Mean Curvature Flow with Surgery

John Head
IMPRS

Supervisor: Prof. Dr. G. Huisken

BMS Days Feb 16, 2010

IMPRS

Heat Equations in Geometry and Topology

Mean Curvature Flow

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International Max Planck Research School

IMPRS for Geometric Analysis, Gravitation and String Theory

is a graduate study program (started in 2004) offered by the

Max Planck Institute for Gravitational Physics
(Albert Einstein Institute)

in collaboration with

- Free University
- Potsdam University
- Humboldt University

The AEI aims in the broadest sense to promote research in the mathematics and physics related to Einstein's general theory of relativity. It consists of several research groups including

- Geometric Analysis and Gravitation (G. Huisken)
- Astrophysical Relativity (B. Schutz)
- Quantum Gravity (H. Nicolai)

as well as additional independent research groups and two divisions in Hannover dedicated to gravitational wave astronomy and observational relativity.

The different avenues of research range from abstract differential geometry and partial differential equations to classical general relativity to reconciling Einstein's theory with quantum mechanics in the framework of unified theories.

The Geometric Analysis and Gravitation division conducts research in pure mathematics together with its applications inside a broader research institute.

The collaboration with the universities includes additional research groups such as

- the geometric analysis group of K. Ecker (FU)
- the nonlinear dynamics group of B. Fiedler (FU)
- the geometry group of C. Bär (Potsdam)

Current IMPRS research topics include

- nonlinear partial differential equations
- Riemannian and Lorentzian geometry
- geometric variational problems
- foundations of general relativity
- asymptotically flat manifolds
- Einstein's equations with matter
- Newtonian limit of general relativity

Heat Equations in Geometry and Topology

What are the fundamental problems in geometry/topology?

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Can we describe the geometric structures they support?

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Uniformisation Theorem: classification of closed surfaces
Poincare & Geometrisation: classification of closed 3-manifolds
Classification of “positive curvature” hypersurfaces
Schoenflies Conjecture: classification of enclosed regions

What is the philosophy behind our approach?

Parabolic geometric evolution equations

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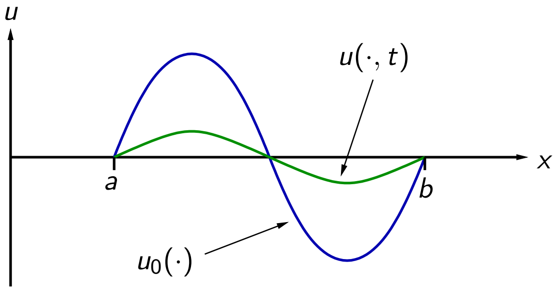
- Recent crowning glory of geometric evolution equations is the Hamilton-Perelman proof of Thurston's Geometrisation Conjecture.

The Heat Equation

$$\begin{cases} \frac{\partial}{\partial t} u(x,t) &= \Delta u(x,t) \\ u(\cdot,0) &= u_0(\cdot) \end{cases}$$



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Harnack inequality:

$$\frac{d}{dt} \log u \geq |D \log u|^2 - \frac{n}{2t}$$

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Definition Given $F_0 : \mathcal{M}^n \to \mathbb{R}^{n+1}$ the evolution of $\mathcal{M}_0^n = F_0(\mathcal{M}^n)$ by MCF is the one-parameter family $F : \mathcal{M}^n \times [0, T) \longrightarrow \mathbb{R}^{n+1}$ satisfying

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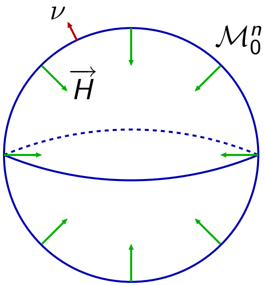

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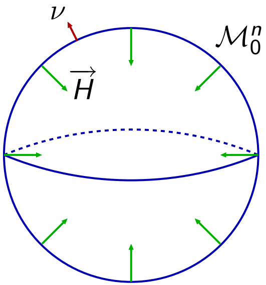

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Let's see how this initial data evolves under (MCF)...

Explicit example: shrinking sphere

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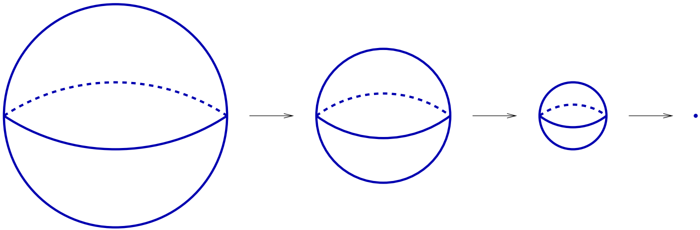

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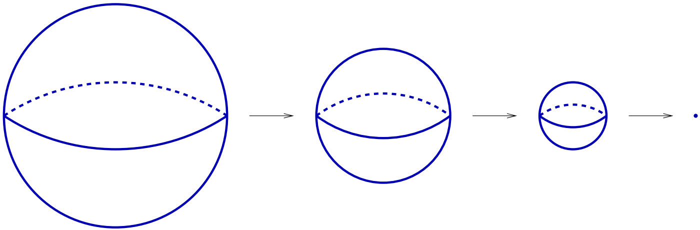

So we see explicitly that the solution $\mathcal{M}_t^n = \partial B_{r(t)}^{n+1}$ exists for $t \in (-\infty, \frac{r_0^2}{2n})$.

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curvature blow-up

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Getting a feel for MCF

What singularities can we expect in general?

1. Positive curvature case:

Convexity: $\lambda_i > 0$

We have already seen one example...

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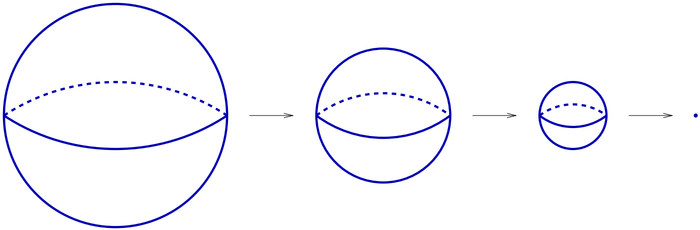

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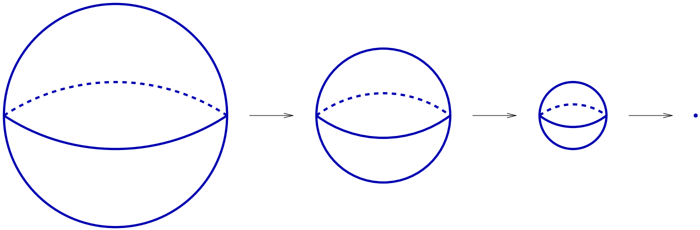

The curvature H^2 blows up like $(T - t)^{-1}$ and is no longer defined at $T := \frac{r_0^2}{2n}$.

This behaviour is typical of any *convex* surface...

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Given any smooth, closed, convex hypersurface, the solution of (MCF) remains smooth, closed and convex and contracts to a point in finite time. After rescaling, the solution converges smoothly to a round sphere in infinite time.

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What if we relax the curvature assumption from *convexity* to *two-convexity*?

2. "Two-positive" curvature case:

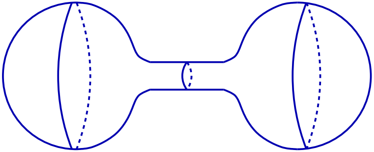
Two-convexity: $\lambda_i + \lambda_j > 0$

How do we expect *general* two-convex hypersurfaces to evolve under MCF?

Neck pinch

Neck pinch

Consider an initial hypersurface which looks like two large spheres connected by a thin cylindrical neck.



We expect the spheres either side to have moved only slightly by the time the neck has *pinched* off.

Shrinking cylinder

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Let $\mathcal{M}_0^n = \partial B_{r_0}^n \times \mathbb{R}$. Then (MCF) reduces to

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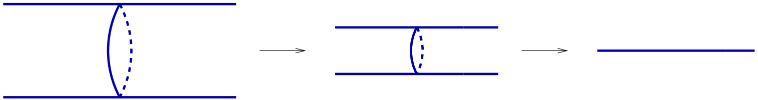

Degenerate neck pinch

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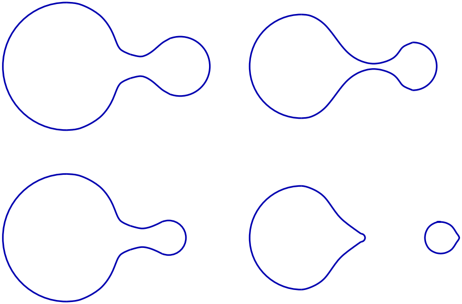
Now consider an asymmetric example of the neck pinch.



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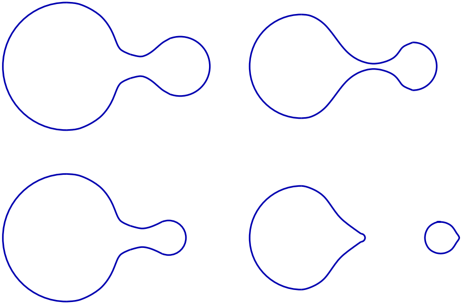

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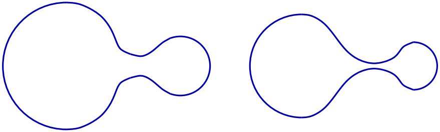

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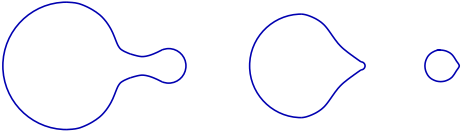

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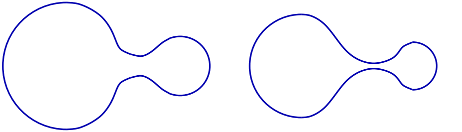

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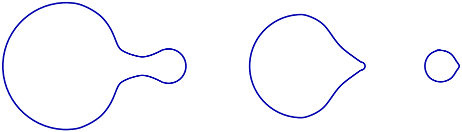

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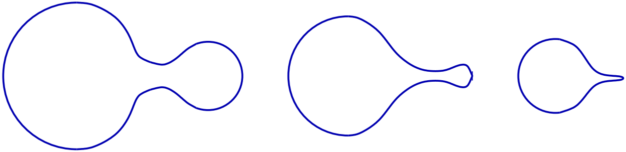
But there exists one additional “in-between” scenario...

Degenerate neck pinch (cont.)

...in which the surface shrinks to a point in finite time, remains two-convex up to the singular time but never becomes convex.



The maximum of the curvature is attained at the tip of the "horn".

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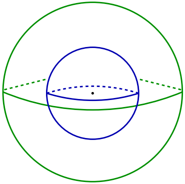

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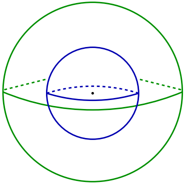

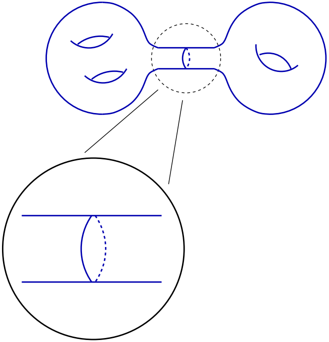

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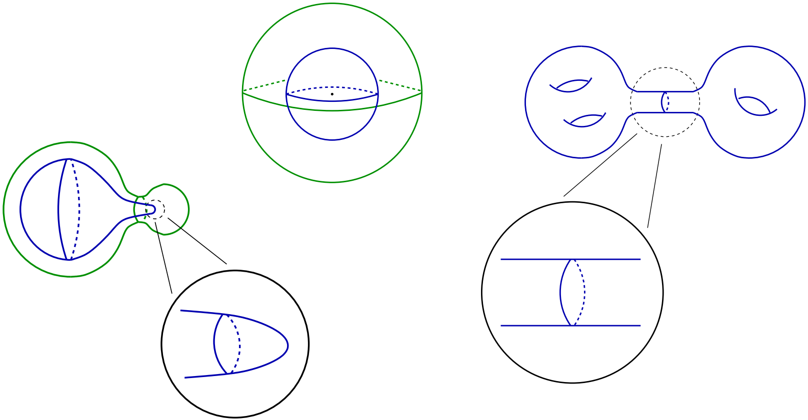
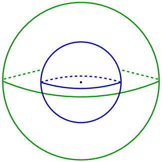
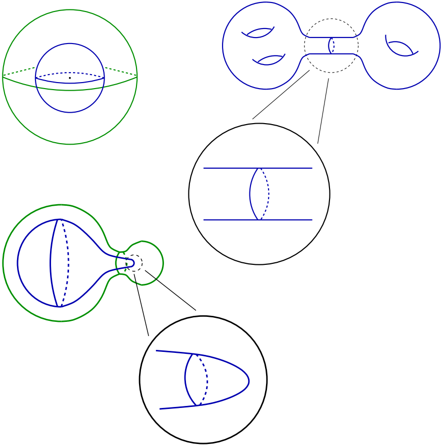


Convex



Two-Convex



Mean Curvature Flow with Surgery

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2) cut out necks and replace with spherical caps (s.t. curvature decreases)
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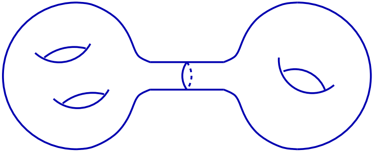

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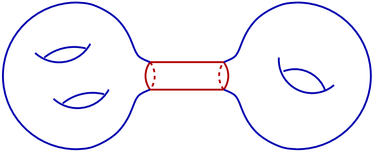

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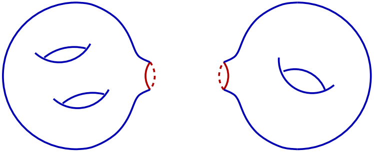

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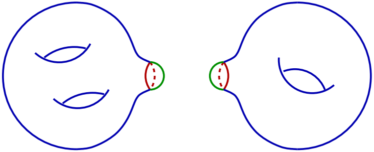

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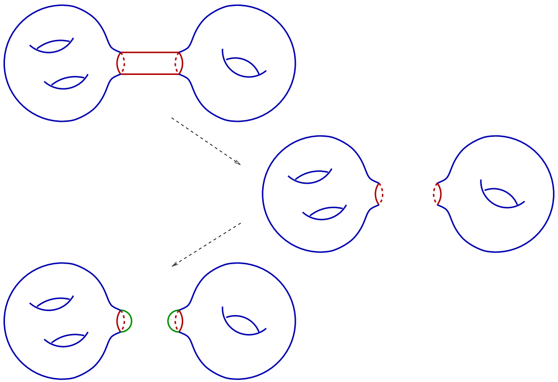
Surgery is the inverse operation of a connected sum.



This must be controlled by a set of parameters depending only on the initial data such that all relevant estimates remain valid.

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It is possible to track changes of topology!

We can therefore use this method to classify initial surface (cf Hamilton/Perelman surgery program for Ricci flow).

Assumption We consider a smooth closed initial immersed hypersurface $\mathcal{M}_0^n$ ($n \geq 3$) which is 2-convex:

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Theorem (Huisken-Sinestrari 2009)

There exists a MCF with surgeries starting from $\mathcal{M}_0^n$ which terminates after a finite number of steps. All surfaces satisfy a uniform curvature bound and all time intervals between surgeries have length bounded from below by a uniform constant.

Corollary 1 (Huisken-Sinestrari 2009) Any smooth closed n-dimensional 2-convex immersed hypersurface $\mathcal{M}^n \hookrightarrow \mathbb{R}^{n+1}$ with $n \geq 3$ is diffeomorphic either to S^n or to a finite connected sum of $S^{n-1} \times S^1$.

Corollary 2 (Huisken-Sinestrari 2009) Any smooth closed n-dimensional 2-convex embedded hypersurface $\mathcal{M}^n \hookrightarrow \mathbb{R}^{n+1}$ ($n \geq 3$) which is simply connected is diffeomorphic to S^n and bounds a ball.

So far we have only painted pictures.

Let's now look at the *a priori* estimates which make this program successful.

The proof of the this theorem requires two key steps.

1) it is possible to perform surgery close to the singular time

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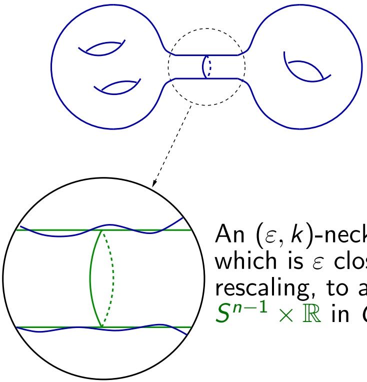
   $\rightsquigarrow$ we have to show that when the curvature blows up we can find a neck (unless the surface is uniformly positively curved)

2) the procedure ends after a finite number of surgeries

   $\rightsquigarrow$ we have to show that surgery results in a fixed decrease in curvature of the surface

Definition



An (ε, k)-neck is a region which is ε close, up to rescaling, to a piece of $S^{n-1} \times \mathbb{R}$ in C^k norm.

Convexity Estimates Nearly singular regions become convex:

For any $\delta > 0$ *there exists* $C_\delta = C_\delta(\mathcal{M}_0^n) > 0$ *s.t.*

$$\lambda_1 \geq -\delta H - C_\delta.$$

Cylindrical Estimates Points where λ_1 is small are "round":

For every $\eta > 0$ *and* $i, j \geq 2$ *there exists* $C_\eta = C_\eta(\mathcal{M}_0^n) > 0$ *s.t.*

$$|\lambda_1| \leq \eta H \Longrightarrow |\lambda_i - \lambda_j|^2 \leq c(n)\eta H^2 + C_\eta.$$

Gradient Estimate We can compare curvature at different points:

There exist constants $c = c(n)$ *and* $C = C(\mathcal{M}_0^n) > 0$ *s.t.*

$$|\nabla A|^2 \leq c(n)|A|^4 + C.$$

To ensure that the flow terminates after finitely many steps, the choice of *times* and *scale* at which we perform the surgery must be carefully controlled by fixed parameters.

Surgery parameters

$$H_0 << H_s << H_1$$

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Fix $H_0 = H_0(\mathcal{M}_0^n) >> 1$ beyond which geometry of necks is controlled by a priori estimates.

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"Surgery" curvature

Fix $H_s = \alpha H_1$, $\alpha = \alpha(\mathcal{M}_0^n) < 1$ to ensure that surgery reduces $\max(H)$ by a fixed factor.

These parameters can be chosen such that the surgery procedure satisfies:

1) surgery is performed at times T_i s.t. $\max H(\cdot, T_i) = H_1$
2) surgery is performed on (ε, k)-necks
3) surgery is performed at cross-sections with mean radius $\frac{n-1}{H_s}$
4) after surgery, curvature drops by fixed amount prescribed by α
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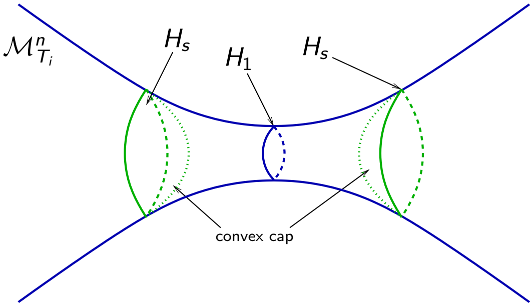

What’s next?

Canonical flow with surgeries / Weak solutions

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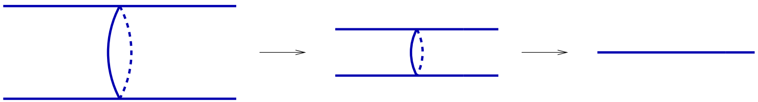

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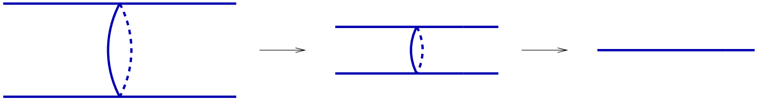

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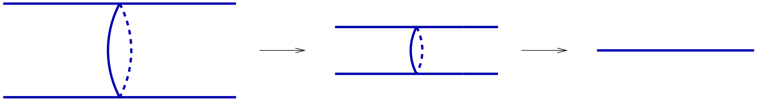

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What happens as $H_1 \longrightarrow \infty$? (Work in progress...)

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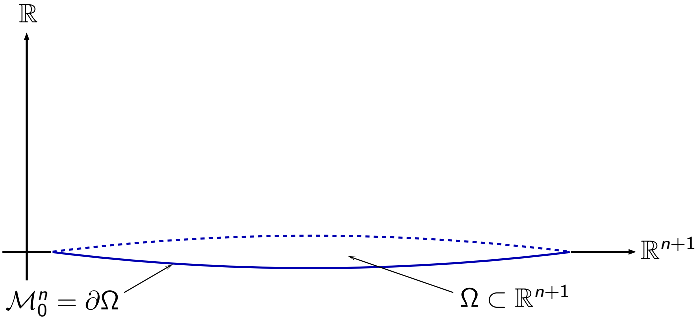
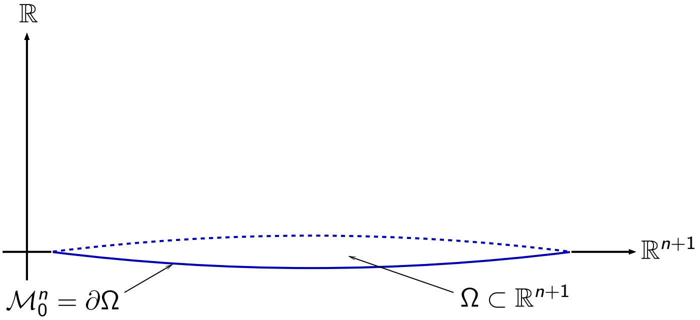


Weak Solutions Let $\mathcal{M}_0^n = \partial\Omega$ where $\Omega \subset \mathbb{R}^{n+1}$, and let $u : \Omega \to \mathbb{R}$ assign to each point $x \in \Omega$ the time t when $x \in \mathcal{M}_t^n$. That is,

$$\mathcal{M}_t^n = \left\{ x \in \Omega \;\middle|\; u(x) = t \right\}$$

where u satisfies the degenerate elliptic equation

$$\operatorname{div}\left(\frac{Du}{|Du|}\right) = -\frac{1}{|Du|}.$$



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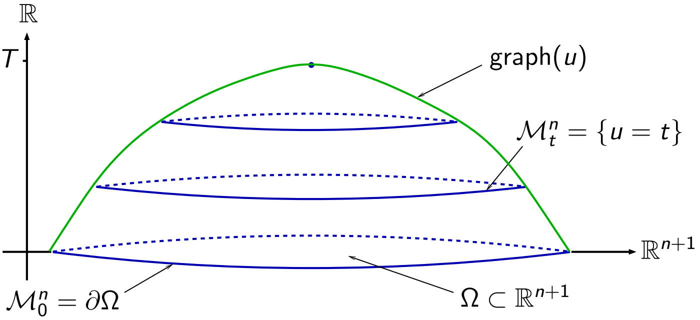


Theorem (Brakke's Clearing Out Lemma)

Let $\mathcal{M}_t^n$ be a solution of (MCF), and suppose that $\mathcal{M}_0^n$ satisfies

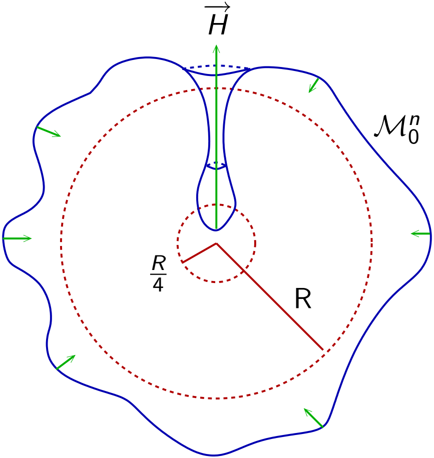
$$|\mathcal{M}_0^n \cap B_R(x_0)| \leq \varepsilon R^n$$

for some $x_0 \in \mathbb{R}^{n+1}$, $R > 0$ and sufficiently small $\varepsilon = \varepsilon(n) > 0$. Then there exist constants $c(n) > 0$ and $0 < \alpha(n) < 1$ such that

$$|\mathcal{M}_t^n \cap B_{\frac{R}{4}}(x_0)| = 0$$

for some $t \leq c\varepsilon^\alpha R^2$.

Example



Thank you for your attention!