Identities in Euclidean and Hyperbolic Geometry which follow from Existence and Uniqueness Theorems

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The purpose of the talk to present some threads of mathematical ideas which I believe are interesting, have applications in various guises and contexts, and allow for further considerations on various levels.

I wish to begin by thanking Professor J. Kramer for the opportunity to speak here today and to A. von Pippich for the assistance with the *What is…*meeting associated to this presentation.

The general theme I will follow is to determine what identities come from assuming *existence and uniqueness* of mathematical quantities, or objects, as characterized by statements such as differential equations with initial conditions.

One of my favorite formulas is

$$\log(w) = \int_{0}^{\infty} \left(e^{-t} - e^{-wt}\right) \frac{dt}{t},$$

which is proved using that the integral $I(w)$ satisfies $I(1) = 0$ and $D_w I(w) = 1/w$, as does $f(w) = \log(w)$.

The existence and uniqueness theorem here is the following: There exists a function $f(w)$ which is uniquely characterized by the conditions that $D_w f(w) = 1/w$ and $f(1) = 1$.

Similarly, one can prove the power series expansion

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

by showing that both functions satisfy the conditions $f(0) = 1$ and $D_x f = f$.

The basic trigonometric functions $\sin x$ and $\cos x$, and their power series realizations, can be proved by studying $D_x^2 f + f = 0$ with various initial conditions.

With the above examples, let us see where we can go with *existence and uniqueness.*

For fixed $x \in \mathbf{Z}_+$ and $t \in \mathbf{R}$, consider the equation

$$t^2 D_t^2 f_x(t) + tD_t f_x(t) - (t^2 + x^2) f_x(t) = 0,$$

with initial conditions which we will worry about later. A solution to the equation is the series

$$I_x(t) = \sum_{n=0}^{\infty} \frac{(t/2)^{2n+x}}{n!(n+x)!}$$

for $x \geq 0$, with $I_{-x} = I_x$, as well as

$$I_x(t) = \frac{1}{\pi} \int_0^{\pi} e^{t \cos\theta} \cos(x\theta) d\theta.$$

From the integral expression, one can show that

$$I_{x-1}(t) + I_{x+1}(t) = 2D_t I_x(t).$$

We can re-package the last expression to the following. For any $f : \mathbf{Z}_+ \to \mathbf{R}$, define

$$\Delta_{\mathbf{Z}} f(x) = 2f(x) - (f(x+1) + f(x-1)).$$

Let

$$K_{\mathbf{Z}}(t;x) = e^{-2t} I_x(2t).$$

Then $K_{\mathbf{Z}}(t;x)$ satisfies the equation

$$(\Delta_{\mathbf{Z}} + D_t) K_{\mathbf{Z}}(t;x) = 0$$

with the initial condition

$$\lim_{t \to 0} K_{\mathbf{Z}}(t;x) = \left\{ \begin{array}{l} 0, \text{ if } x \neq 0 \\ 1, \text{ if } x = 0 \end{array} \right\}.$$

Do these equations uniquely characterize $K_{\mathbf{Z}}(t;x)$? Yes.

How can this get to be more interesting? Look at functions which are also periodic with respect to $x \mapsto x + N$ for $N \in \mathbf{Z}_+$.

Symbolically, one uses the notation that we are studying functions of $t \in \mathbf{R}$ and with $x \in (N\mathbf{Z})\backslash\mathbf{Z}$. Can we name some functions on $(N\mathbf{Z})\backslash\mathbf{Z}$ which satisfy the differential equation? Yes

The first functions we can name are

$$\sum_{j=-\infty}^{\infty} e^{-2t} I_{jN+x}(2t).$$

To replace x by $x+N$ amounts to rearranging the sum. The differential equation follows since are can apply the operator term by term.

The second functions we can name are

$$\phi_{k,N}(t;x) = e^{-\lambda_{k,N}\cdot t}e^{2\pi ikx/N}$$

where

$$\lambda_{k,N} = 2 - 2\cos(2\pi k/N),$$

and with $k = 0, \cdots, N-1$. If we have the right uniqueness theorem, then we would arrive at the identity

$$\frac{1}{N}\sum_{k=0}^{n-1} e^{-\lambda_{k,N}\cdot t}e^{2\pi ikx/N} = \sum_{j=-\infty}^{\infty} e^{-2t}I_{jN+x}(2t).$$

The uniqueness theorem is true, and the relation holds for any positive integer N and $t \in \mathbf{R}_+$.

If $N = 1$, the space $\mathbf{Z}\backslash\mathbf{Z}$ is a single point! The *semi-group* property

$$\sum_{x=0}^{N-1} K_{N\mathbf{Z}\backslash\mathbf{Z}}(t_1, y-x)K_{N\mathbf{Z}\backslash\mathbf{Z}}(t_2, x) = K_{N\mathbf{Z}\backslash\mathbf{Z}}(t_1+t_2, y)$$

which is easy to prove on the spectral side is non-trivial on the I-Bessel side.

The above computations come from an article by Karlsson and Neuhauser and can be viewed as the study of heat kernels on discrete circles. Other identities follow by taking various integral transforms of the *theta function* identity.

Take the product of the theta inversion formula with itself r times with $x = 0$, subtract e^{-t}, integrate against dt/t for $t > 0$, and send $N \to \infty$. The resulting identity is the following formula:

Let

$$F(x_1, \cdots, x_r) = \log\left(2d - 2\cos(2\pi x_1) - \cdots - 2\cos(2\pi x_r)\right).$$

Then

$$\int\limits_{\mathbf{Z}^r\backslash\mathbf{R}^r} F(x_1, \cdots, x_r)dx_1 \cdots dx_r = -\int\limits_0^\infty (e^{-2dt}I_0(2t)^d - e^{-t})\frac{dt}{t}.$$

The technical issue comes from verifying that each *non-identity* term in the sum goes to zero, and that the convergence is uniform (which can been done).

Imagine the problem of numerically estimating the left-hand-side for large d where convergence issues render the integral impossible to encode, yet the convergence of the right-hand-side improves when d grows.

Similar considerations, with a few tricks, allow one to relate integrals of functions of the form

$\log\left(A - B_1\cos(2\pi x_1) - \cdots - B_r\cos(2\pi x_r)\right)$

to a single integral of a product of I-Bessel functions. When using the power series expansion of I_0, the integral of Bessel functions can be shown to equal

$$\sum_{n=1}^{\infty} \frac{a_n(B_1, \cdots, B_r)}{n^A},$$

where the coefficients a_n can be explicitly evaluated from the power series expansions of I_x, even with

$$a_n(B, \cdots, B) = B^n \cdot c_n.$$

The previous results were built from studying the differential equation

$$(\Delta_{\mathbf{Z}} + D_t)K_{\mathbf{Z}}(t; x) = 0$$

for $t > 0$ and x being a point in the discrete circle $(N\mathbf{Z})\backslash\mathbf{Z}$.

Let us now look at *real* circles, beginning by studying solutions to

$$(\Delta_{\mathbf{R}} + D_t)K_{\mathbf{R}}(t; x) = 0$$

for $t > 0$, $x \in \mathbf{R}$ and

$$\Delta_{\mathbf{R}} = D_x^2 = -\frac{d^2}{dx^2}.$$

If we add the initial conditions

$$\lim_{t \to 0} \int\limits_{\mathbf{R}} K_{\mathbf{R}}(t; x) f(x) dx = f(0),$$

then an *existence and uniqueness* theorem can be proved, showing that

$$K_{\mathbf{R}}(t; x) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/(4t)}.$$

Now look at functions which are invariant under $x \mapsto x+2\pi$, with the same properties as above, namely the differential equation and initial conditions. One function which satisfies the equations is

$$\sum_{n=-\infty}^{\infty} K_{\mathbf{R}}(t; x + 2\pi n).$$

Another function which satisfies the conditions is

$$\sum_{m=-\infty}^{\infty} e^{-m^2 t} \cdot \frac{1}{2\pi} e^{mix} \cdot 1.$$

With the right existence and uniqueness theorem, we arrive at the equality

$$\sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-(x+2\pi n)^2/(4t)} = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{-m^2 t} e^{mix}.$$

The issue of numerical evaluation becomes interesting, as can be seen, for example, by taking $t = 1000$ and $x = 0$: The left-hand-side is slow to converge, but the right-hand-side converges to hundreds of decimal places after the $m = 0$ term.

Multiply by $g(t)$, for most any g, and integrate dt on $\mathbf{R}_+$. Define

$$f(x) = \int_0^\infty g(t) e^{-x^2 t} dt.$$

For any h, define

$$\mathfrak{F}(h)(x) = \int_{\mathbf{R}} h(u) e^{ixu} du.$$

By interchanging the order of integration, together with an elementary integral, one can show that

$$\mathfrak{F}(f)(x) = 2\pi \cdot \int_0^\infty g(t) \frac{1}{\sqrt{4\pi t}} e^{-x^2/(4t)} dt.$$

In other words,

$$\sum_{m=-\infty}^{\infty} f(m) = \sum_{n=-\infty}^{\infty} \mathcal{F}(f)(n).$$

This is the Poisson summation formula, which holds for a reasonable general class of functions.

The case when $f(x) = e^{-x^2 t}$ is both a special case of the Poisson summation formula and the special case from which the general formula can be derived.

It is only a question of notation to generalize the above discussion to $\mathbf{R}^r$ and the discrete torus $\Lambda\backslash\mathbf{R}^r$, for some lattice Λ.

A more interesting problem is to compare the discrete circle identity, involving I-Bessel functions, to the real circle problem, involving Gaussians. The notation of quotient spaces suggests the way in which a comparison can be made. Symbolically, we write

$$\text{Discrete circle} = (N\mathbf{Z})\backslash\mathbf{Z} = \mathbf{Z}\backslash(\frac{1}{N}\cdot\mathbf{Z}).$$

Take the differential equation involving the operator $\Delta_{\mathbf{Z}}+D_t$ and divide by N^2 in order to match the factor of N^{-2} in the re-scaling of the topological space. We then need to re-scale time by a factor of N^2. We are drawn to study the functions

$$Ne^{-2N^2t}I_{Nx}(2N^2t).$$

Heuristically, it seems that we could make sense out of the notion

$$\frac{1}{N}\cdot\mathbf{Z}\to\mathbf{R},$$

which is leads us to hypothesize that

$$\lim_{N\to\infty} Ne^{-2N^2t}I_{Nx}(2N^2t)=\frac{1}{\sqrt{4\pi t}}e^{-x^2/(4t)},$$

which indeed is true. Thus, the Poisson summation formula can be derived from analysis on discrete circles.

How far does this go? This is hard to tell. For example, the equality of the integrals

$$\int\limits_{\mathbf{Z}^r\backslash\mathbf{R}^r} F(x_1,\cdots,x_r)dx_1\cdots dx_r=-\int\limits_0^\infty(e^{-2dt}I_0(2t)^d-e^{-t})\frac{dt}{t}$$

contains an error term which is an automorphic form coming from the Kronecker limit problem applied to an Epstein zeta function, thus relating to automorphic forms which are $\mathrm{GL}(r,\mathbf{Z})$ invariant.

Let's now exit the setting of $\mathbf{R}$, with Euclidean geometry, to $\mathbf{h}$, with hyperbolic geometry, where

$$\mathbf{h} = \{z = x + iy \in \mathbf{C} \mid y > 0\}.$$

The Laplacian is

$$\Delta_{\mathbf{h}} = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) = y^2 \Delta_{\mathbf{R}^2}.$$

By following the setting from $\mathbf{R}$, we look at the *heat operator* $\Delta_{\mathbf{h}} + D_t$ and a *heat kernel,* which means a function $K_{\mathbf{h}}(t, z, w)$ for $t \in \mathbf{R}_+$ and $z, w \in \mathbf{h}$ such that

$$\left(\Delta_{\mathbf{h}} + D_t\right) K_{\mathbf{h}}(t, z, w) = 0$$

and

$$\lim_{t \to 0} \int\limits_{\mathbf{h}} K_{\mathbf{h}}(t, z, w) f(z) d\mu_{\mathbf{h}}(z) = f(w),$$

where $d\mu_{\mathbf{h}}(z) = (\mathrm{d}x\mathrm{d}y)/y^2$. By now, the notion of *existence and uniqueness* becomes a battle cry.

After somewhat detailed, but structurally clear, computations (involving the spherical transform), one can show that

$$\int_{\mathbf{h}} K_{\mathbf{h}}(t,z,w) = \frac{\sqrt{2}e^{-t/4}}{(4\pi t)^{3/2}} \int_{\rho}^{\infty} \frac{re^{-r^2/(4t)}}{\sqrt{\cosh(r) - \cosh(\rho)}} \mathrm{d}\rho$$

where $\rho = d_{\mathbf{h}}(z,w)$.

As before, let us study functions which are periodic with respect to some *translation.* This turns our attention to hyperbolic geometry, Riemann surfaces, and the uniformization theorem.

Let Γ be a Fuchsian group of the first kind acting on $\mathbf{h}$ without fixed points such that the quotient $\Gamma\backslash\mathbf{h}$ is compact. General theory applies to tell us there exists a *complete* orthonormal basis of functions ϕ_n such that

$$\Delta_{\mathbf{h}}\phi_n = \lambda_n\phi_n$$

for some $\lambda_n \geq 0$.

The functions ϕ_n and constants λ_n are very mysterious. However, we seem to have the following identity:

$$\sum_{n=0}^{\infty} e^{-\lambda_n \cdot t} \phi_n(z)\phi_n(w) = \sum_{\gamma\in\Gamma} K_{\mathbf{h}}(t, z, \gamma w).$$

The previous identities, namely on $(N\mathbf{Z})\backslash\mathbf{Z}$ and $(2\pi\mathbf{Z})\backslash\mathbf{R}$, can be shown to have the same structure as above, when taking into account that the quotient spaces are translation invariant.

One of the next steps is to set $z = w$ and integrate over $\Gamma\backslash\mathbf{h}$. Immediately, we have that

$$\sum_{n=0}^{\infty} e^{-\lambda_n t} = \int_{\Gamma\backslash\mathbf{h}} \sum_{\gamma\in\Gamma} K_{\mathbf{h}}(t, z, \gamma z) d\mu_{\mathbf{h}}(z).$$

To evaluate the right-hand-side, a few computations a needed, most of which were first understood by Huber and Selberg. In roughly 3 pages, one can prove that

$$\sum_{n=0}^{\infty} e^{-\lambda_n t} = \sum_{n=1}^{\infty} \sum_{\gamma \in H(\Gamma)} \frac{c_{n,\ell_\gamma}}{\sqrt{4\pi t}} e^{-t/4} e^{-(n\ell_\gamma)^2/(4t)}$$

$$+\mathrm{vol}(\Gamma\backslash\mathbf{h}) K_{\mathbf{h}}(t,0),$$

where $H(\Gamma)$ is the set of primitive, inconjugate, hyperbolic geodesics (the *primes*),

$$c_{n,\ell_\gamma} = \frac{\ell_\gamma}{\sinh(n\ell_\gamma/2)},$$

and $K_{\mathbf{h}}(t,0)$ is the heat kernel on $\mathbf{h}$ evaluated whenever $z = w$.

The computations yielding the Poisson summation formula applies, after we multiply by $e^{t/4}$, so then we arrive at the identity

$$\sum_{n=0}^{\infty} f(\mu_n) = \sum_{n=1}^{\infty} \sum_{\gamma \in H(\Gamma)} c_{n,\ell_\gamma} \cdot \mathfrak{F}(f)(\ell_\gamma) \quad .$$

$$+ \text{ trivial term coming from } K_{\mathbf{h}}(t,0).$$

where $\mu_n = \sqrt{(\lambda_n - 1/4)}$. The above identity is the *Selberg trace formula* for compact, hyperbolic Riemann surfaces.

At this point, we can use one of my other favorite formulas, namely

$$2s\int\limits_0^\infty e^{-s^2t}e^{-w^2/(4t)}\frac{dt}{\sqrt{4\pi t}} = e^{-sw}.$$

What does this suggest? Take the theta inversion formula, multiply both sides by $2se^{-(s^2-1/4)t}$, then integrate over $\mathbf{R}_+$. There is a little, easy to handle, issue regarding convergence, together with a rather enjoyable expansion involving the geometric series and the Taylor series for $\log(1-x)$, until we arrive at the following statement:

The function

$$Z(s) = \prod_{k=0}^{\infty}\prod_{\gamma\in H(\Gamma)}\left(1-e^{-(s+k)\ell_\gamma}\right)$$

converges for $\mathrm{Re}(s) > 1$, *admits a meromorphic continuation to all* $s \in \mathbf{C}$, *has a functional equation, and satisfies the Riemann hypothesis (almost).*

The non-abelian discrete setting is under investigation (together with Chinta and Karlsson). However, things are quite promising. The heat kernel on the Cayley graph associated the finitely generated free group can be written in terms of I-Bessel functions. The Cayley graph associated to a finite group and a fixed set of generators can be viewed as a quotient of the Cayley graph of a finitely generated free group.

The steps described above apply without difficulty, yielding the analogue of the Selberg trace formula as well as all key properties of the Ihara zeta function. We are now looking at the prospect of a naturally defined *denseness* analogues to discrete tori converging to real tori.

In the past few years, I have pursued a very enjoyable and productive research collaboration with Professor Kramer. In effect, our current program of study begins with a new identity which itself is proved using an existence and uniqueness theorem (in this case, for a particular metric).

The above problems and methods can certainly be expanded to other geometric settings, such as hyperbolic manifolds of finite volume or co-finite quotients of rank one Lie groups. The type of *degeneration* problems (with discrete tori converging to real tori) has a wealth of results in the literature, and many problems exist, in some cases where the geometry has arithmetic interpretation as well as cases where no limiting geometry is evident.

We have not pointed out any of the identities which stem from existence and uniqueness theorems in other fields. However, the theme remains the same: Mathematical objects are uniquely characterized by specific properties, and existence of the mathematical object is sought.

More generally, there is no shortage of mathematical problems, and I encourage all of you to *keep your eyes wide open* when it comes to seeing new problems.