Bruhat-Tits buildings and their compactifications

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Bruhat-Tits buildings: The setting

Basic data

K non-Archimedean local field

$v : K^x \to \mathbb{Z}$ discrete valuation
$\mid x \mid = c^{-v(x)}$ $(c > 1)$ absolute value on K
$R_K = \{x \in K : \mid x \mid \leqq 1\}$ valuation ring
$m_K = \{x \in K : \mid x \mid < 1\}$ valuation ideal

K is complete, the residue field R_K/m_K is finite

Example $K = \mathbb{Q}_p =$ completion of $\mathbb{Q}$ with respect to $\mid x \mid_p = p^{-v_p(x)}$.

Bruhat-Tits buildings: The setting

$\mathbf{G}/K$ semisimple algebraic group,
$\mathbf{G} \hookrightarrow \mathbf{GL_{n,K}}$ closed subgroup such that the rad $(\mathbf{G})$ (=biggest connected solvable normal subgroup) $= 1$
$G = \mathbf{G}(K)$

Example: $\mathbf{SL_{n,K}}, \mathbf{PGL_{n,K}}, \mathbf{Sp_{2n,K}}, \mathbf{SO_{n,K}}$
$\mathbf{SL_n}(\mathbf{D})$, D central division algebra over K.

Bruhat-Tits building $\mathcal{B}(\mathbf{G}, K)$

- complete metric space with continuous $G-$action
- polysimplicial structure

Why are Bruhat-Tits buildings useful ?

- $\mathcal{B}(\mathbf{G}, K)$ is a "nice" space on which G acts,
- $\mathcal{B}(\mathbf{G}, K)$ encodes information about the compact subgroups of G.
- $\mathcal{B}(\mathbf{G}, K)$ is a non-Archimedean analogue of a Riemann symmetric space $G_{\mathbb{C}}/H$ of non-compact type. ($G_{\mathbb{C}}$ a complex semisimple algebraic group, H a maximal compact subgroup)
- Buildings can be used to prove results for symmetric spaces (e.g. Kleiner-Leeb)
- Representation theory of G (Schneider-Stuhler)
- Cohomology of arithmetic groups (Borel-Serre)

How is $\mathcal{B}(\mathbf{G}, K)$ constructed ?

General case:

$\mathbf{S} \subset \mathbf{G}$ maximal $k-$split torus

$\mathbf{Z} \subset \mathbf{N}$ centralizer/normalizer of $\mathbf{S}$ in $\mathbf{G}$.

$N = \mathbf{N}(K), Z = \mathbf{Z}(K)$

$W = N/Z$ Weyl group,

W acts on

$X_*(\mathbf{S}) = \mathrm{Hom}_{K-\text{groups}}(\mathbb{G}_m, \mathbf{S})$

$\mathbf{G} = \mathbf{PGL}_{n,K} = \mathbf{GL}_{n,K} / \text{center}$

$S = \left\{ \begin{pmatrix} s_1 & & \\ & \ddots & \\ & & s_n \end{pmatrix} \right\} / K^x$

$Z = S$

$N = \{\text{permutation matrices}\}$

$W \simeq$ symmetric group $\mathcal{S}_{n-1}$

W acts by permutation of the elements

$$\eta_i : x \mapsto \text{diag}\,(1, \ldots, \underset{\substack{\uparrow \\ i}}{x}, \ldots, 1)$$

on $X_*(\mathbf{S}) = \bigoplus_{i=1}^{n} \mathbb{Z}\eta_i / \mathbb{Z}(\sum_{i=1}^{n} \eta_i)$.

How is $\mathcal{B}(\mathbf{G}, K)$ constructed ?

$A = X_*(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{R} / X_*(\mathbf{C}) \otimes_{\mathbb{Z}} \mathbb{R}$

$\mathbf{C}$ = connected center of $\mathbf{G}$

is called an apartment

$$A = \bigoplus_{i=1}^{n} \mathbb{R}\eta_i / \mathbb{R}(\sum_{i=1}^{n} \eta_i)$$

The action of W on A can be extended to an action of N

$N = S \rtimes W$

diag $(s_1, \ldots, s_n) \in S$ acts by

$$\sum_{i=1}^{n} x_i \eta_i \mapsto \sum_{i=1}^{n} \big(x_i - v(s_i)\big)\eta_i$$

How is $\mathcal{B}(\mathbf{G}, K)$ constructed ?

$\Phi = \Phi(\mathbf{S}, \mathbf{G})$ root system

$\Phi = \{a_i/a_j : i \neq j\}$ with
$a_i : \text{diag}\,(s_1, \ldots, s_n) \mapsto s_i$

For $a \in \Phi$ we have a root group $\mathbf{U}_a$

$U_{a_i/a_j} = \left\{ \begin{pmatrix} 1 & & x \\ & \ddots & \\ & & 1 \end{pmatrix} \right\}$ with x in the $i-$th row and $j-$th column

$\ell \begin{pmatrix} 1 & & x \\ & \ddots & \\ & & 1 \end{pmatrix} = v(x)$

$\ell : \quad U_a \backslash \{1\} \to \mathbb{R}$,
$U_{-a}\ u\ U_{-a} \cap N = \{m(u)\}$
$m(u)$ acts on A as reflection at some affine hyperplane
$H_{a,u} = \{x \in A : a(x) + \ell(m) = 0\}$

Affine hyperplanes:
$\{\sum_{i=1}^{n} x_i \eta_i \in A : x_i - x_j = k\}$ for all $i \neq j, k \in \mathbb{Z}$

How is $\mathcal{B}(\mathbf{G}, K)$ constructed ?

Bruhat-Tits blackbox

For every $x \in A$ define a subgroup $P_x \subset G$

$x = 0$

$P_x = PGL_n(R_K)$

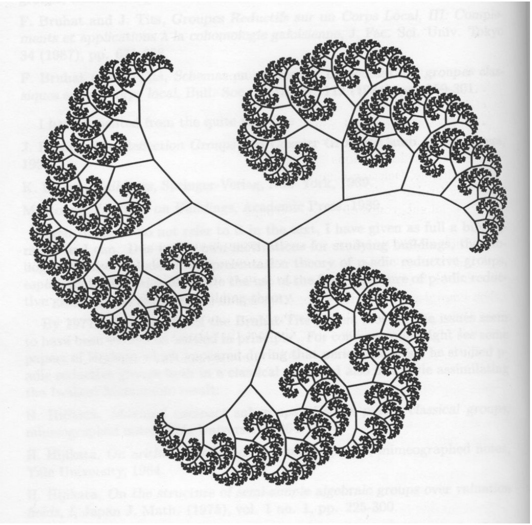
Define an equivalence relation $\sim$ on $G \times A$ by

$$(g, x) \sim (h, y) \Leftrightarrow \exists n \in N : nx = y \text{ and } g^{-1}hn \in P_x$$

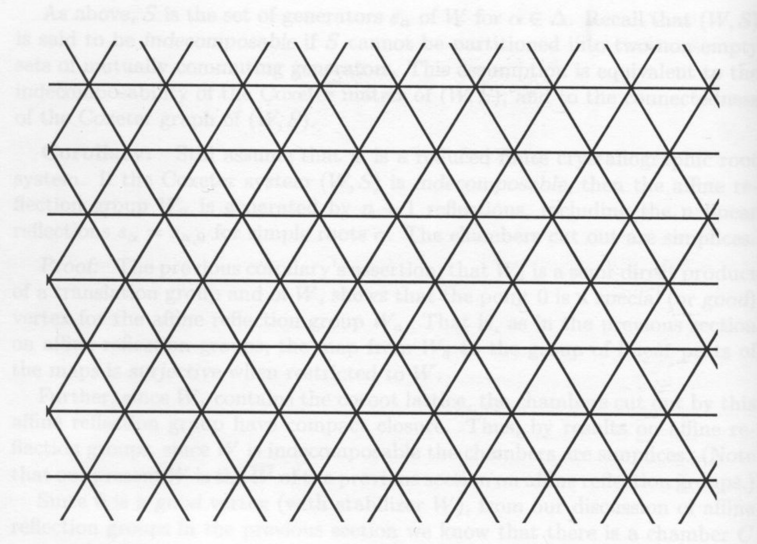
$$\mathcal{B}(\mathbf{G}, K) = G \times A / \sim$$

$\mathcal{B}(\mathbf{PGL_n}, K) \simeq$ {non-Archimedean norms on K^n} modulo scaling (Goldman-Iwahori)

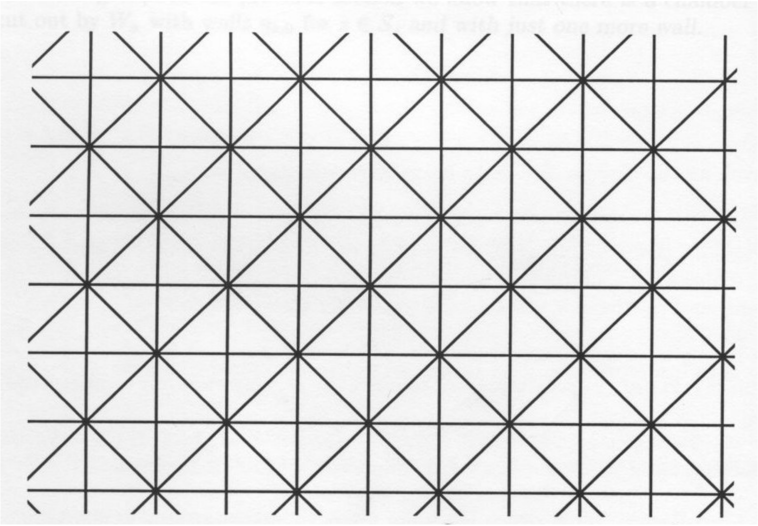
$\mathcal{B}(PGL_2, \mathbb{Q}_2)$



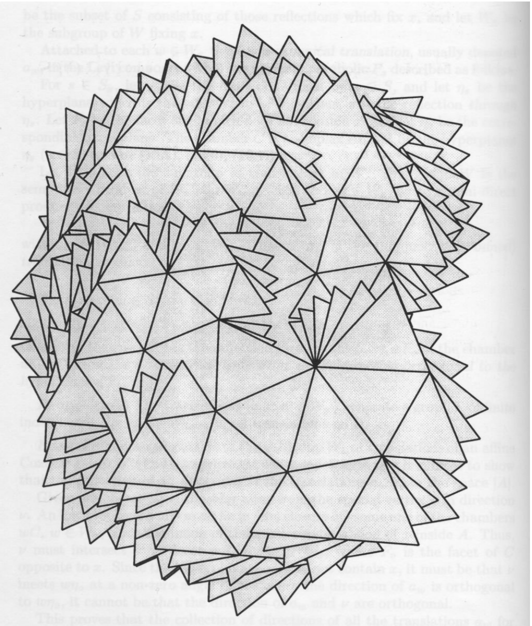
Apartment for PGL_3



Apartment for Sp_4



Some part of $\mathcal{B}(PGL_3, \mathbb{Q}_p)$



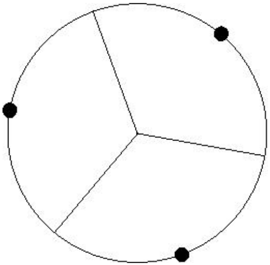
Compactifications of Bruhat-Tits buildings

- Borel-Serre compactification (1976):
  Fix some $x \in \mathcal{B}(G, K)$ and
  add an endpoint to every geodesic ray starting at x.
  (This works in the same way for $CAT(0)$ spaces).

- Polyhedral compactification (Landvogt 1996):
  analogue of the maximal Satake compactification
  for Riemann symmetric spaces.

Compactifications of Bruhat-Tits buildings

- Concrete compactification for $\mathbf{PGL_n}$ (A.W. 2001/2004)



  - $\overline{A}$ = compactification of one apartment
  - refine the Bruhat-Tits blackbox to define P_x for all $x \in \overline{A}$
  - $\overline{\mathcal{B}}(\mathbf{PGL_n}, K) = PGL_n(K)x\overline{A}/\sim$

Theorem 1: The Goldman-Iwahori identification $\mathcal{B}(\mathbf{PGL_n}, K) \simeq (\{$non-Archimedean norms on $K^n\}$ modulo scaling) can be continued to a $PGL_n(K)-$ equivariant homeomorphism $\overline{\mathcal{B}}(\mathbf{PGL_n}, K) \simeq (\{$ non-Archimedean seminorms on $K^n\}$ modulo scaling)

Theorem 2: $\overline{\mathcal{B}}(\mathbf{PGL_n}, K)$ can be embedded in the Berkovich projective space $(\mathbb{P}^{n-1})^{an}$.

What are Berkovich spaces ?

A Berkovich space is a $p-$adic analytic space with good topological properties.

X/K smooth, projective variety (projective manifold). The set of points $X(K)$ inherits a non-Archimedean topology from K with bad topological properties (e.g. it is totally disconnected).

Berkovich adds “a lot of” new points to fill in the gaps.

More precisely:

$X = \text{Spec } B$ affine variety $(B = K[x_1, \ldots, x_n] \diagup \mathfrak{a})$

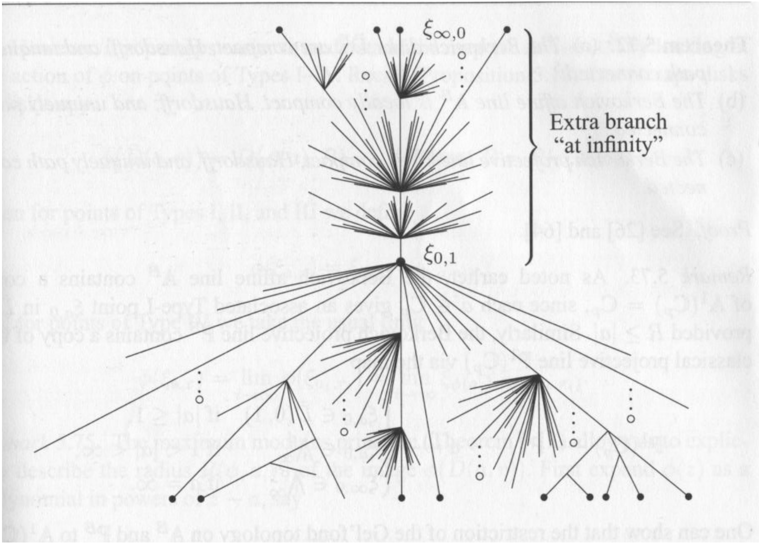
$$X^{an} = \left\{ \begin{array}{l} \text{multiplicative seminorms on } B \\ \text{extending } | \quad |_K \end{array} \right\}$$

Let $x \in X(K)$, i.e. $x = (x_1, \ldots, x_n) \in K^n$ such that $f(x) = 0$ for all $f \in \mathfrak{a}$.

X defines a multiplicative seminorm on B as follows:
$| \, g \, |_x = | \, g(x_1, \ldots, x_n) \, |_K$.

There are many more multiplicative seminorms on B.

The Berkovich projective line



Compactification of Bruhat-Tits buildings with Berkovich spaces

(Joint work with Bertrand Rémy and Amaury Thuillier (Lyon))

We define a continuous map

$$\vartheta : \mathcal{B}(\mathbf{G}, K) \to \mathbf{G}^{an}$$

as follows:

- For every $x \in \mathcal{B}(\mathbf{G}, K)$ there exists a unique $K-$affinoid subgroup $G_x \subset \mathbf{G}^{an}$ such that for all non-Archimedean field extensions L/K

$$G_x(L) = \text{Stabilizer of } x \text{ in } \mathcal{B}(\mathbf{G}_L, L)$$

- $\vartheta(x) =$ Shilov boundary point given by G_x (a certain multiplicative seminorm on the algebra of $\mathbf{G}$). This generalizes results of Berkovich in the split case.

Compactification of Bruhat-Tits buildings with Berkovich spaces

- Choose a parabolic subgroup $\mathbf{P}$ in $\mathbf{G}$
  (If $\mathbf{G} = \mathbf{SL}_n$, the parabolic subgroups are precisely the stabilizers of flags in K^n)
- $\vartheta_P : \mathcal{B}(\mathbf{G}, K) \xrightarrow{\vartheta} \mathbf{G}^{an} \rightarrow \mathbf{G}^{an} \big/ \mathbf{P}^{an}$ is a continuous, $G-$equivariant map.
  It may forget some almost simple factors of $\mathbf{G}$.
  ϑ_P only depends on the type of $\mathbf{P}$.
- **Definition** $\overline{\mathcal{B}_P}(\mathbf{G}, K) =$ closure of $\vartheta_P\big(\mathcal{B}(\mathbf{G}, K)\big)$ in the Berkovich space $\mathbf{G}^{an}/\mathbf{P}^{an}$.

Compactification of Bruhat-Tits buildings with Berkovich spaces

Theorem $\overline{\mathcal{B}}_P(\mathbf{G}, K) = \bigcup_{\mathbf{Q} \text{ “good” parabolic}} \mathcal{B}(\mathbf{Q}_{ss}, K)$

Theorem Any two points x, y in $\overline{\mathcal{B}}p(\mathbf{G}, K)$ are contained in one compactified apartment.

Theorem (Mixed Bruhat decomposition)
Let $x, y \in \overline{\mathcal{B}}p(\mathbf{G}, K)$ with stabilizers $P_x, P_y \subset G$. Then $G = P_x N P_y$.

Compactification of Bruhat-Tits buildings with Berkovich spaces

Example:

1. **P** Borel subgroup
   (If $\mathbf{G} = \mathbf{SL}_n$, a Borel subgroup is the stabilizer of a maximal flag in K^n)
   Then all parabolic subgroups are "good" and $\overline{\mathcal{B}}_P(\mathbf{G}, K)$ is Landvogt's polyhedral compactification.
2. $\mathbf{G} = \mathbf{PGL_n}$ $\quad \mathbf{P} = \left\{ \begin{pmatrix} * & \cdots & * & * \\ \vdots & & \vdots & \vdots \\ * & \cdots & * & * \\ 0 & \cdots & 0 & * \end{pmatrix} \right\}$ the stabilizer of a hyperplane
   Then $\overline{\mathcal{B}}_P(\mathbf{G}, K)$ is the seminorm compactification described above.

$\overline{\mathcal{B}}_P(\mathbf{G}, K)$ is a non-Archimedean analogue of Satake's compactifications for Riemann symmetric spaces.