Network Design Games

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Motivation

- Understand formation of large networks
- Internet is product of many selfish agents
- Emerge from distributed uncoordinated spontaneous actions
- How costly is lack of coordination?

Network design games

- Model 1

  Edges may be laid down between any pair of network nodes.

  Fabrikant, Luthra, Maneva, Papadimitriou, Shenker PODC'03

- Model 2

  Edges must be chosen from a given set of edges.

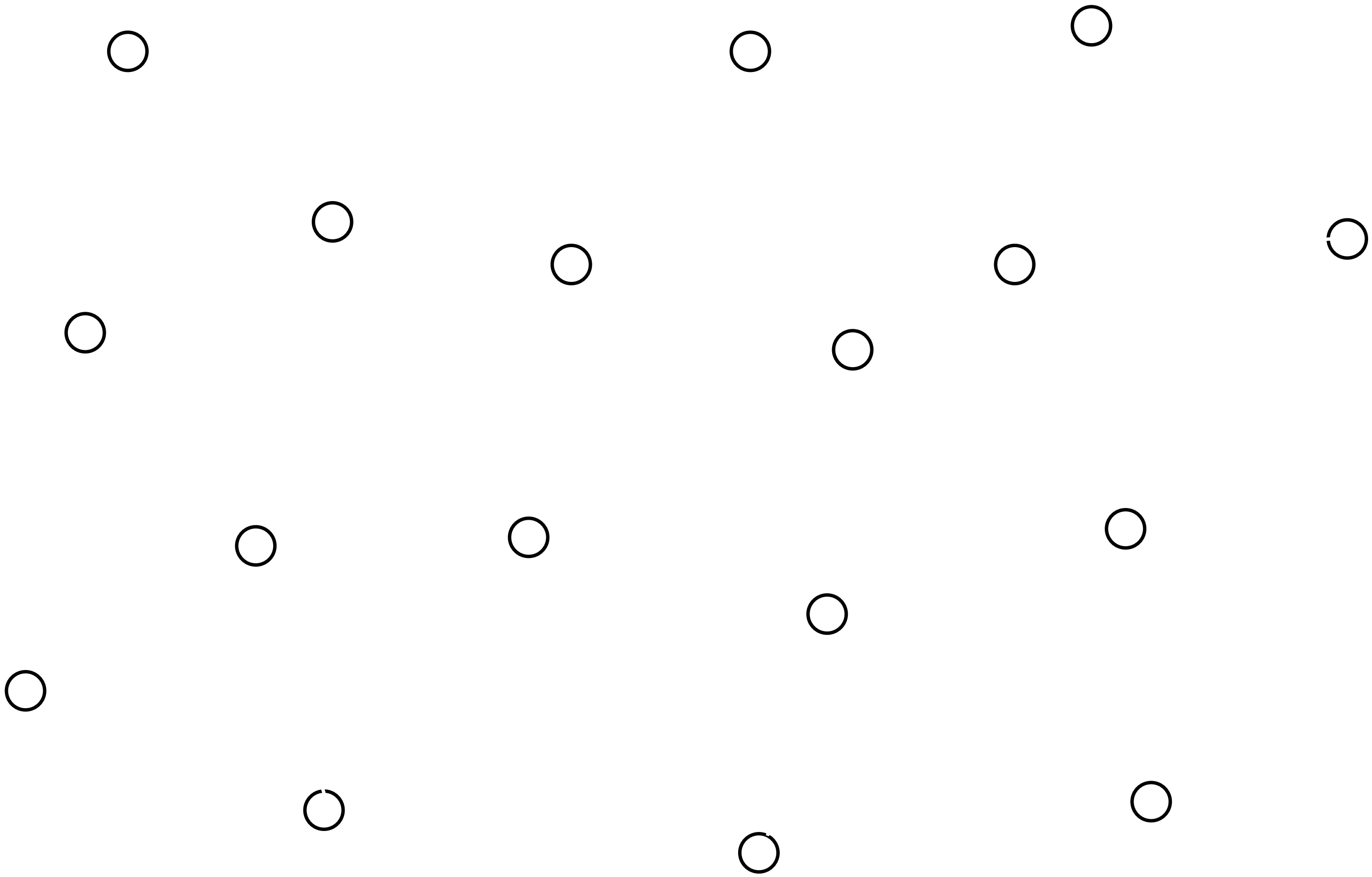
  Anshelevich, Dasgupta, Tardos, Wexler STOC'02

First network design game

- n agents build connected network
- Agent i lays out set of edges to other agents
- Edges may by used in both directions
- Hardware cost   QoS cost

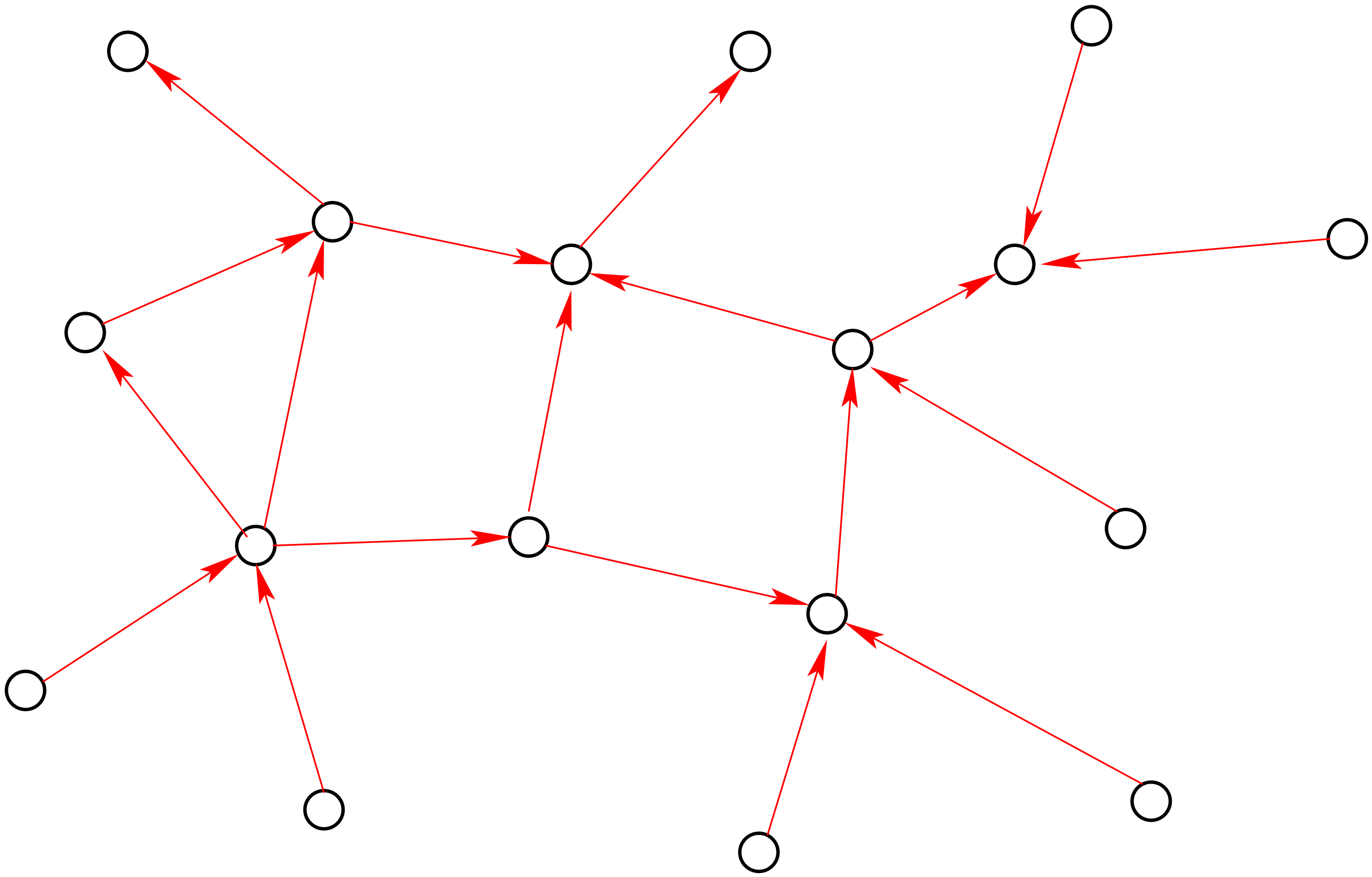
Fabrikant, Luthra, Maneva, Papadimitriou, Shenker PODC'03

First network design game



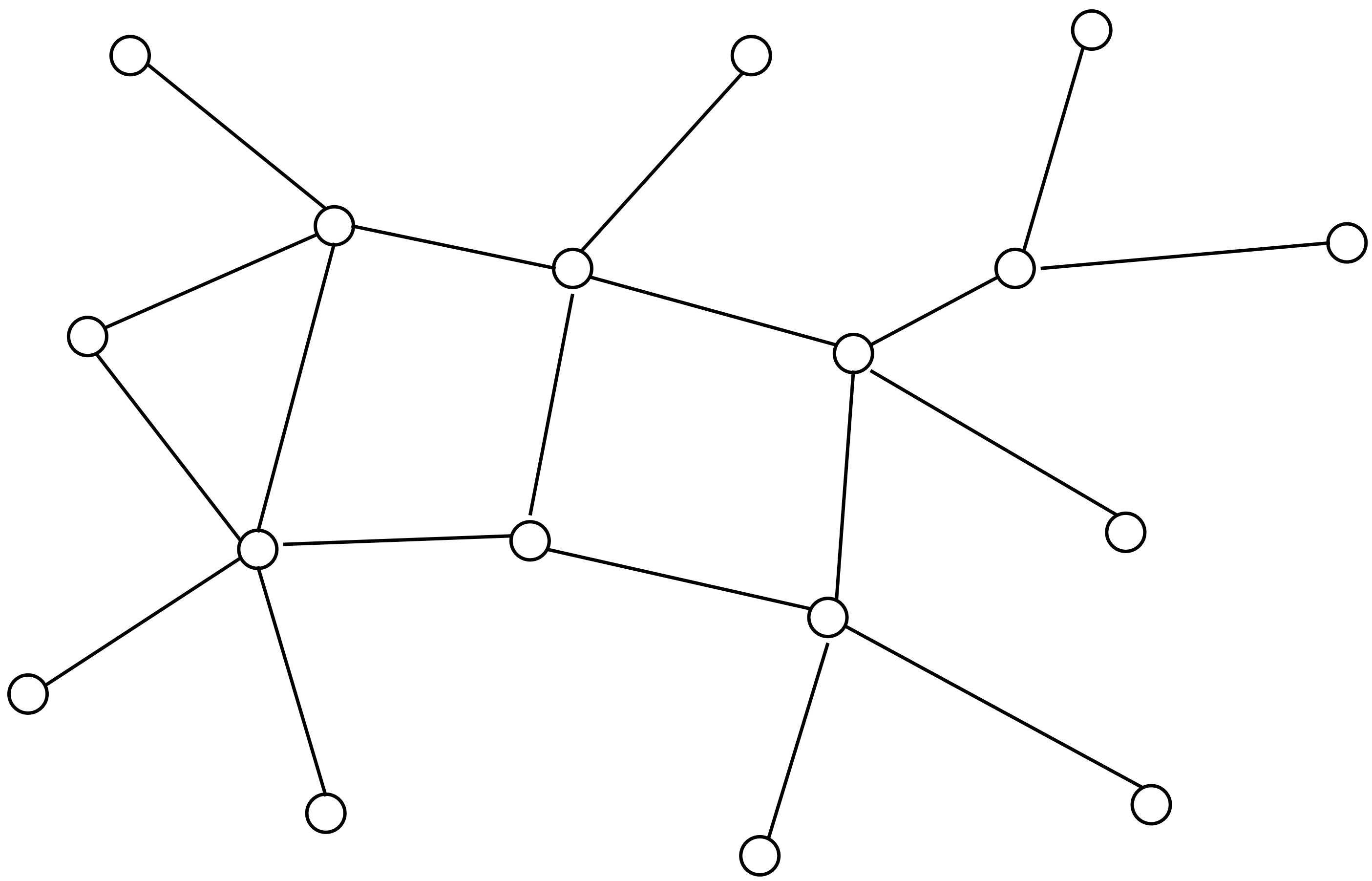
n agents have to build connected network.

First network design game



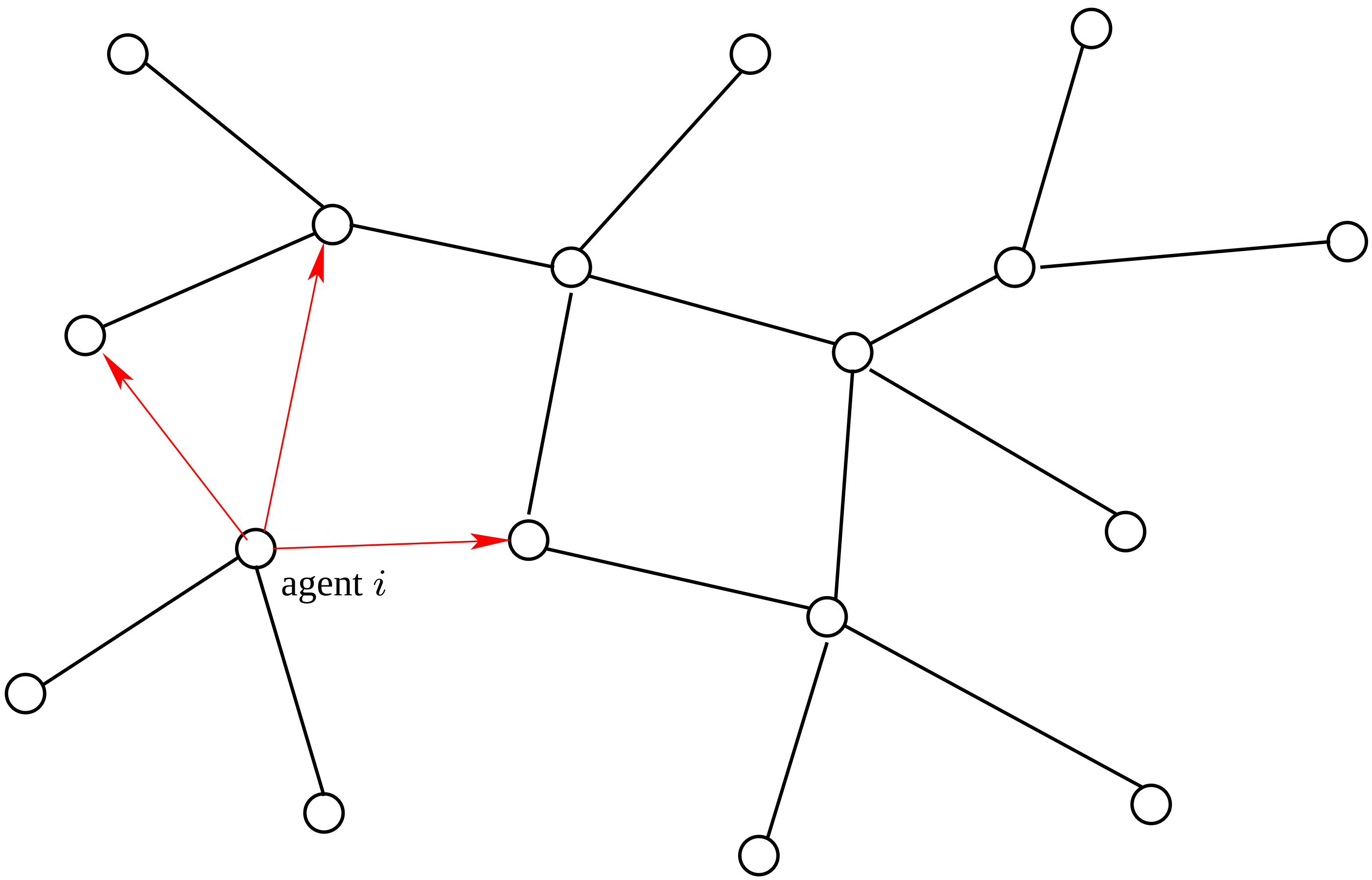
n agents have to build connected network.

First network design game



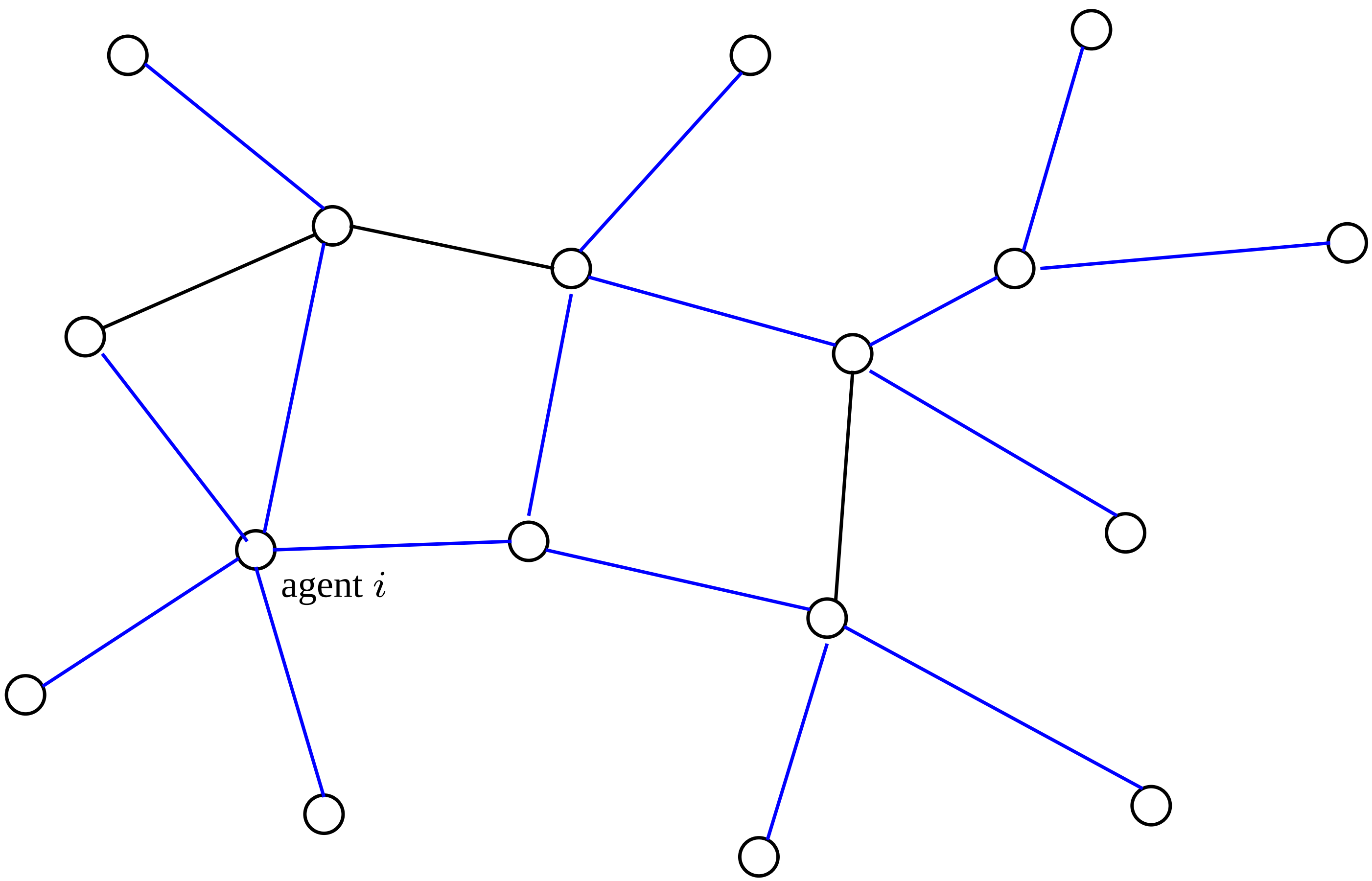
n agents have to build a connected network.

Hardware cost



Cost of $\alpha > 0$ for each edge.

QoS cost



Shortest path distance to agent j, for all $j \neq i$.

Applications

- Information services network
  Nodes represent facilities containing data storage.

- Social networks
  Set of nodes represents a community for disseminating information; links represent phone calls.

- Postal and delivery services
  Nodes represent mail office branches; a link indicates that mail can reach destination directly.

Problem

Agents $V = \{1, \ldots, n\}$

Strategy of agent i $\quad S_i \subseteq V \setminus \{i\}$

Combination of strategies $\quad \vec{S} = (S_1, \ldots, S_n)$

$$
\begin{aligned}
G &= (V, E) \\
E &= \bigcup_{i \in V} \bigcup_{j \in S_i} \{i, j\}
\end{aligned}
$$

$$
\begin{aligned}
\text{Cost}(i, \vec{S}) &= \alpha\, |S_i| + \sum_{\substack{j \in V \\ j \neq i}} \text{Dist}(i, j) \\
\text{Cost}(\vec{S}) &= \sum_{i=1}^{n} \text{Cost}(i, \vec{S})
\end{aligned}
$$

Nash equilibria

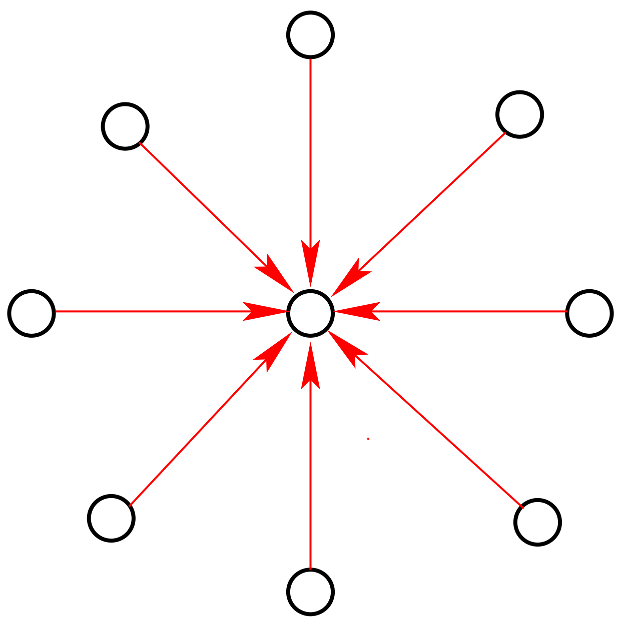
$\vec{S}$ forms Nash equilibrium if, for all i,

$$\text{Cost}(i, \vec{S}) \leq \text{Cost}(i, \vec{S'})$$

for all $\vec{S'}$ that differ from $\vec{S}$ only in i-th component

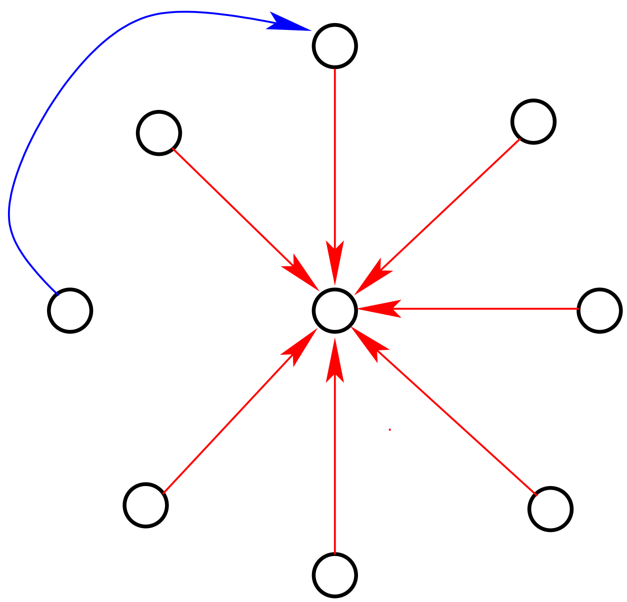
Nash equilibrium

$\alpha > 1$



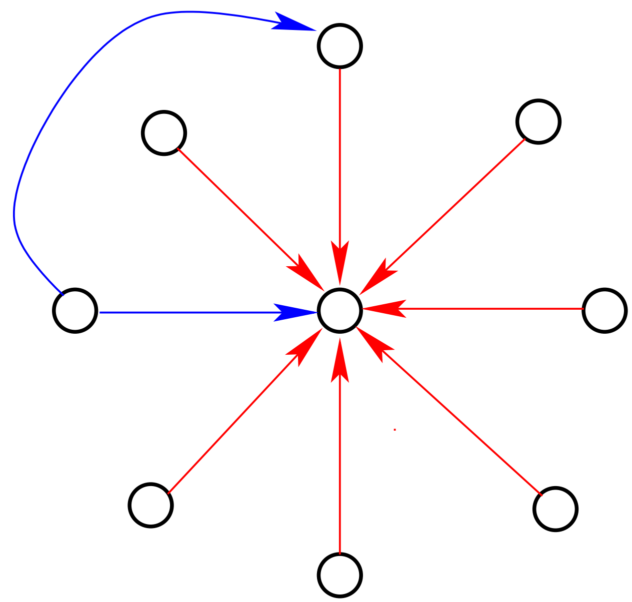
Nash equilibrium

$\alpha > 1$



Nash equilibrium

$\alpha > 1$



Price of anarchy

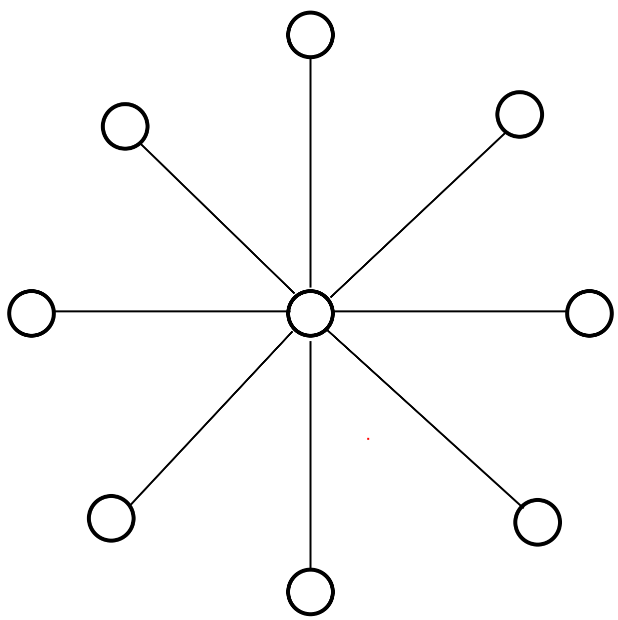
$$P = \max_{\vec{S} \text{ Nash eq.}} \frac{\text{Cost}(\vec{S})}{\text{Cost(OPT)}}$$

Cost(OPT): global optimum

Koutsoupias, Papadimitriou ’99

Global optimum

$\alpha > 2$



Previous results

Fabrikant, Luthra, Maneva, Papadimitriou, Shenker PODC'03

- Computing optimal strategy for an agent is NP-hard
- $\alpha < 1, \quad \alpha > n^2$: $\quad P$ is constant
- $1 \leq \alpha \leq n^2$: $\quad P$ is bounded by $O(\sqrt{\alpha})$
- Lower bound: $P \geq 3$
- Tree-conjecture: $\exists C \; \forall \alpha > C$ every Nash equilibrium is tree.
  If tree-conjecture holds, P is constant, for all α.

Our results

Tree-conjecture is wrong: $\forall n_0\ \exists$ graphs built by $n \geq n_0$ agents that contain cycles and form Nash equilibirum for $1 < \alpha \leq \sqrt{n/3}$

$P = O(1 + (\min\{\frac{\alpha^2}{n}, \frac{n^2}{\alpha}\})^{1/3})$ P constant for $\alpha \geq 12n \log n$

$\alpha \in O(\sqrt{n})$:	P is constant
$\alpha \in \Omega(\sqrt{n})$, $\alpha \in O(n)$:	P increasing, bounded by $O(n^{1/3})$
$\alpha \in \Omega(n)$:	P decreasing, constant for $\alpha \geq 12n \log n$

Albers, Eilts, Even-Dar, Mansour, Roditty 2006

Our results

Upper bounds can be extended to:

Cost sharing: agent can pay for a fraction of an edge

Weighted game: $t_{ij} =$ traffic sent from agent i to j

$\text{Cost}(i, \vec{S}) = \alpha |S_i| + \sum_{j \neq i} t_{ij} \text{Dist}(i, j)$

Nash equilibrium representing a chordal graph is transient.

Such Nash equilibria exist for any n.

In any Nash equilibrium hardware cost is at most $2\text{Cost}(OPT)$.

Further work

Improved bounds:

- $\alpha = O(n^{1-\epsilon})$ $\quad P$ is constant
- P bounded by $2^{O(\sqrt{\lg n})}$

Demaine, Hajiaghayi, Mahini, Zadimoghaddam 2007

Bilateral Model:

Link is formed if both endpoints agree

Corbo, Parkes 2005

Nonuniform interests:

Each agent is interested in subset of agents

Halevi, Mansour 2008

General graphs:

Edge set restricted

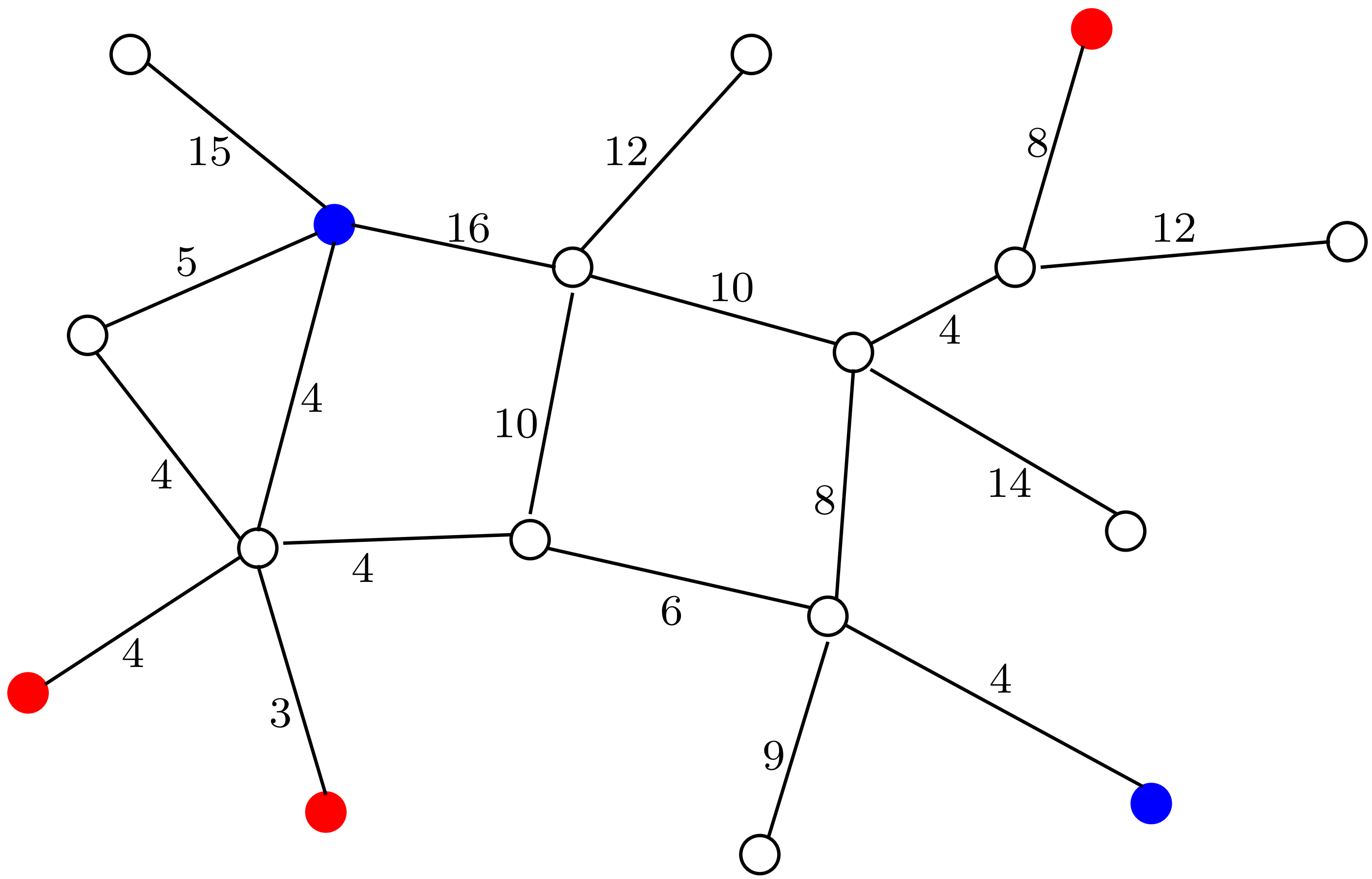
Demaine, Hajiaghayi, Mahini, Zadimoghaddam 2009

Second network design game

- $G = (V, E)$ directed o. undirected graph
- Edge cost $c : E \longrightarrow \mathbf{R}^0_+$
- n agents; agent i has to connect $V_i \subseteq V$
- Strategy agent i $S_i \subseteq E$ connecting V_i
- Cost of used edges must be fully covered

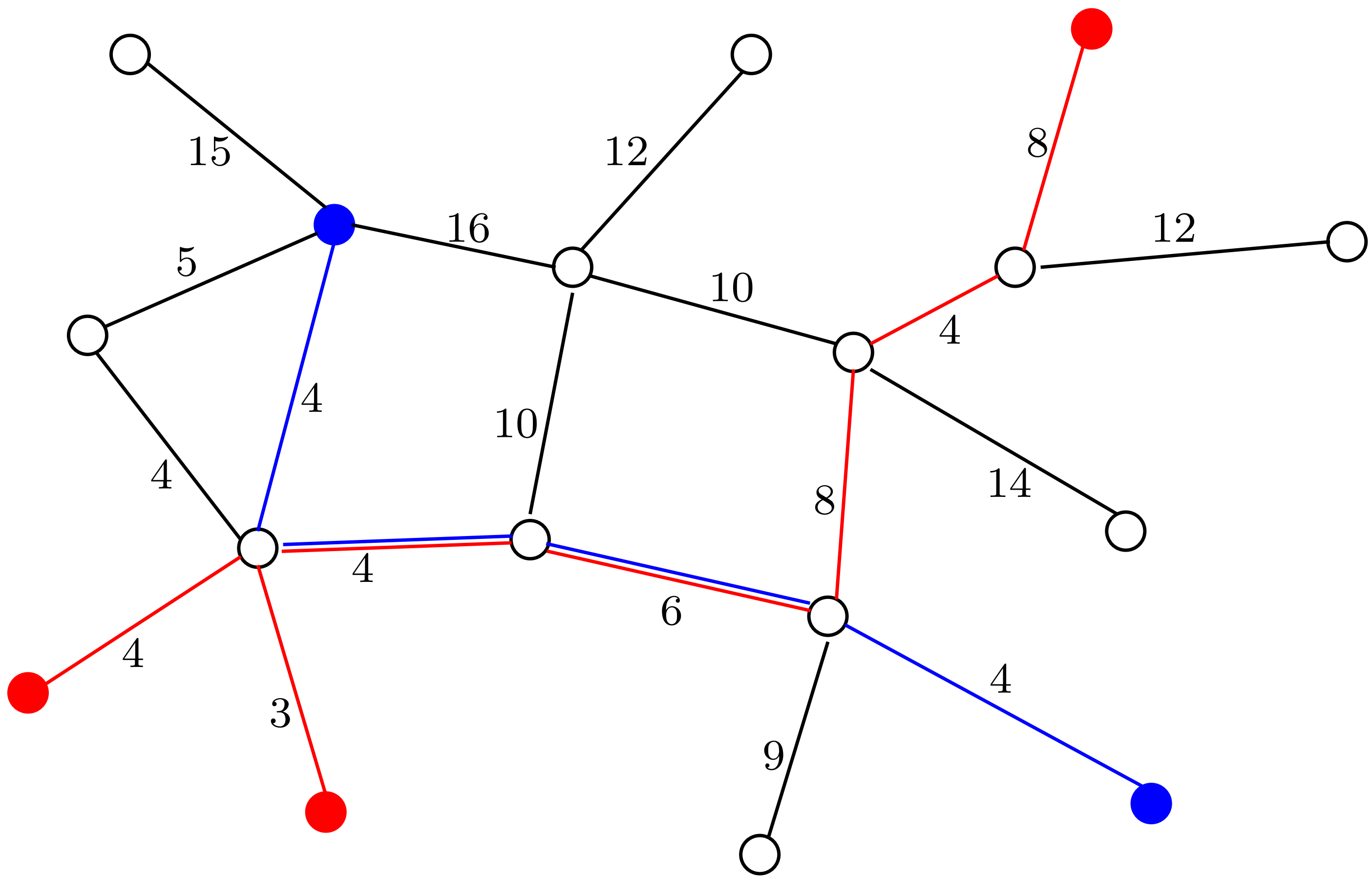
Anshelevich, Dasgupta, Tardos, Wexler STOC'02

Second network design game



n agents have to connect desired nodes.

Second network design game



n agents have to connect desired nodes.

Shapley cost sharing

Cost of an edge is split equally among agents using the edge.

Combination of strategies $\vec{S} = (S_1, \ldots, S_n)$

$$\text{Cost}(i, \vec{S}) = \sum_{e \in S_i} \frac{c(e)}{|\{j \,:\, e \in S_j\}|}$$

Anshelevich, Dasgupta, Kleinberg, Tardos, Wexler, Roughgarden FOCS'04

Previous results

- Price of anarchy can be as high as n.
- Price of stability

$$P' = \min_{\vec{S} \text{ Nash eq.}} \frac{\text{Cost}(\vec{S})}{\text{Cost(OPT)}}$$

$P' \leq H_n \qquad H_n = \sum_{i=1}^{n} 1/i \qquad H_n \approx \ln n$

$P' \geq H_n$ in directed graphs

Anshelevich, Dasgupta, Kleinberg, Tardos, Wexler, Roughgarden FOCS'04

Weighted games

Agent i has a weight w_i

$$\text{Cost}(i, \vec{S}) = \sum_{e \in S_i} c(e) \frac{w_i}{W_e}$$

W_e = total weight of agents using e

Directed graphs

$P' = \Omega(\log W)$ $\qquad W = \sum_{i=1}^{n} w_i$

Price of stability H_n

$$\Phi(\vec{S}) = \sum_{e \in E} c(e) H_{n_e}$$

n_e = #agents using e

While $\vec{S}$ is no NE

Improvement move by agent i: $\Delta\Phi$ = cost saving of i

Price of stability H_n

$$\Phi(\vec{S}) = \sum_{e \in E} c(e) H_{n_e}$$

n_e = #agents using e

While $\vec{S}$ is no NE

Improvement move by agent i: $\Delta\Phi$ = cost saving of i

Start process at OPT

$$\Phi(OPT) = \sum_{e \in OPT} c(e) H_{n_e} \leq \text{Cost}(OPT) H_n$$

Price of stability H_n

$$\Phi(\vec{S}) = \sum_{e \in E} c(e) H_{n_e}$$

Improvement move by agent i: $\Delta\Phi$ = cost saving of i

- e is dropped

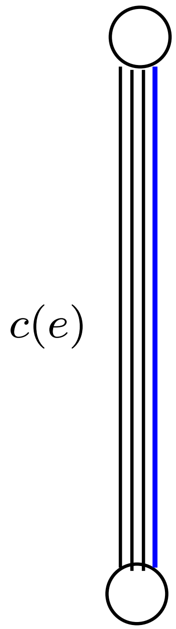
  $\Delta\Phi$: $c(e)H_{n_e} - c(e)H_{n_e - 1}$

  cost saving: $c(e)\frac{1}{n_e}$

- e is added

  $\Delta\Phi$: $c(e)H_{n_e+1} - c(e)H_{n_e}$

  cost increase: $c(e)\frac{1}{n_e+1}$



Coordination among agents

Albers 2008

Motivation

- Agents may communicate and discuss possible strategies.
- Agents form coalitions taking strategic actions that are beneficial to all members.

Questions

- Is it possible to achieve significant improvements?
- How do stable states rank relative to the best ones without coordination?

Strong Nash equilibria

Aumann 1959

No coalition can change strategy s.t. all agents of coalition strictly improve.

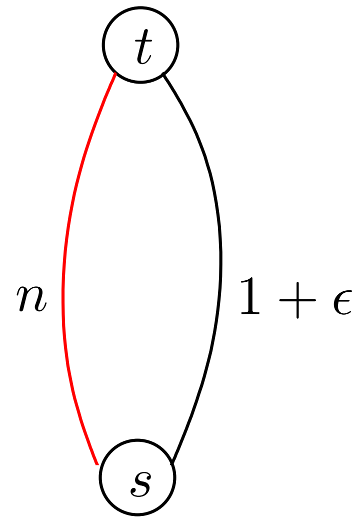
$\vec{S}$ forms strong Nash equilibrium if for any coalition I and strategy change $\vec{S}\prime$,

$$\text{Cost}(i, \vec{S}) \leq \text{Cost}(i, \vec{S}\prime)$$

for at least one $i \in I$.

Extendable to coalitions of bounded size

No strong NE



n agents

Our results

$\exists$ graphs not admitting strong Nash equilibria

Coalition size max. $k \implies \exists$ α-approximate strong Nash eq. for $\alpha \geq H_k$

Upper bounds

$P \leq H_n$ coalitions of any size $H_n = \sum_{i=1}^{n} 1/i$

$P \leq \frac{n}{k} H_k$ coalition size max. k $H_n \approx \ln n$

Lower bounds

$P \geq H_n$ directed graphs

$P = \Omega(\sqrt{\log n})$ undirected graphs

For α-approximate strong Nash eq. all bounds multiply by α.

Albers 2008

α-approximate strong NE

$\vec{S}$ is α-approximate strong NE, where $\alpha \geq 1$, if for any coalition I and strategy change $\vec{S'}$,

$$\frac{1}{\alpha}\text{Cost}(i, \vec{S}) \leq \text{Cost}(i, \vec{S'})$$

for at least one $i \in I$.

Our results

Weighted games

$\exists$ α-approximate strong Nash eq. for $\alpha \geq 1 + \ln W$

Upper bounds

$P \leq 1 + \ln W$ $\qquad W = \sum_{i=1}^{n} w_i$

Lower bounds

$P = \Omega(\log W)$ directed graphs

$P = \Omega(\sqrt{\log W})$ undirected graphs

For α-approximate strong Nash eq. all bounds multiply by α.

Results extendable to coalitions of bounded size or weight.

Our results

Undirected graphs, price of stability

$P' = \Omega(\frac{\log W}{\log \log W})$

Conclusions:

Strong Nash eq. achieve exponential improvements in the PoA relative to classical Nash eq.

Strong Nash eq. always as good as the best classical Nash eq.

unweighted games, directed graphs	$P = H_n$	$P' = H_n$
weighted games	$P \leq 1 + \ln W$	$P' = \Omega(\log W)$
		$P' = \Omega(\frac{\log W}{\log \log W})$

Upper bound $P \leq H_n$

Consider strong Nash eq. $\vec{S}$

K = coalition consisting of all agents

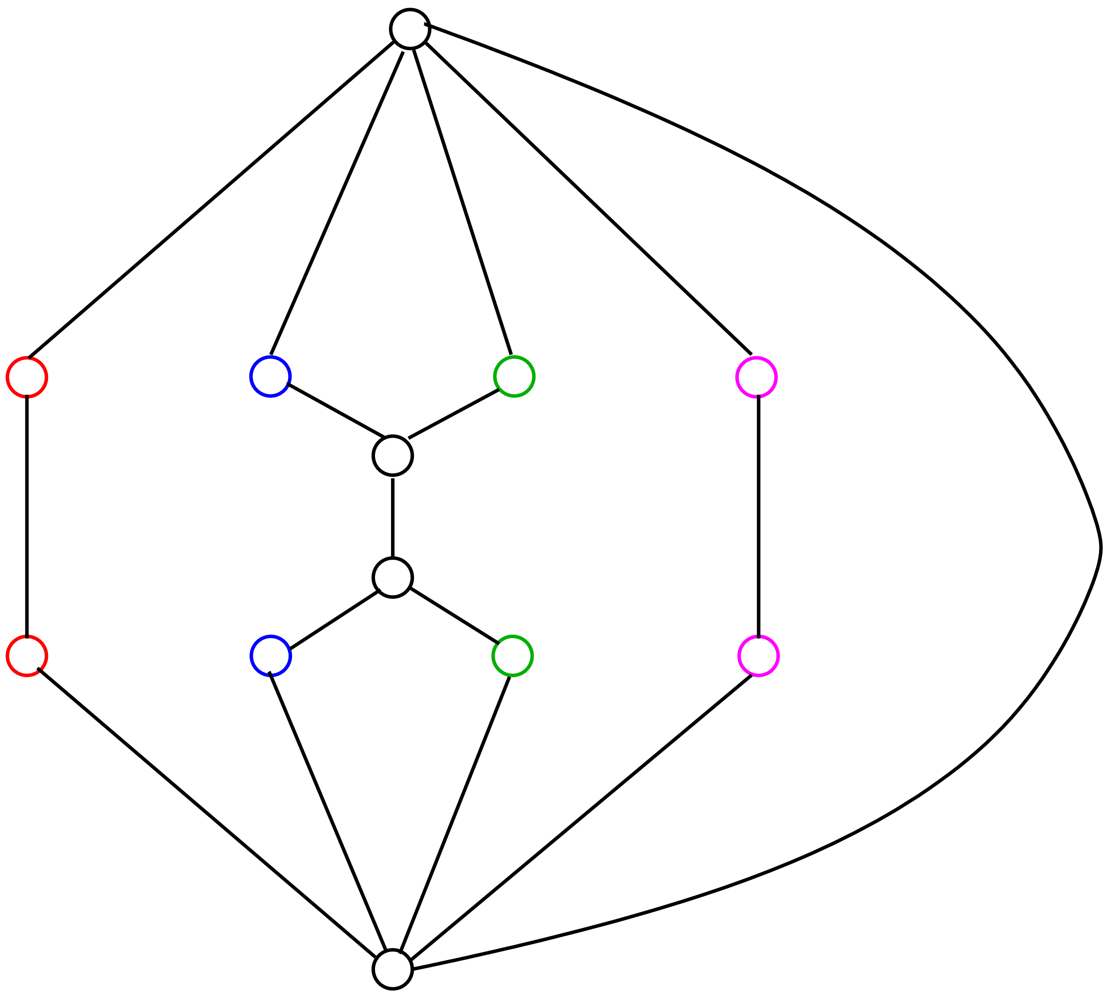
repeat

strategy change K purchases OPT;

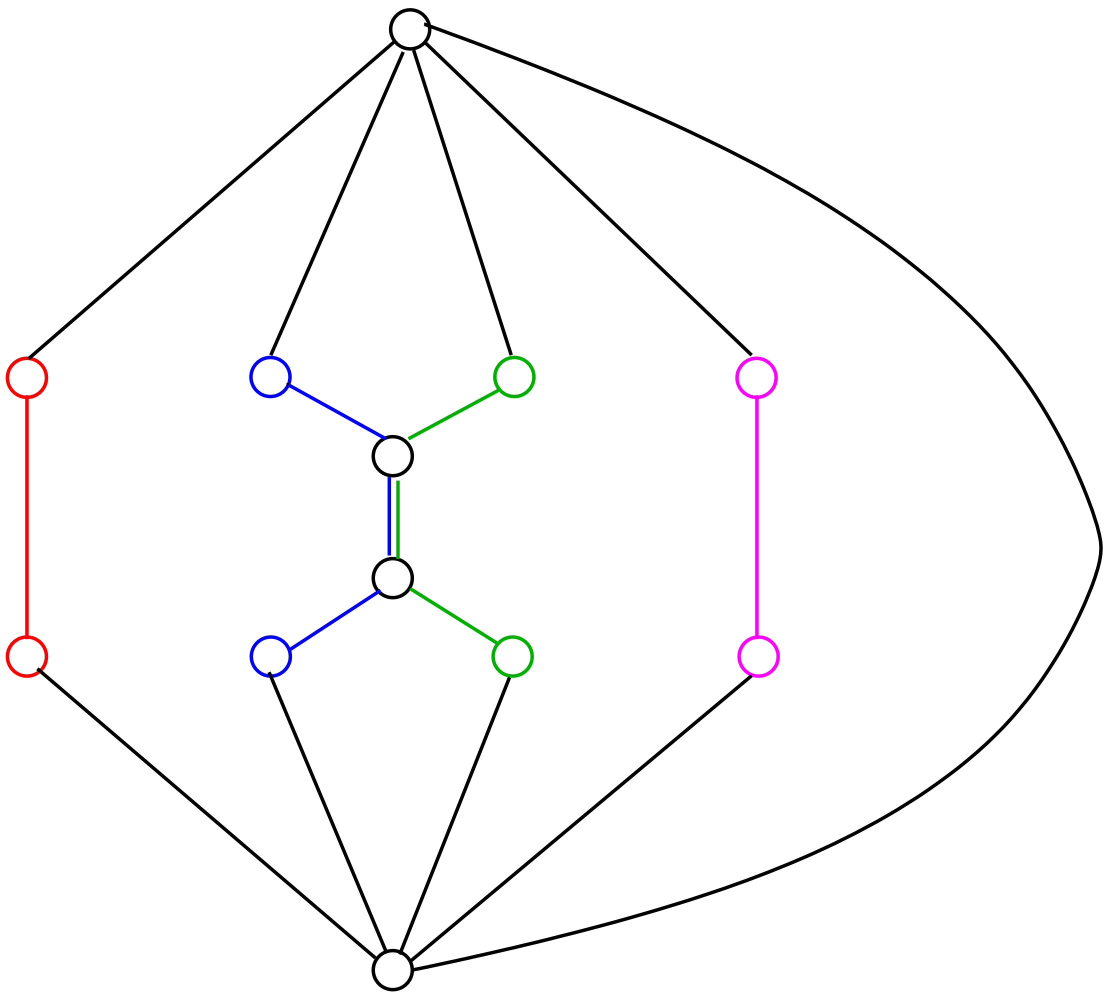
agent i not satisfied $K := K \setminus \{i\}$;

until $K = \emptyset$;

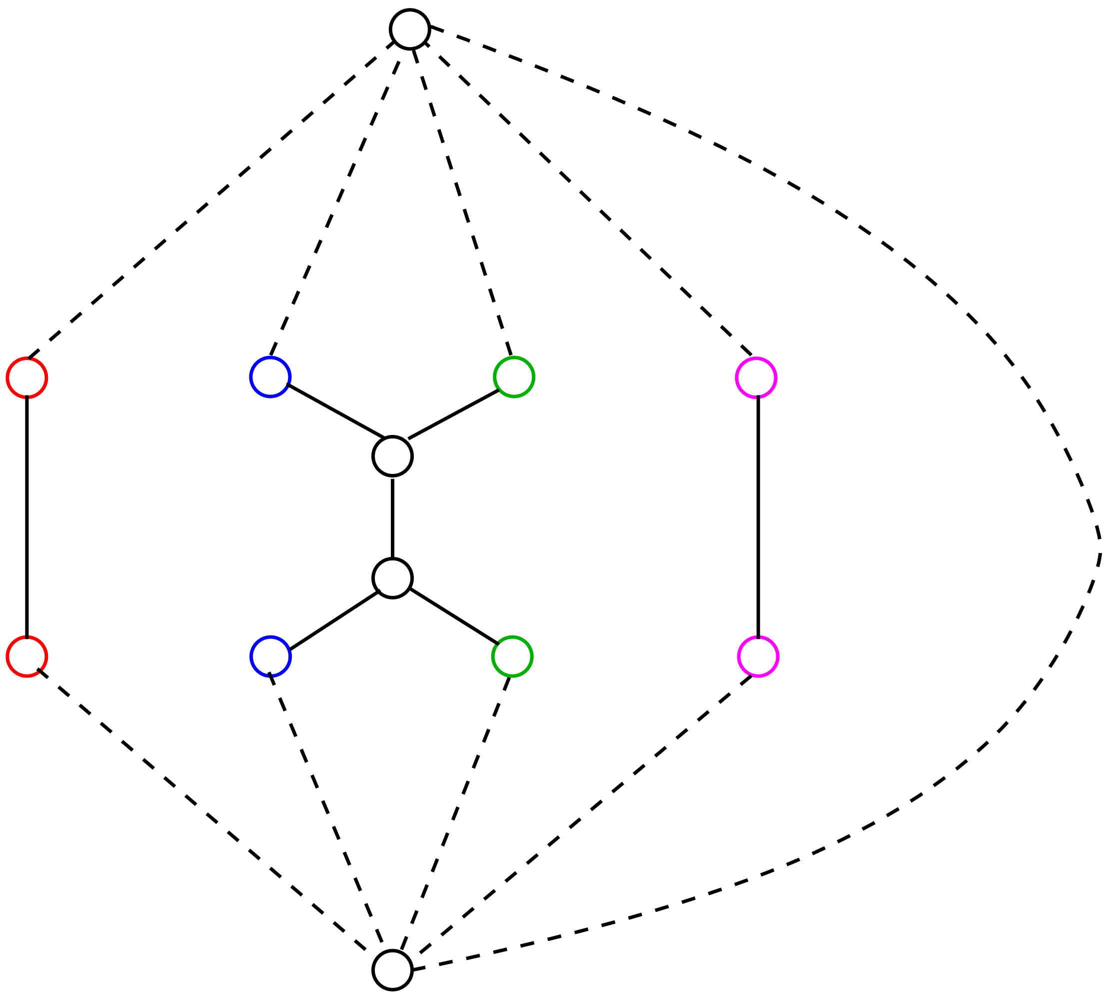
Upper bound $P \leq H_n$



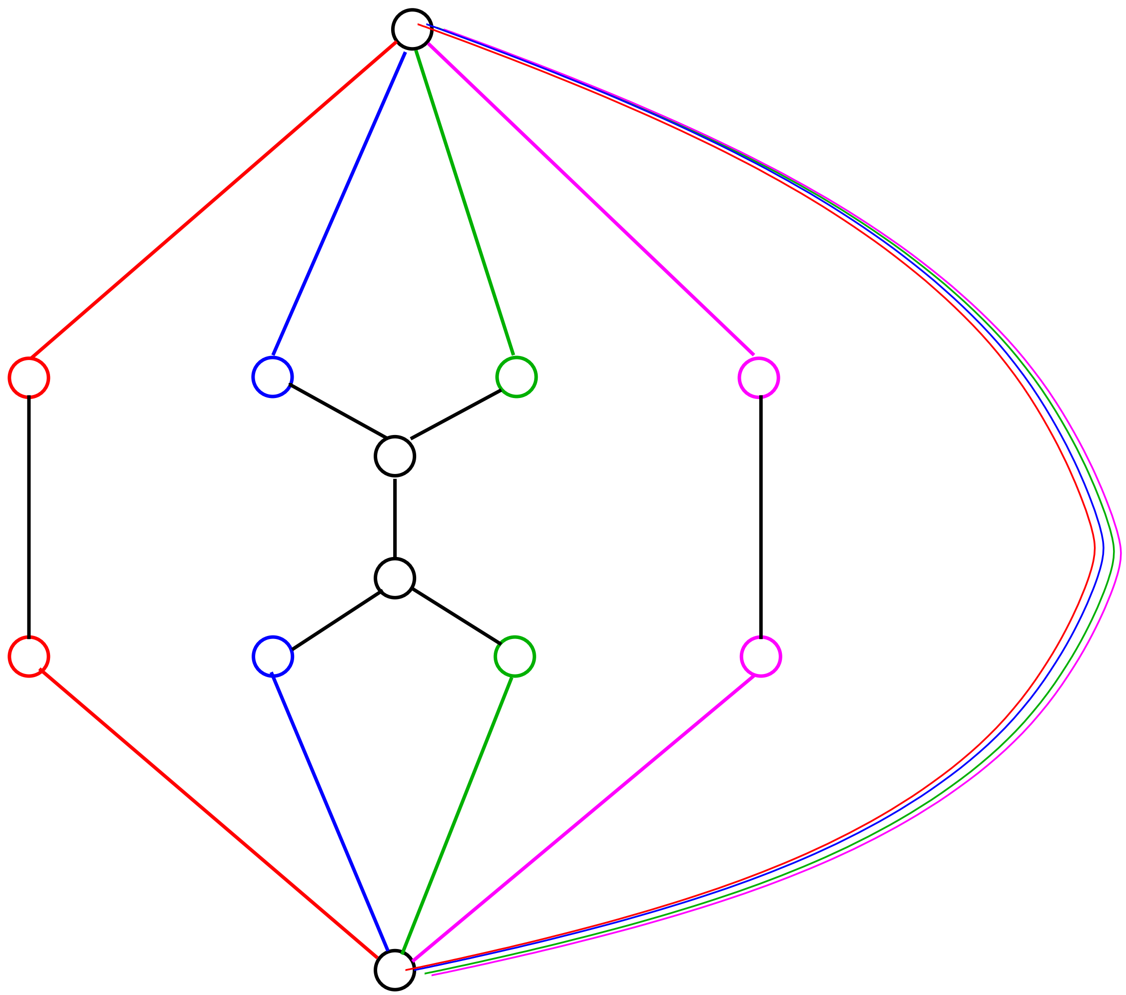
Strong Nash equilibrium



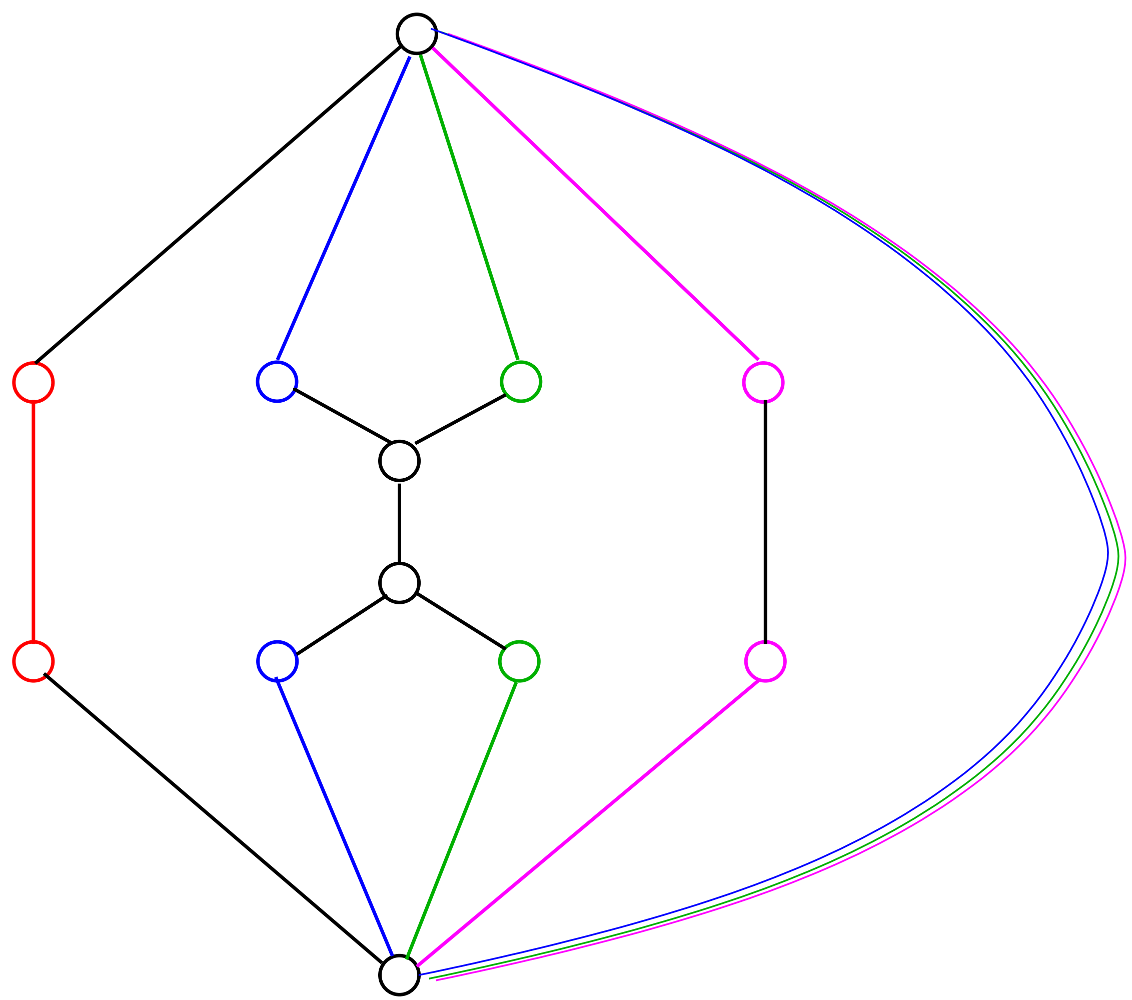
Optimal solution



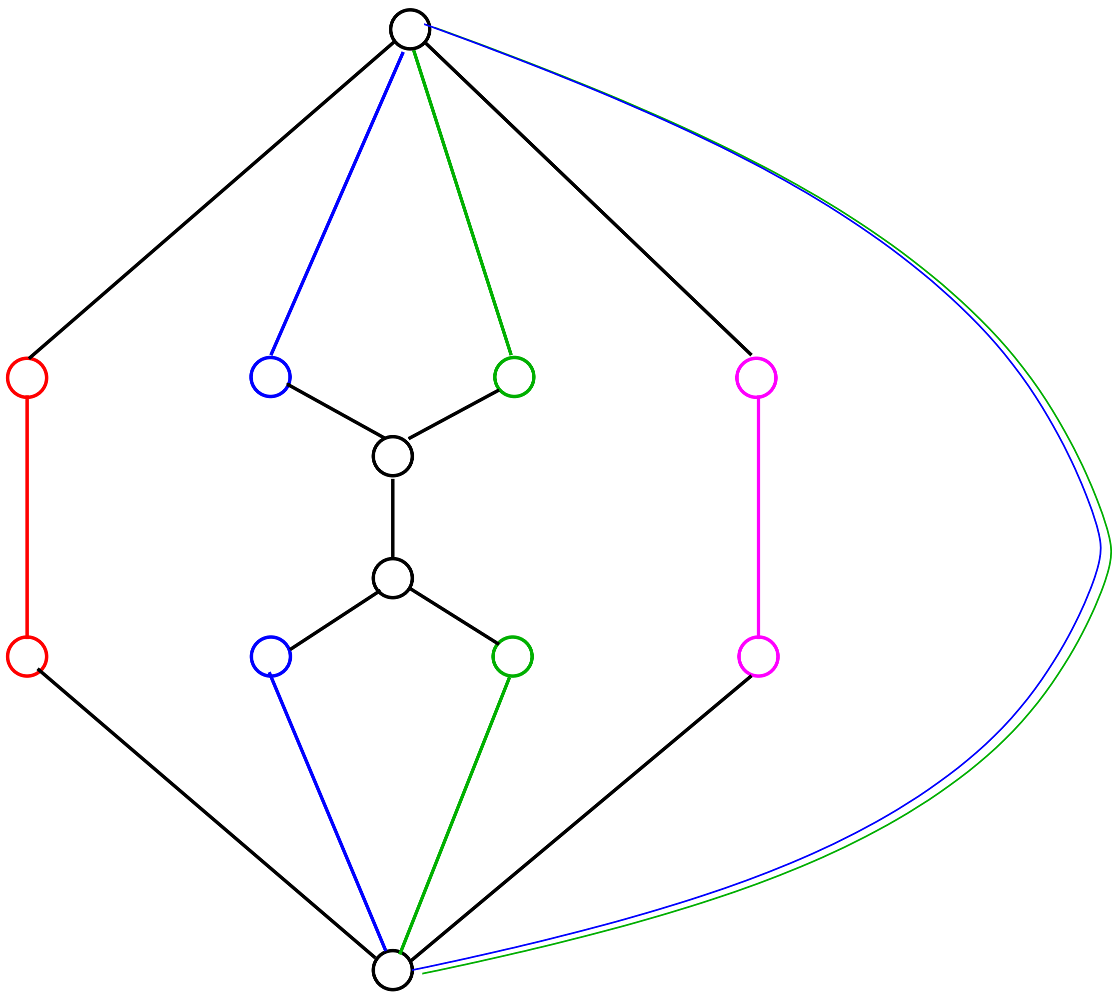
Strategy change 1



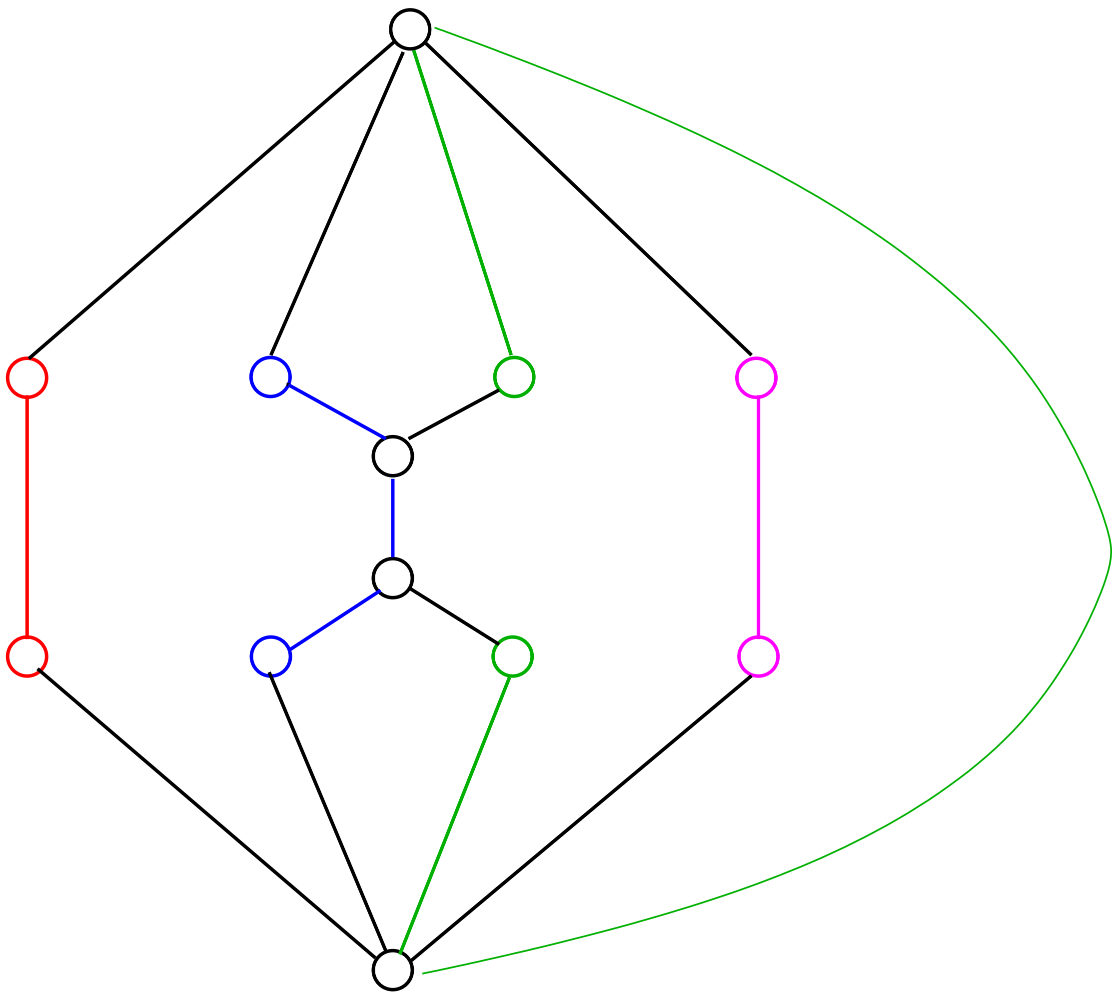
Strategy change 2



Strategy change 3



Strategy change 4



Cost analysis

In each iteration i: Agent i leaving coalition

$$\begin{array}{c}\text{Cost of } i \\ \text{in strong Nash eq.}\end{array} \leq \begin{array}{c}\text{Cost of } i \\ \text{in OPT in stategy change } i\end{array}$$

Cost analyse

In each iteration i: Agent i leaving coalition

$$\begin{array}{c}\text{Cost of } i \\ \text{in strong Nash eq.}\end{array} \leq \begin{array}{c}\text{Cost of } i \\ \text{in OPT in stategy change } i\end{array}$$

$$\text{Cost}(\vec{S}) \leq \sum_{i=1}^{n} \text{Cost of } i \text{ in OPT in strategy change } i$$

Cost analysis

In each iteration i: Agent i leaving coalition

$$\begin{array}{c}\text{Cost of } i \\ \text{in strong Nash eq.}\end{array} \leq \begin{array}{c}\text{Cost of } i \\ \text{in OPT in strategy change } i\end{array}$$

$$\text{Cost}(\vec{S}) \leq \sum_{i=1}^{n} \text{Cost of } i \text{ in OPT in strategy change } i$$

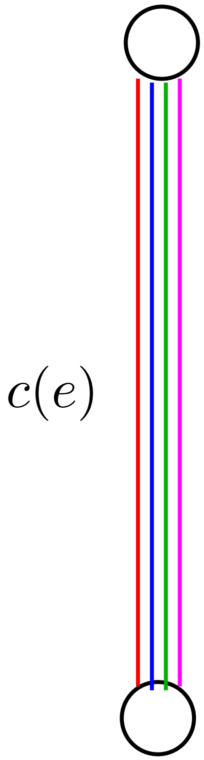
$$= \sum_{i=1}^{n} \sum_{e \in OPT} \text{Cost of } i \text{ on } e \text{ in strategy change } i$$

Edge e in OPT

ℓ = $\#$ agents i using e in OPT

Cost share

$\frac{1}{\ell}c(e)$



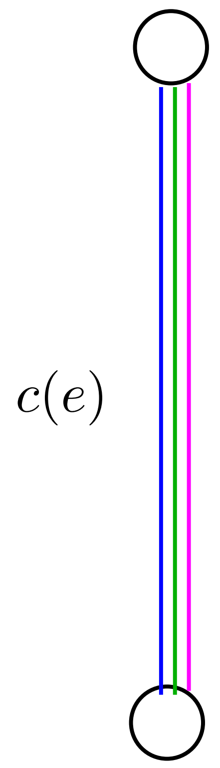
Edge e in OPT

ℓ = $\#$ agents i using e in OPT

Cost share

$\frac{1}{\ell}c(e)$

$\frac{1}{\ell-1}c(e)$



Edge e in OPT

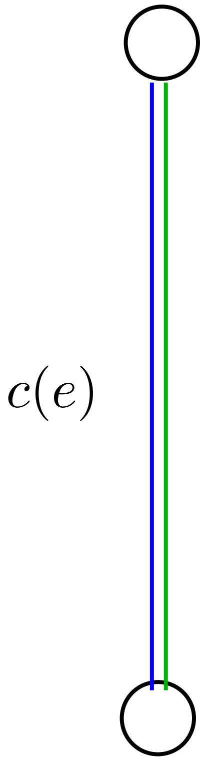
ℓ = $\#$ agents i using e in OPT

Cost share

$\frac{1}{\ell}c(e)$

$\frac{1}{\ell-1}c(e)$

$\frac{1}{\ell-2}c(e)$



Edge e in OPT

ℓ = # agents i using e in OPT

Cost share

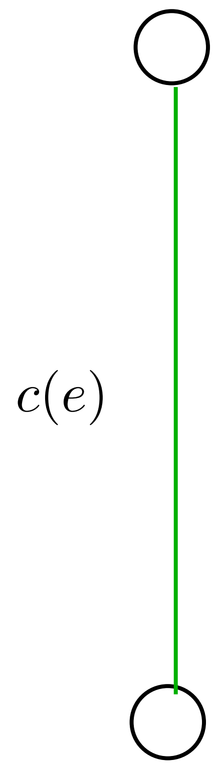
$\frac{1}{\ell}c(e)$

$\frac{1}{\ell-1}c(e)$

$\frac{1}{\ell-2}c(e)$

$\vdots$

$\frac{1}{1}c(e)$



Edge e in OPT

ℓ = # agents i using e in OPT

Cost share

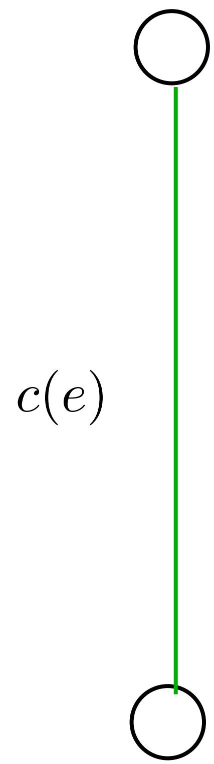
$\frac{1}{\ell}c(e)$

$\frac{1}{\ell-1}c(e)$

$\frac{1}{\ell-2}c(e)$

$\vdots$

$\frac{1}{1}c(e)$



In total : $(\frac{1}{\ell} + \frac{1}{\ell-1} + \frac{1}{\ell-2} + \ldots + \frac{1}{3} + \frac{1}{2} + 1)c(e)$

Cost analysis

Edge $e \in OPT$: cost contribution $\leq c(e)(\frac{1}{n} + \frac{1}{n-1} + \ldots + \frac{1}{2} + 1) = c(e)H_n$

$$\text{Cost}(\vec{S}) \leq \sum_{e \in OPT} c(e)H_n = H_n\text{Cost(OPT)}$$

Open problems

First network design game

Bounded degree graphs.

Edges incur cost but also yield benefit.

Second network design game

Improve lower bounds.

Price of stability in unweighted games and undirected graphs.

Formation of coalitions is restricted.