Number theory of binary quadratic and cubic forms

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The theory of binary quadratic forms deals with homogeneous quadratic polynomials $ax^2 + bxy + cy^2$ in two variables over the rational integers. A fundamental question is to describe the set of values assumed by such a quadratic form as x and y vary over all integers. It was known already to Fermat that every prime $p \equiv 1\,(4)$ is a sum of two squares, but no prime $p \equiv 3\,(4)$. For the forms $ax^2 + bxy + cy^2$ it is possible to formulate similar necessary conditions, but there is in general no simple complete description of the integers represented by a single form. The theory of binary quadratic forms is closely connected to the theory of quadratic number fields, i. e. the field extensions of the rational number field generated by quadratic irrationalities. A central role is played by the class number of quadratic forms of a fixed discriminant $D = b^2 - 4ac$. This number determines also whether or not there is a unique decomposition into prime factors in a ring of quadratic irrationalities. We will study in the case of negative discriminants both the problem of determining bounds for the class number as well as the question of its average behavior. The latter can by studied by elementary geometric arguments and by considering the corresponding generating Dirichlet series, i. e. the zeta function introduced by Shintani. There is an analogous but slightly more complicated theory for positive discriminants in which the class number is replaced by the product of class number and regulator. A second part of the course will consider the theory of binary cubic forms which is connected to the theory of cubic number fields.

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