is a random cloud of points without cluster points (configuration of network components) with the following properties:
  1. Point clouds in disjoint areas are stochastically independent.
  2. The number of points in an area $A \subset \mathbb{R}^d$ is Poisson distributed with parameter $\lambda \mathrm{Vol}(A)$:

$$\mathbb{P}_\lambda(X \text{ has } k \text{ points in } A) = e^{-\lambda \mathrm{Vol}(A)} \frac{(\lambda \mathrm{Vol}(A))^k}{k!}$$



Gilbert graph

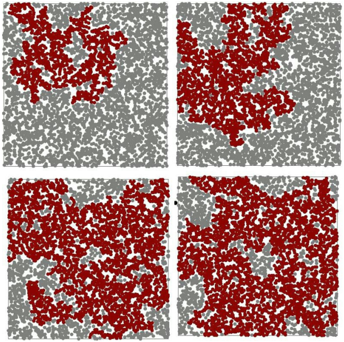
- Gilbert (1961): First network model $g_r(X)$ based on Poisson point porcess X.
- Two network components x, y can communicate if their distance is smaller then a connectivity parameter $r > 0$: $|x - y| < r$



Percolation

- Quality of network connectivity measured via size of connected components: clusters
- Existence of infinite cluster is called percolation

$$\mathbb{P}_\lambda(g_r(X) \text{ percolates}) > 0$$



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Phase transition

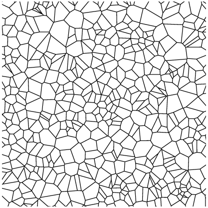
- Percolation is a phase-transition phenomena in the intensity parameter λ.
- There exists $0 < \lambda_{\mathsf{c}} < \infty$ with the property that

$$\lambda_{\mathsf{c}} = \lambda_{\mathsf{c}}(r) = \inf\{\lambda : \mathbb{P}(g_r(X) \text{ percolates}) > 0\}.$$

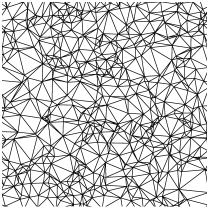
- In the sub-critical regime $\lambda < \lambda_{\mathsf{c}}$ we have local communication.
- In the super-critical regime $\lambda > \lambda_{\mathsf{c}}$ global communications is possible.
- There is no known closed form expression for λ_{c} as a function of r.
- Numerical approximations suggest that $\lambda_{\mathsf{c}} \approx 1.436$ for $r = 1$.

Poisson tesselations

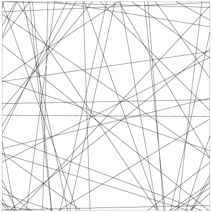
Based on the Poisson point process, a large number of tessellations can be defined.



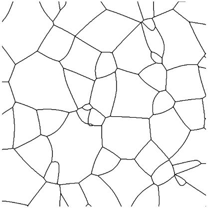
Voronoi tessellation



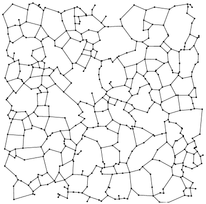
Delaunay tessellation



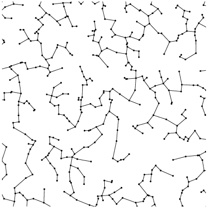
Line tessellation



Johnson-Mehl tessellation



Relative neighborhood graph



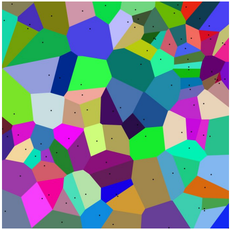
Minimum spanning forest

Poisson-Voronoi tesselation

- Also other names: Voronoi diagram, Voronoi decomposition, Voronoi partition, Dirichlet tessellation or Thiessen Polygon
- Formal definition: The Voronoi cell around the point $x \in X$ is given by

$$Z(x) = \{z \in \mathbb{R}^d : |z - x| < |z - y| \text{ für alle } y \in X \setminus \{x\}\}.$$

- Used in a great number of scientific fields (Algorithmic geometry, material sciences, ...) and applications (biology, chemistry, meteorology, crystallography, architektur, ...).



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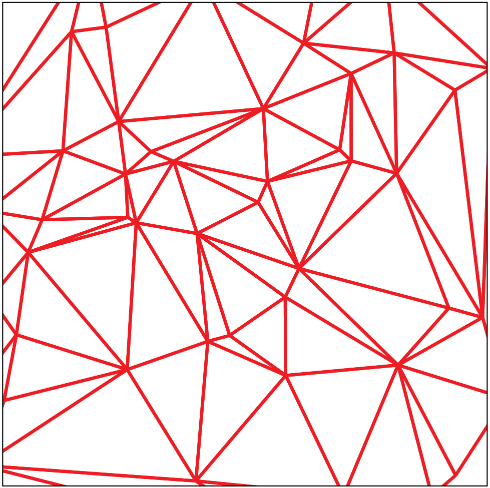
Poisson tesselations for urban networks



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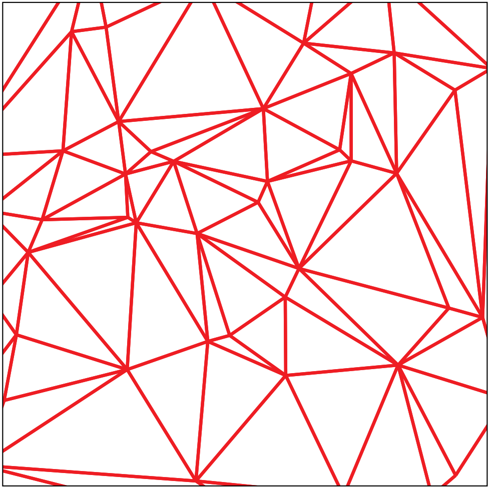

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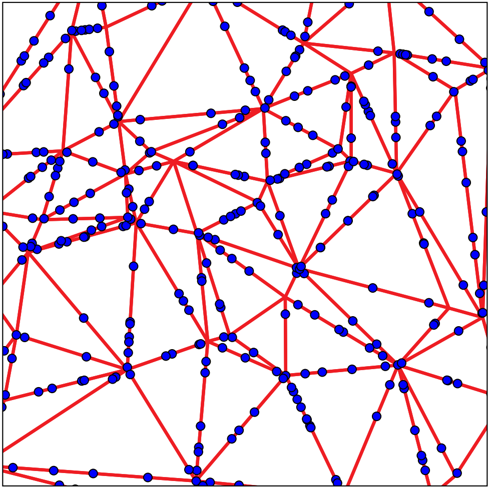

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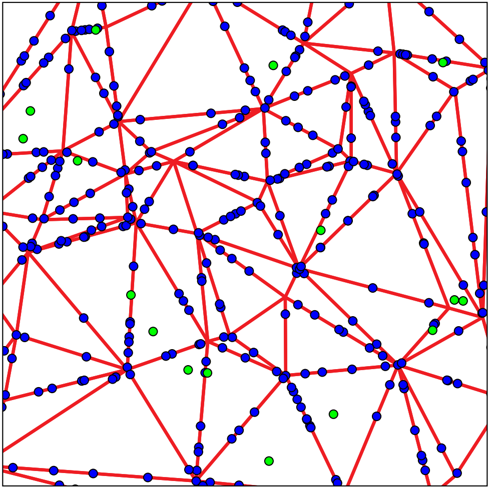

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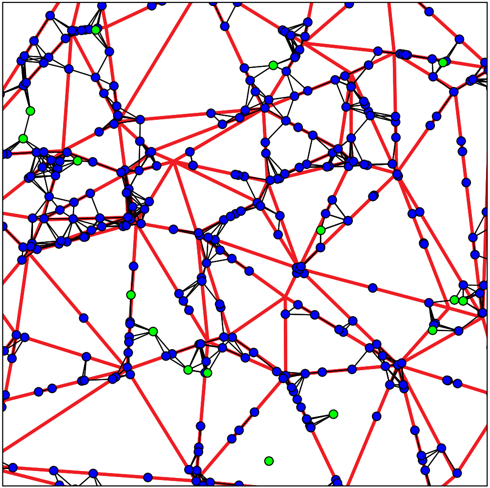


Fundamental network characteristics

1. What is the probability that two users are connected dependent on their distance?

$$p_s = \mathbb{P}(o \leftrightsquigarrow se_1)?$$

2. What is the proportion of pairs of connected users?

$$\pi_s = \mathbb{E}(\#(X \leftrightsquigarrow Y) \in B_s(o))?$$

3. What is the probability that two users are connected if the number of hops is constraint?

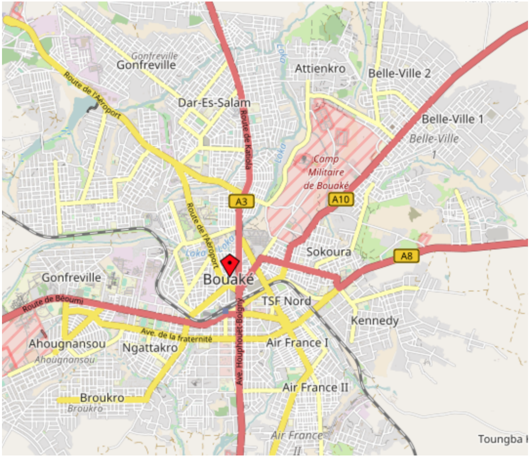
$$\hat{p}_s = \mathbb{P}(o \leftrightsquigarrow se_1 | \#\,\text{Hops} < \alpha s)?$$

Lets get to it.

Voronoi structure: Avignon (France)

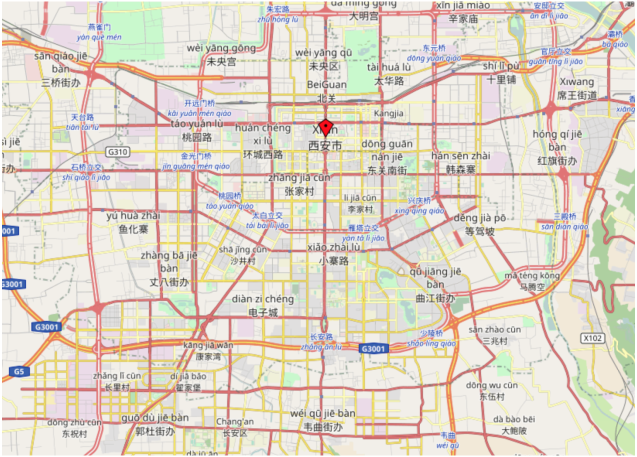
From: Open street maps

Manhattan grid structure: Bouake (Ivory Coast)



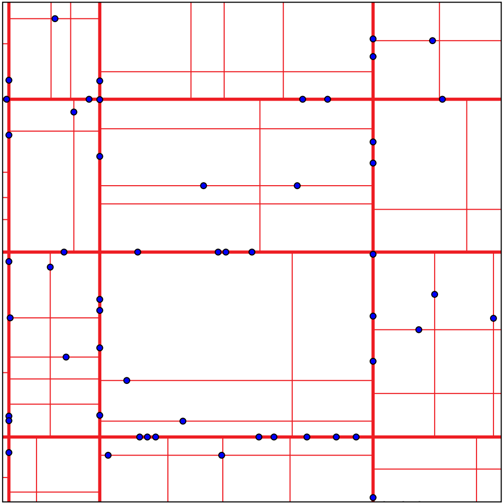
From: Open street maps

Manhattan grid structure: Xian (China)



From: Open street maps

Nested Manhattan grid structure with users



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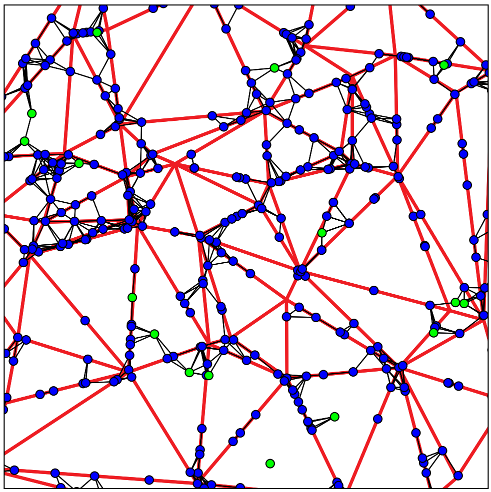

Essentially asymptotically connected Cox point processes

Theorem

If the random intensity measure is essentially asymptotically connected, then $0 < \lambda_c < \infty$.

- Examples are Poisson Voronoi tessellation (PVT) or the Poisson Delaunay tessellation.
- Manhattan and nested Manhattan grids are not stabilizing and proofs for non-triviality should be much harder.
- Continuum percolation for general Cox processes can exhibit pathological effects, for example $\lambda_c = 0$ (see Błaszczyszyn & Yogeshwaran, 2013).

Approximations for λ_c

- Users form Cox point process X with random intensity $\lambda|du \cap S|$ where
- S realization of a street system, e.g., PVT
- PVT is characterized by length intensity $\gamma = \mathbb{E}[|S \cap [-1/2, 1/2]^2|]$

Approximations for λ_c

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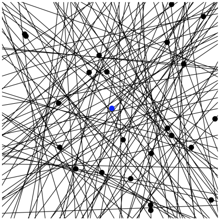
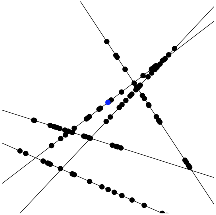
- Dense streets: approximate X by 2D Poisson point process with spatial intensity $\gamma\lambda$
  - $4.51\pi^{-1}r^{-2}$ is the approximate critical intensity for percolation of the Boolean model
  - Approximation I:
    $\lambda_c \approx 4.51\pi^{-1}\gamma^{-1}r^{-2}$ becomes exact for $\gamma \uparrow \infty$ with $\lambda\gamma$ fixed.



Approximations for λ_c

- Users form Cox point process X with random intensity $\lambda|du \cap S|$ where
- S realization of a street system, e.g., PVT
- PVT is characterized by length intensity $\gamma = \mathbb{E}[|S \cap [-1/2, 1/2]^2|]$
- Sparse streets: approximate X by inhomogenous Bernoulli bond percolation with parameter b^l where l is edge length
  - b_{crit} is critical percolation threshold. For PVT with distance parameter 1, by simulations $b_{\text{crit}} \approx 0.725$
  - Approximation II:
    $\frac{\lambda_c}{\gamma}\exp(-\lambda_c r) \approx -\log(b_{\text{crit}})$ becomes exact for $\gamma \downarrow 0$ with $\frac{\lambda}{\gamma}\exp(-\lambda r)$ fixed.



Fundamental network characteristics

1. What is the probability that two users are connected dependent on their distance?

$$p_s = \mathbb{P}^0(o \leftrightsquigarrow se_1)?$$

2. What is the expected number of pairs of connected users?

$$\pi_s = \mathbb{E}(\#(X \leftrightsquigarrow Y) \in B_s(o))?$$

3. What is the probability that two users are connected if the number of hops is constraint?

$$\hat{p}_s = \mathbb{P}^0(o \leftrightsquigarrow se_1 | \# \text{ Hops} < \alpha s)?$$

Connection probability as a function of distance

- Palm calculus to ensure o on streets: Define Palm version of S via

$$\mathbb{E}^0 f(S) = \frac{1}{\gamma}\mathbb{E}\int_{Q_1(o)\cap S} f(S-u)\mathrm{d}u$$

- define device connection probability at relative position B via

$$p_B(\lambda, r, \gamma) = \frac{\mathbb{E}^0 \int_{S\cap B} \mathbb{1}\{o \leftrightsquigarrow v \text{ in } g_r(X \cup \{v\})\}\mathrm{d}v}{\mathbb{E}^0|S\cap B|}$$

where $\mathbb{E}^0$ denote the Palm measure for S and X.

Theorem (Scaling invariance)

Let $\lambda, r, \gamma > 0$ be arbitrary. Then, for every $a > 0$,

$$p_{aB}(a^{-1}\lambda, ar, a^{-1}\gamma) = p_B(\lambda, r, \gamma).$$

Large distance approximation

- $p_s = p_{Q_1(se_1)}$ converges to the square of the percolation probability

$$\theta(\lambda, r, \gamma) = \mathbb{P}^0\big(o \leftrightsquigarrow \infty \text{ in } g_r(X)\big)$$

Theorem

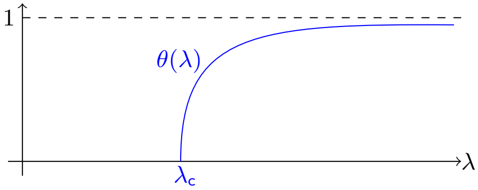
Let $\lambda, r, \gamma > 0$ be arbitrary. Assume some cluster uniqueness and vacancy condition, then

$$\lim_{s \uparrow \infty} p_s = \theta^2.$$

More precisely, there exists $c > 0$ such that $|p_s - \theta^2| \le \exp(-cs)$ for all sufficiently large s.

Percolation approximation w.r.t. λ - universality

- no closed form expression available for θ



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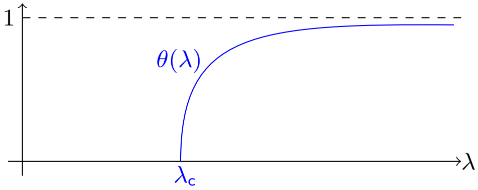

Conjecture

Let $r, \gamma > 0$. *Then,* $\theta(\lambda) \approx (\lambda - \lambda_c)^{5/36}$ *as* $\lambda \to \lambda_c$.

More precisely, $\lim_{\lambda \to \lambda_c} \frac{\log \theta(\lambda)}{\log(\lambda - \lambda_c)} = 5/36$.

- Smirnov and Werner 2001 for the triangular lattice
- Assumed to be universal, i.e., depend only on the local structure of the graph and the dimension

Percolation approximation w.r.t. λ - large deviations

- finite box crossing: $\theta_K(\lambda) = \mathbb{P}^0(o \leftrightsquigarrow X_i$ for some $X_i \in X \setminus Q_K(o))$

Theorem (large λ_{U})

Let $r, \gamma > 0$ and $K > 2r_{\mathrm{U}}$ be arbitrary. Then,

$$\theta_K(\lambda) \approx 1 - \mathit{exp}(-2r\lambda)$$

as $\lambda \to \infty$. More precisely,

$$\lim_{\lambda \to \infty} \lambda^{-1} \log(1 - \theta_K(\lambda)) = -2r.$$

- expect statement to hold also for θ

Thank you for your attention.