Topological Rigidity

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Outline

Conjecture (Borel Conjecture)

Two aspherical closed manifolds are homeomorphic if and only if their fundamental groups are isomorphic.

- Explain this conjecture. Put it into a general context. Report on its status.

Theorem (Bartels-Lück (2008))

The Borel Conjecture is true if the fundamental group is hyperbolic or CAT(0).

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Some basic notions

Definition (Homeomorphism)

A *homeomorphism* $f\colon X \to Y$ between topological spaces is a (continuous) map such that there exists a (continuous) map $g\colon Y \to X$ with $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.

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Definition (Manifold)

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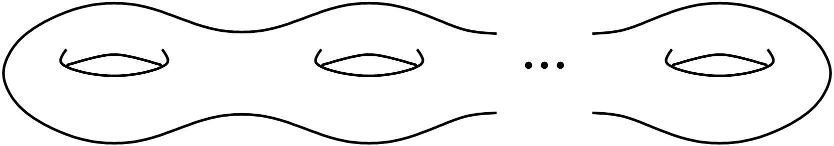
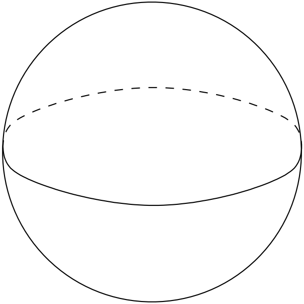
- A 2-dimensional orientable closed manifold is homeomorphic to the standard surface of genus g for precisely one g.



- Let $f\colon \mathbb{R}^m \to \mathbb{R}^n$ be a smooth map and $y \in \mathbb{R}^n$ be a regular value. Then the preimage $f^{-1}(y)$ is a manifold.

- An example is the *n-dimensional sphere*

$$S^n = \left\{ (x_0, x_2, \ldots, x_n) \in \mathbb{R}^{n+1} \,\middle|\, \sum_{i=0}^{n} x_i^2 = 1 \right\}$$



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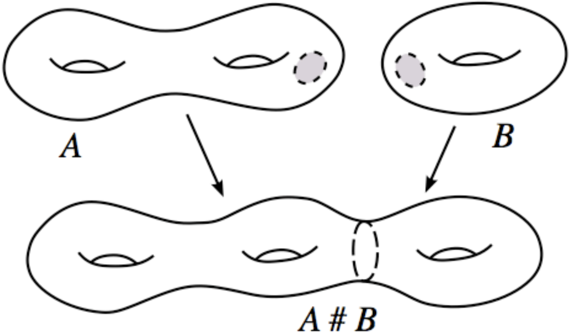
- The connected sum $A \# B$ of two n-dimensional manifolds A and B is again one.



Definition (Homotopy)

Two maps $f_0, f_1 \colon X \to Y$ between topological spaces are called *homotopic* $f_0 \simeq f_1$, if and only if there is *homotopy* between them, i.e., a map

$$h \colon X \times [0,1] \to Y$$

satisfying $h(x,0) = f_0(x)$ and $h(x,1) = f_1(x)$ for all $x \in X$.

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- A homeomorphism is a homotopy equivalence.
- The converse is not true in general. For instance $\mathbb{R}^n$ and $\mathbb{R}^m$ are homotopy equivalent for all m, n, but they are homeomorphic if and only if $m = n$.

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Definition (Contractible space)

A space is called *contractible* if the projection $X \to \{\bullet\}$ to the one-point-space is a homotopy equivalence.

- A space is contractible if and only if the identity is homotopic to a constant map.
- Any convex or star-shaped subset of $\mathbb{R}^n$ is contractible. In particular $\mathbb{R}^n$ is contractible.
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Let (X, x) be a pointed space. Its *fundamental group* $\pi_1(X, x)$ has as elements pointed homotopy classes of loops with base point x, i.e., pointed maps $(S^1, 1) \longrightarrow (X, x)$.

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- If X is path-connected, the fundamental group is independent of the choice of the base point up to group isomorphism.
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- Let X be a path-connected space. Its universal covering $\widetilde{X} \to X$ is the unique covering for which the total space is simply-connected.
- Its group of deck transformations can be identified with $\pi_1(X)$. In particular we rediscover X from $\widetilde{X}$ by

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- *Suppose that $k + d \neq 3$. Then $S^k \times S^d$ is topologically rigid if and only if both k and d are odd.*
- *Every closed 3-manifold with torsionfree fundamental group is topologically rigid.*
- *Let M and N be closed manifolds of the same dimension $n \geq 5$ with torsionfree fundamental groups. If both M and N are topologically rigid, then the same is true for their connected sum $M \# N$.*

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Let M^{4k+3} be a closed oriented smooth manifold for $k \geq 1$ whose fundamental group has torsion. Then M is not topologically rigid.

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- For instance, there are smooth manifolds M which are homeomorphic to T^n but not diffeomorphic to T^n.
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The Kaplansky Conjecture says for a torsionfree group G and an integral domain R that 0 and 1 are the only idempotents in RG.

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If G is of type FP, then G is already of type FF.

Conjecture (Whitehead group)

If G is torsionfree, then the Whitehead group $\operatorname{Wh}(G)$ *vanishes.*

Conjecture (Novikov Conjecture)

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If G satisfies both the K-theoretic and L-theoretic Farrell-Jones Conjecture (for any additive G-category as coefficients), then all the conjectures mentioned above (and further conjectures) will follow for G.

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- The Borel Conjecture (for dim ≥ 5),
- Kaplansky Conjecture (for R a field of characteristic zero),
- Vanishing of $\widetilde{K}_0(RG)$ and $\mathrm{Wh}(G)$,
- Serre's Conjecture,
- Novikov Conjecture (for dim ≥ 5),
- other conjecture, e.g., the ones due to Bass and Moody, the one about Poincaré duality groups (for dim ≥ 5) and the one about the homotopy invariance of L^2-torsion.

The status of the Farrell-Jones Conjecture

Theorem (Bartels-Lück (preprint will be available in the beginning of 2009))

Let $\mathcal{FJ}$ be the class of groups for which both the K-theoretic and the L-theoretic Farrell-Jones Conjectures holds (in his most general form, namely with coefficients in any additive G-category) has the following properties:

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- *If* G_1 *and* G_2 *belong to* $\mathcal{FJ}$*, then* $G_1 \times G_2$ *and* $G_1 * G_2$ *belong to* $\mathcal{FJ}$*;*
- *If* H *is a subgroup of* G *and* $G \in \mathcal{FJ}$*, then* $H \in \mathcal{FJ}$*;*
- *Let* $\{G_i \mid i \in I\}$ *be a directed system of groups (with not necessarily injective structure maps) such that* $G_i \in \mathcal{FJ}$ *for* $i \in I$*. Then* $\operatorname{colim}_{i \in I} G_i$ *belongs to* $\mathcal{FJ}$*.a*

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- There are many constructions of groups with exotic properties which arise as colimits of hyperbolic groups.
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- However, our results show that these groups do satisfy the Farrell-Jones Conjecture in its most general form and hence also the other conjectures mentioned above.

- Mike Davis (1983) has constructed exotic closed aspherical manifolds using hyperbolization techniques. For instance there are examples which do not admit a triangulation or whose universal covering is not homeomorphic to Euclidean space.
- However, in all cases the universal coverings are CAT(0)-spaces and hence the fundamental groups are CAT(0)-groups.
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- There are still many interesting groups for which the Farrell-Jones Conjecture in its most general form is open. Examples are:
  - Amenable groups;
  - $Sl_n(\mathbb{Z})$ for $n \geq 3$;
  - Mapping class groups;
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Boundaries of hyperbolic groups

Theorem (Bartels-Lück-Weinberger (2009?))

Let G be a torsionfree hyperbolic group and let n be an integer ≥ 6. Then the following statements are equivalent:

- *The boundary ∂G is homeomorphic to S^{n-1};*
- *There is a closed aspherical topological manifold M such that $G \cong \pi_1(M)$, its universal covering $\widetilde{M}$ is homeomorphic to $\mathbb{R}^n$ and the compactification of $\widetilde{M}$ by ∂G is homeomorphic to D^n.*

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