Noncommutative Geometry and Fundamental Physics: Constructing the Spectrum of Space, Time and Matter

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Overview

1. Basic Ideas
2. Geometry
3. Physics
4. Conlusions

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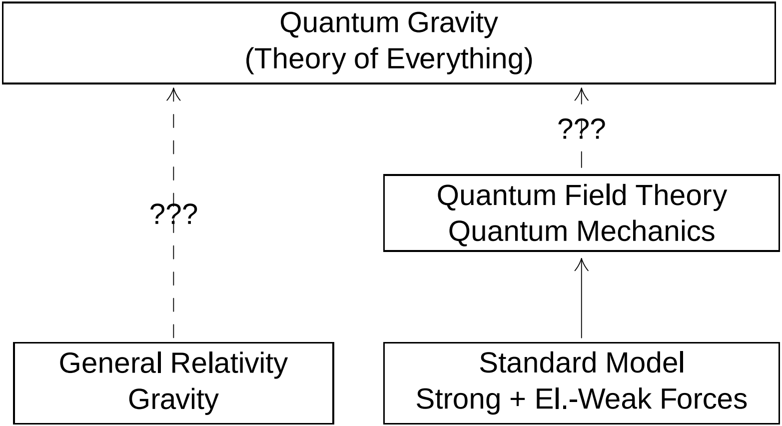
Schematic Structure of modern Physics:



Quantum Gravity
(Theory of Everything)

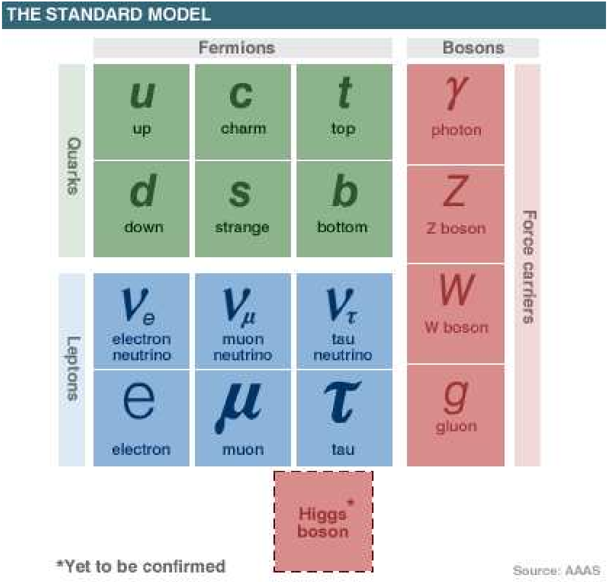
???

???

Quantum Field Theory
Quantum Mechanics

General Relativity
Gravity

Standard Model
Strong + El.-Weak Forces



THE STANDARD MODEL

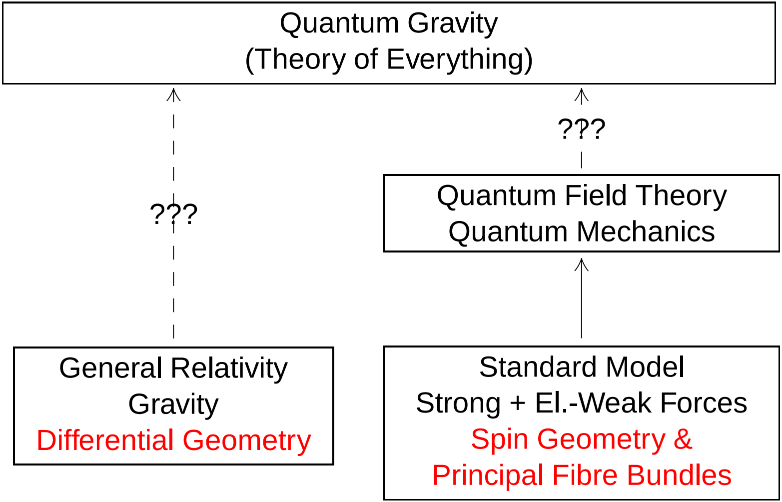
	Fermions			Bosons	
Quarks	u up	c charm	t top	γ photon	Force carriers
	d down	s strange	b bottom	Z Z boson	
Leptons	ν_e electron neutrino	ν_μ muon neutrino	ν_τ tau neutrino	W W boson	
	e electron	μ muon	τ tau	g gluon	

Higgs boson*

*Yet to be confirmed

Source: AAAS

Schematic Structure of modern Physics:



Quantum Gravity
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Quantum Field Theory
Quantum Mechanics

General Relativity
Gravity
Differential Geometry

Standard Model
Strong + El.-Weak Forces
Spin Geometry &
Principal Fibre Bundles

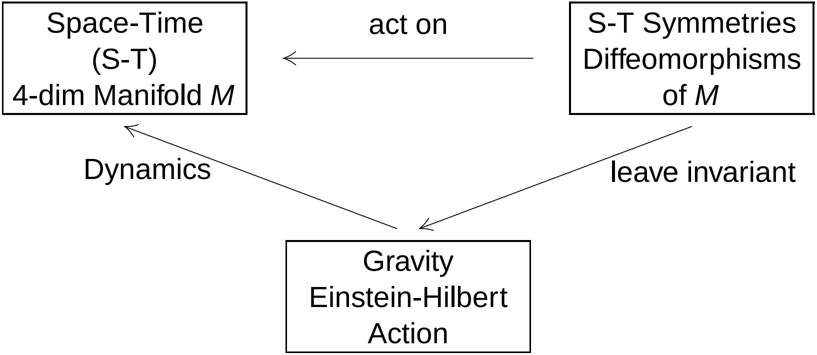
Aim of Noncommutative Geometry in Physics:

Aim: To unify general relativity (GR) and the standard model of particle physics (SM) on the same geometrical level.
This means to describe gravity and the electro-weak and strong forces as gravitational forces of a generalised “space-time”.
(Get predictions!)

Idea: Find generalised “space-time” with symmetries of General Relativity and the Standard Model (SM).

Pos.Sol.: Try a special type of noncommutative geometry (A. Connes)

Schematic Structure of GR:
Gravity emerges as a pseudo-force associated to the space-time symmetries, i.e. the diffeomorphisms of the manifold M.



Space-Time
(S-T)
4-dim Manifold M

act on

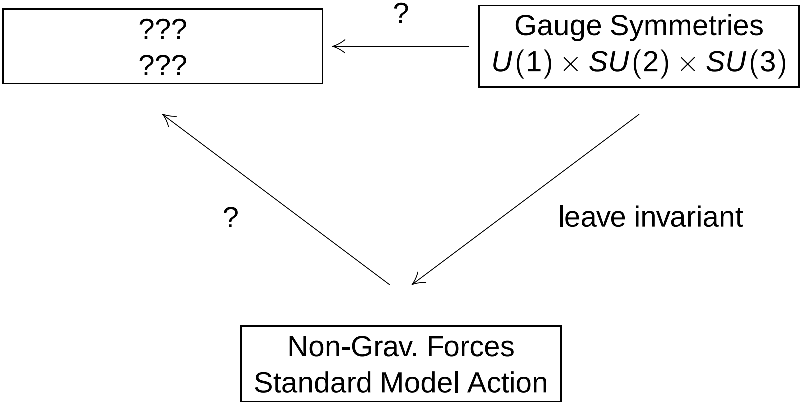
S-T Symmetries
Diffeomorphisms
of M

Dynamics

leave invariant

Gravity
Einstein-Hilbert
Action

How to find an underlying geometric structure for the Standard Model, which is equivalent to space time:



???
???

?

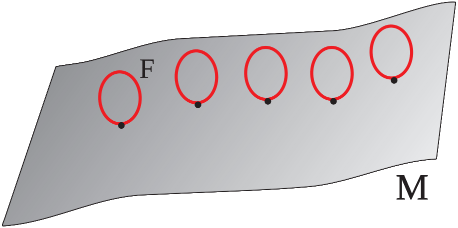
Gauge Symmetries
$U(1) \times SU(2) \times SU(3)$

?

leave invariant

Non-Grav. Forces
Standard Model Action

Analogy: Almost-comm. geometry ↔ Kaluza-Klein space

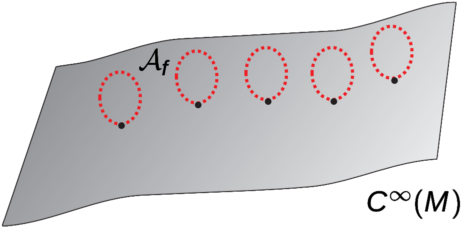


Idea: $M \to C^\infty(M)$,

$F \to$ some "finite space",

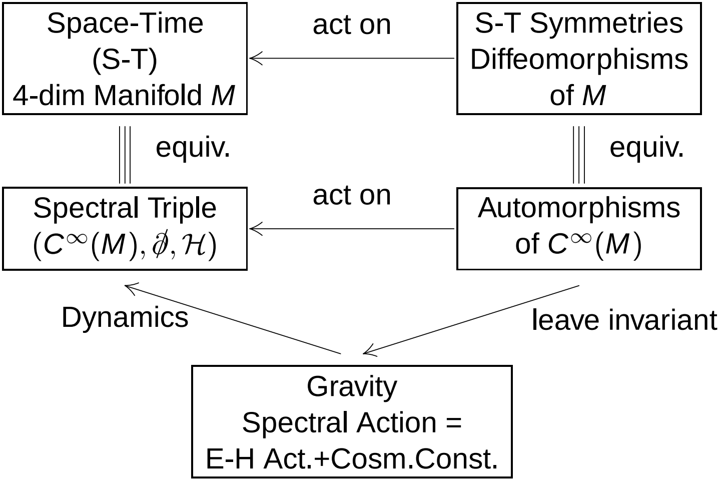
differential geometry → spectral triple

Almost-commutative Geometry



”finite space” → $\mathcal{A}_f = M_1(\mathbb{K}) \oplus M_2(\mathbb{K}) \oplus \ldots$
Kaluza-Klein space → almost-com. geometry, $\mathcal{A} = C^\infty(M) \otimes \mathcal{A}_f$
Notion of classical “point” looses its meaning!

Euclidean space-time!



Space-Time
(S-T)
4-dim Manifold M

act on

S-T Symmetries
Diffeomorphisms
of M

equiv.

equiv.

Spectral Triple
$(C^\infty(M), \partial\!\!\!/, \mathcal{H})$

act on

Automorphisms
of $C^\infty(M)$

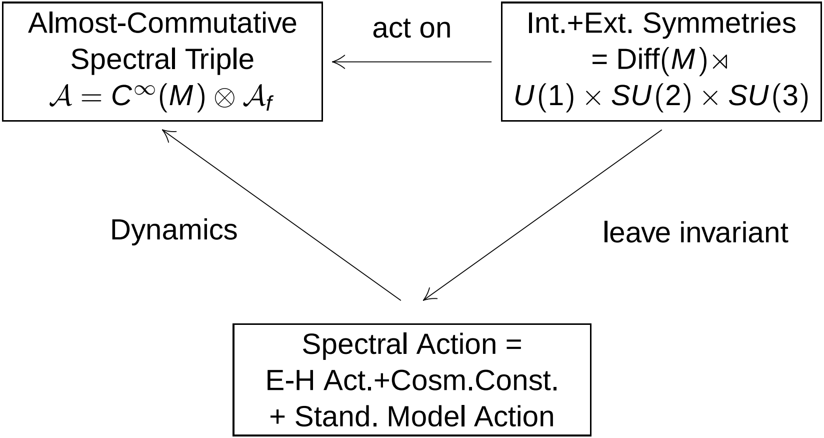
Dynamics

leave invariant

Gravity
Spectral Action =
E-H Act.+Cosm.Const.

Add finite Space: Almost-Commutative Standard Model

(A.Chamseddine, A.Connes):



Almost-Commutative
Spectral Triple
$\mathcal{A} = C^{\infty}(M) \otimes \mathcal{A}_f$

act on

Int.+Ext. Symmetries
= Diff$(M)\rtimes$
$U(1) \times SU(2) \times SU(3)$

Dynamics

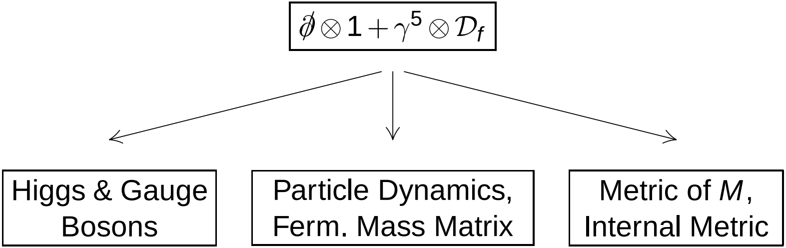
leave invariant

Spectral Action =
E-H Act.+Cosm.Const.
+ Stand. Model Action

The almost-commutative standard model automatically produces:

- The combined Einstein-Hilbert and standard model action
- A cosmological constant
- The Higgs boson with the correct quartic Higgs potential

The Dirac operator plays a multiple role:



$$\boxed{\not{\partial} \otimes 1 + \gamma^5 \otimes \mathcal{D}_f}$$

Higgs & Gauge Bosons

Particle Dynamics, Ferm. Mass Matrix

Metric of M, Internal Metric

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An even, real spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$**;**
the ingredients (A. Connes):

- A real, associative, unital pre-C^*-algebra $\mathcal{A}$
- A Hilbert space $\mathcal{H}$ on which the algebra $\mathcal{A}$ is faithfully represented via a representation ρ
- A self-adjoint, first order operator $\mathcal{D}$ with compact resolvent, the Dirac operator
- An anti-unitary operator J on $\mathcal{H}$, the real structure
- A unitary operator γ on $\mathcal{H}$, the chirality

The ‘classical” conditions or axioms of noncommutative geometry (A. Connes 1996):

Condition 1: Classical Dimension *n* (we assume *n* even)
Condition 2: Regularity (Algebra)
Condition 3: Finiteness (Hilbert space)
Condition 4: First Order of the Dirac Operator
Condition 5: Poincaré Duality
Condition 6: Orientability
Condition 7: Reality

Some important Theorems

- Gelfand-Naimark Theorem:
  Commutative, unital C*-Algebra $\leftrightarrow$ topological Space
- Weyl Asymptotics for $\mathcal{D}$:
  Growth of Eigenvalues of $\mathcal{D}$ $\leftrightarrow$ Dimension of Manifold
- Connes' Distance Formula:
  $d(p,q) = sup\{f(p) - f(q) | f \in C^\infty(M),\ ||[\mathcal{D}, f]|| = |grad\, f| \leq 1\}$
- Connes' Reconstruction Conjecture (Theorem):
  Commutative Spectral Triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$
  $\leftrightarrow$ compact Riemannian Spin Manifold

Almost-commutative geometry:

An almost-commutative spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is defined as the tensor product of a spectral triple $(\mathcal{A}_M = C^\infty(\mathcal{M}), \mathcal{H}_M = L^2(\Sigma_M), \mathcal{D}_M = \not{\partial})$ (for space-time $n_M = 4$) and a finite spectral triple $(\mathcal{A}_f, \mathcal{H}_f, \mathcal{D}_f)$ with metric dimension $n_f = 0$.

$$\mathcal{A} = \mathcal{A}_M \otimes \mathcal{A}_f, \quad \mathcal{H} = \mathcal{H}_M \otimes \mathcal{H}_f,$$

$$J = J_M \otimes J_f, \quad \gamma = \gamma_M \otimes \gamma_f,$$

$$\mathcal{D} = \mathcal{D}_M \otimes 1_f + \gamma_M \otimes \mathcal{D}_f$$

$$\mathrm{Aut}(\mathcal{A}_M \otimes \mathcal{A}_f) = \mathrm{Diff}(M) \rtimes \mathrm{U}(\mathcal{A}_f)$$

The noncommutative setup imposes constraints:

- mathematical axioms
  => Restrictions on particle content
- symmetries of finite space
  => determines gauge group
- representation of matrix algebra
  => representation of gauge group
- Dirac operator => allowed mass terms / Higgs fields

Almost-comm. geom. $\leftrightarrow$ Fibre bundle formalism

Gauge sector: $Spin(4) \times G_{SM}-$principle bundle

Fermions: sections Ψ in assoc. vect. bundle $\Sigma \otimes H_f \to M$

Higgs boson: section φ in assoc. vect. bundle $V_{Higgs} \to M$

generalised twisted Dirac operator: $\mathcal{D} = D(\tilde{\nabla}) + \gamma_5 \otimes \Phi(\varphi)$

Physics (action functional): follows from spectrum of $\mathcal{D}$

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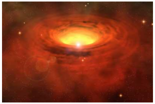
How to construct the Action?
Observation/Experiment $\longleftrightarrow$ Analysing Spectra



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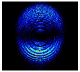
Star/Galaxy

Detector/Telescope

$\downarrow$
$\downarrow$

$\downarrow$
$\downarrow$



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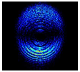

Fermion

Photon

Fermion

$\downarrow$

$\downarrow$

$\downarrow$

$\sim$ Section in Spinorbundle

$\sim$ Eigenvalues of the Dirac Operator

$\sim$ Section in Spinorbundle

The Spectral Action (A. Connes, A. Chamseddine 1996):

$$(\Psi, \mathcal{D}\Psi) + S_\Lambda(\mathcal{D}) \quad \text{with } \Psi \in \mathcal{H}$$

- $(\Psi, \mathcal{D}\Psi)$ = fermionic action
  includes Yukawa couplings
  & fermion–gauge boson interactions
- $S_\Lambda(\mathcal{D})$ = bosonic action
  = $\sharp$ eigenvalues of $\mathcal{D}$ up to cut-off Λ
  = Einstein-Hilbert action + Cosm. Const.
  + full bosonic SM action
- constraints => less free parameters than classical SM

Consequences from the SM constraints:

Input:

- Big Desert ↔ choice of finite geometry
- $g_2(m_Z) = 0.6514,\ \ g_3(m_Z) = 1.221$ (from Experiment)
- renormalisation group equations (not so well defined)
- ($m_{top} = 171.2 \pm 2.1$ GeV) (from Experiment)

Output:

- $g_2^2(\Lambda) = g_3^2(\Lambda)$ at $\Lambda = 1.1 \times 10^{17}$ GeV
- $m_{Higgs} = 168.3 \pm 2.5$ GeV
- $m_{top} < 190$GeV

Waiting for CERN and TEVATRON results!

Classification of the finite spaces:

$$\mathcal{A}_f = M_1(\mathbb{K}) \oplus M_2(\mathbb{K}) \oplus \ldots$$

- with respect to the number of summands in the algebra
- with respect to physical criteria

Physicist's "shopping list" (B. Iochum, T. Schücker, C.S. 2003):

- minimal internal Hilbert space
- have a non-degenerate Fermionic mass spectrum

=> The Standard Model is a "minimal" noncom. space

Some areas of Mathematics connected to Noncommutative Geometry

Spectral triples:

- Spin Geometry (Clifford Algebras/Moduls)
- Operator Algebras (von Neumann algebras)
- K-Theory, Morita equivalence
- ...

Spectral action:

- Spectral Geometry (Eigenvalues of Dirac Operators)
- Heat kernel asymtotics, Laplace transforms
- Statistical Physics, Stochastics
- ...

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A little To-do-List:

- Classify finite Spaces
  => Models beyond the Standard Model
  (LHC signature and/or dark matter?)
- Spectral triples with Lorentzian signature
- Action principle in Lorentzian signature
- Spectral triples for Supermanifolds
- Connection to Statistical Physics

Bibliography:

- Noncommutative Geometry, Quantum Fields and Motives
  A. Connes & M. Marcolli
  http://www.alainconnes.org/en/downloads.php
- Forces from Connes' geometry, T. Schücker
  Lect.Notes Phys.659:285-350,2005, hep-th/0111236
- Elements of Noncommutative Geometry
  J. Gracia-Bondia, H. Figueroa & J. C. Varilly
  Birkhäuser Advanced Texts
- Nichtkommutative Geometrie, C. Bär
  http://geometrie.math.uni-potsdam.de
- An Introduction to Noncommutative Spaces and their Geometry
  R. Wulkenhaar
  wwwmath.uni-muenster.de/u/raimar