Power variation of stable Lévy processes and application to paleoclimatic data

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IRTG Stochastic Models of Complex Processes

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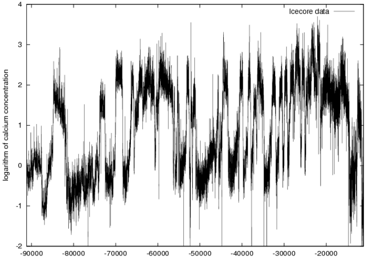
Motivation - data from Greenland ice



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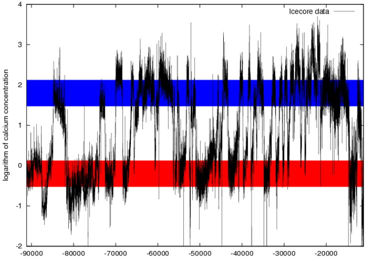

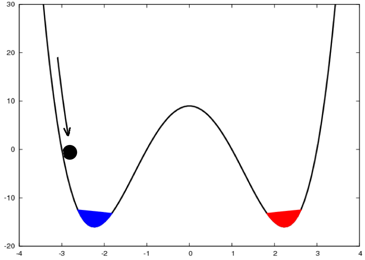

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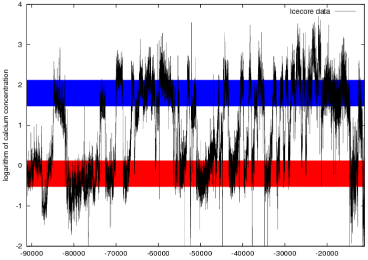

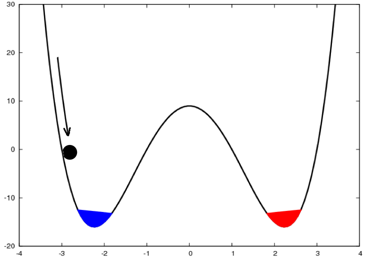

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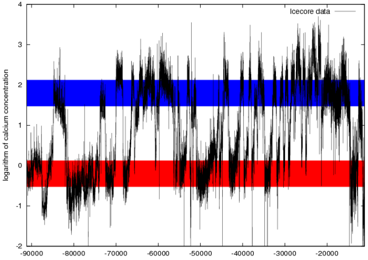

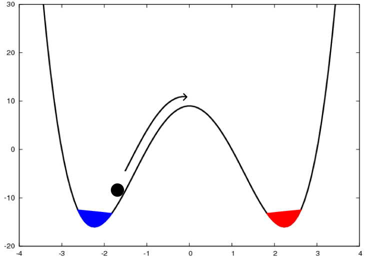

$$X_t = x_0 - \int_0^t U'(X_s)\,ds + L_t$$

Brownian Motion – Stable Lévy process

$\alpha = 2$ (BM)	$\alpha \in (0, 2)$
$\mathbb{E}e^{i\lambda B_t} = e^{(-1/2)t\|\lambda\|^2}$	$\mathbb{E}e^{i\lambda L_t} = e^{ct\|\lambda\|^\alpha}$

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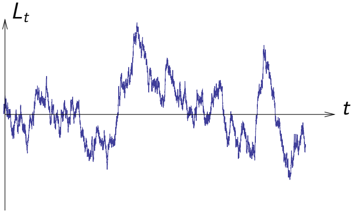
continuous	infinite jumps in every intervall

Stable Lévy process



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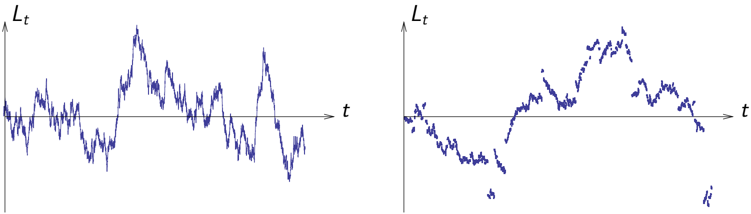

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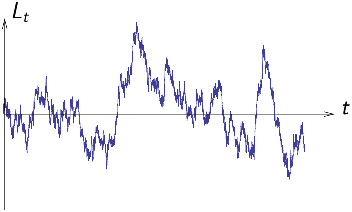
Stable Lévy process



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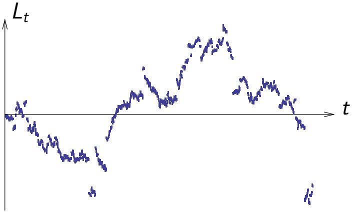

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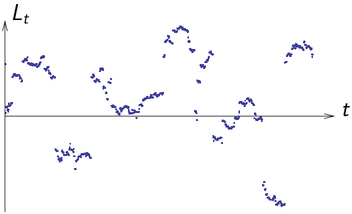

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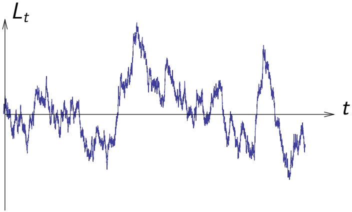

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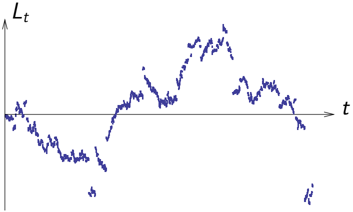

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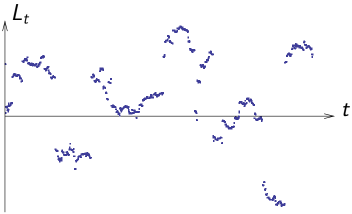

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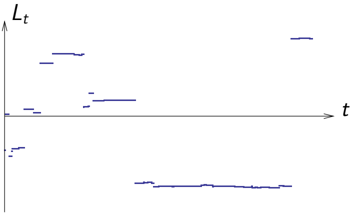

$\alpha = 0.5$

Power variation

Definition

For a stochastic process $X = (X_t)_{t \geq 0}$ define

$$V_p(X)_t := \lim_{n \to \infty} V_p^n(X)_t := \lim_{n \to \infty} \sum_{i=1}^{[nt]} |X_{\frac{i}{n}} - X_{\frac{i-1}{n}}|^p.$$

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- If $V_p(X)_t < \infty$ for a $p > 0$ then $V_{p'}(X)_t < \infty$ for all $p' \geq p$.
- $\inf\{p > 0 : \ V_p(X)_t < \infty\}$?

Power variation of stable processes

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- Then the power variation of $Y + L$ converges:

$$(V_p^n(L+Y)_t)_{t\geq 0} \xrightarrow{\mathscr{D}} (L'_t)_{t\geq 0}, \quad n \to \infty$$

where L' is a spectrally positive α/p-stable Lévy Process.

Relevance for SDEs

- For $p > 1$
$$V_p^n\left(\int_0^{\cdot} f(s)\,ds\right)_t \to 0$$
for any integrable function f and $t \geq 0$

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- So for any $p > \max\{1, \alpha\}$

$$(V_p^n(X)_t)_{t\geq 0} = \left(V_p^n\left(x_0 - \int_0^{\cdot} U'(X_s)ds + L\right)_t\right)_{t\geq 0} \xrightarrow{\mathscr{D}} (L'_t)_{t\geq 0}$$

where L' is an α/p-stable Lévy process and just depends on L.

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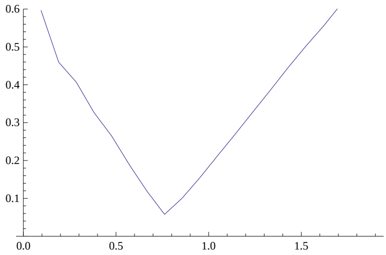
- Minimum should be at $\alpha/p = 1/2$, so calculate $\alpha = p/2$

Application to data



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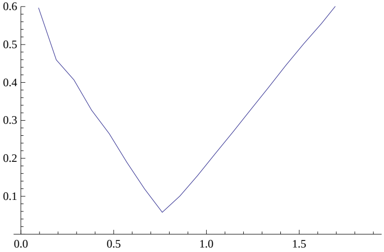

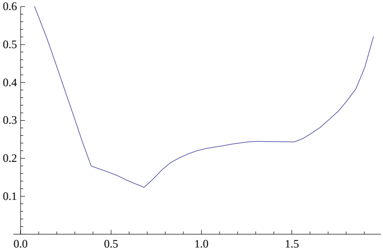

Thank you