Discrete Groups: A Story of Geometry, Complexity, and Imposters

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University of Oxford

BMS Colloquium, Berlin, 29 June 2012.

Outline

What kind of mathematics do you want to do?

Study X

Decide on nature of maps $X \to Y$

$$\mathrm{Aut}(X)$$

symmetries = automorphisms of X

• X just a set, $\mathrm{Aut}(X)$ is the group of bijections $X \to X$
• X a vector space, $\mathrm{Aut}(X) = \mathrm{GL}(X)$ linear bijections $X \to X$
• X a metric space, maybe $\mathrm{Aut}(X) = \mathrm{Isometries}(X)$, or bi-Lipschitz maps $X \to X$, or ...
• X a topological space, $\mathrm{Aut}(X) = \{\mathrm{self-homeos}\}$ (or maybe homotopy equivalences mod homotopy),...

$\mathrm{Aut}(X)$ is ALWAYS a group!!

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Figure: $\langle a, b \mid a^6 = 1,\ b^2 = 1,\ bab = a^{-1} \rangle$

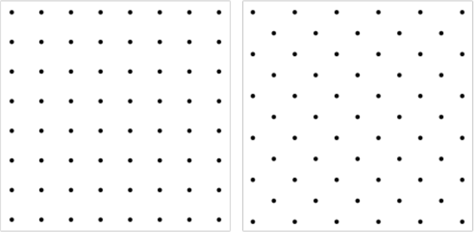


Figure: Some isometries $\langle \alpha, \beta \mid \alpha\beta = \beta\alpha \rangle$

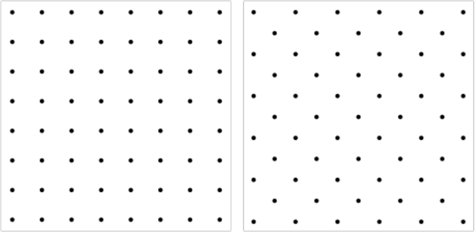
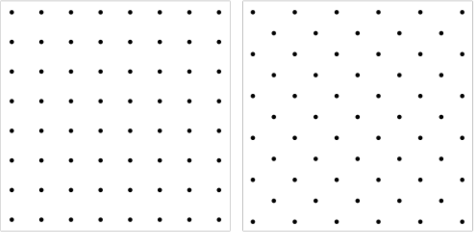


Figure: linear automorphisms $\langle A, B, J \mid J^2 = 1,\ A^2 = J,\ B^3 = J \rangle$

$$J = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$



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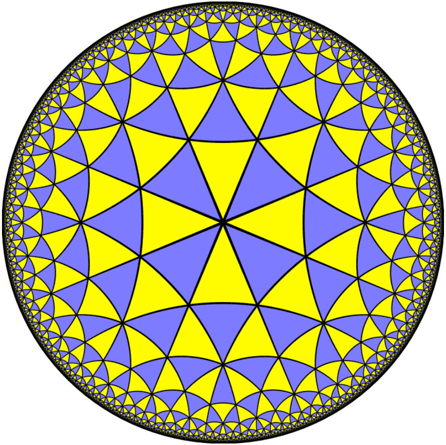

Figure: $\langle a, b, c \mid abc = 1,\ a^3 = b^3 = c^4 \rangle$

Finitely presented groups

$$\Gamma \cong \langle a_1, \dots, a_n \mid r_1, \dots, r_m \rangle \equiv \mathcal{P}$$

The a_i are the generators and the r_j are the relators (defining relations).
A word in the symbols $a_i^{\pm 1}$ is a relation, ie equals $1 \in \Gamma$ if and only if it is a consequence of the r_j, i.e

$$w \overset{\text{free}}{=} \prod_{k=1}^{N} x_i^{-1} r_{j(k)}^{\pm 1} x_i.$$

in other words, there is a short exact sequence

$$1 \to \langle\langle r_j \rangle\rangle \to \mathrm{Free}(a_i) \to \Gamma \to 1.$$

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Hopeless goal: understand the universe of all finitely presented groups.

Why "finitely presented groups"?

ANSWERS: This is a compactness condition that controls the level of pathology

Higman: all recursively presented groups are subgroups of finitely presented groups

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We want action!

If we just write down a group

$$\Gamma \;\cong\; \langle a_1, \dots, a_n \mid r_1, \dots, r_m \rangle$$

what objects X might exist with $\Gamma \cong \mathrm{Aut}(X)$?.

Where might Γ ACT? Look for homomorphisms $\Gamma \to \mathrm{Aut}(Y)$??

Qu: If $\Gamma \neq 1$, is there always a non-trivial action of Γ on a finite set?

Qu: ... on a vector space? Is there a non-trivial $\rho : \Gamma \to \mathrm{GL}(n, \mathcal{C})$?

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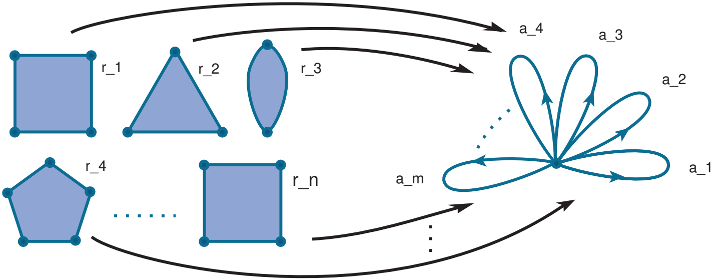

Figure: The standard 2-complex $K(\mathcal{P})$

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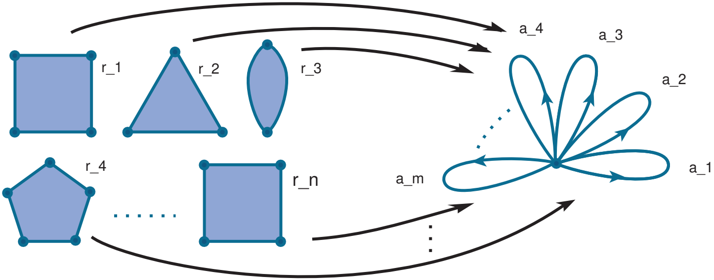

Figure: The standard 2-complex $K(\mathcal{P})$

The group springing into action

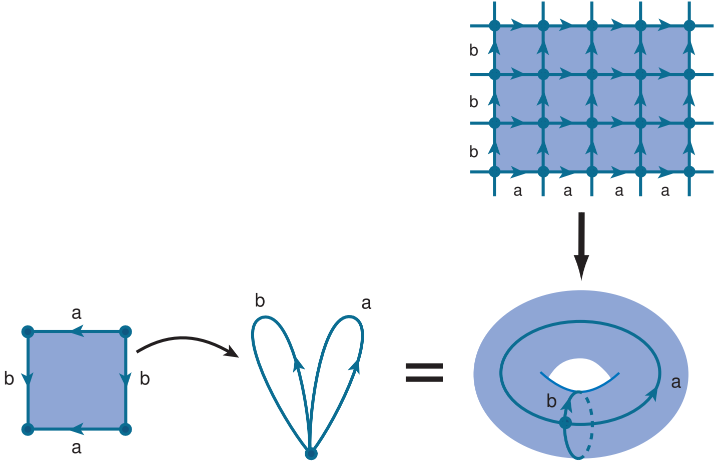

Figure: The 2-complex and Cayley graph for $\langle a, b \mid ab = ba \rangle$

Basic Topological Models

Recall that the universal cover of a space X is a 1-connected space $\tilde{X}$ on which a group Γ acts freely and properly with quotient X.

Such universal covers exist for all reasonable spaces (eg cell complexes, manifolds), and Γ is called the fundamental group of X.

Theorem

A group is finitely presented if and only if it is the fundamental group of a compact 2-dimensional cell complex, and of a compact 4-dimensional manifold (space-time).

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Better models, Distinguished Classes

- Add more cells to make $K(\mathcal{P})$ highly connected, towards a $K(\Gamma, 1)$
  - Finiteness properties (F_n, $\mathrm{FP}_n, \dots$ etc.)
- **Manifold Models:** 4-manifold; symplectic; $\mathbb{C}; \dots$
- Uniqueness issues (Borel conjecture etc.)
- Geometric conditions such as non-positive curvature
- Special Classes Arising:
  - 3-manifold groups; Kähler groups; $\mathrm{PD}(n)$ groups; 1-relator groups; Thompson groups; CAT(0) groups;...

Groups refusing to act nicely

A space is contractible if it can be continuously deformed to a point. (So $\mathbb{R}^2$ is contractible but $\mathbb{S}^2$, although simply-connected, is not.)

There are invariants that obstruct groups from acting freely and discretely on contractible, finite-dimensional spaces,

e.g If $H_n(\Gamma, \mathbb{Z}) \neq 0$, then Γ cannot act freely and discretely on a contractible space of dimension $< n$ – e.g finite groups

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IDEA: Try to study all (??) finitely presented groups by building them up from finite groups and groups that act nicely on contractible spaces.

- Level 0: Finite groups
- Level 1: groups that act nicely on finite-dimensional, contractible spaces, with finite (level 0) isotropy (point-stabilizers)
- Level n: groups that act as above with isotropy at level $(n-1)$.

NB: Actions on trees are allowed, so the above incorporates amalgamated free products and HNN extensions.

Kropholler-Mislin: There exist finitely presented groups that do not appear in this hierarchy.

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ABJECT GROUPS (Arzhantseva, Bridson, Januszkiewicz, Leary, Minasyan, Swiatkowski)

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Linear Realisations

Question

Can every finitely presented group be realised as a group of matrices?

e.g. $\Gamma \hookrightarrow \mathrm{GL}(n, \mathbb{C})$?? Or, at least, is there non-trivial $\Gamma \to \mathrm{GL}(n, \mathbb{C})$?

Obstruction (Malcev): Finitely generated subgroups of $\mathrm{GL}(n, \mathbb{C})$ are residually finite:

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Three Groups

The following group acts on $\mathbb{R}$

$$G_2 = \langle A, B \mid BAB^{-1} = A^2 \rangle$$

by

$$A(x) = x + 1 \quad B(x) = 2x$$

and thus one sees that it is infinite.

One of the following groups is trivial and one is an infinite group with no finite quotients

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Combinatorial Group Theory (Dehn 1912)

$$\Gamma \;\cong\; \langle a_1, \ldots, a_n \mid r_1, \ldots, r_m \rangle$$

"The general discontinuous group is given [as above]. There are above all three fundamental problems.

- The identity [word] problem
- The transformation [conjugacy] problem
- The isomorphism problem

[...] One is already led to them by necessity with work in topology. Each knotted space curve, in order to be completely understood, demands the solution of the three"

Higman Embedding (1961): Every recursively presented group is a subgroup of a finitely presented group.

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Undecidability

This has nothing to do with religion or Schrödinger's cat!

Fix a finite set A. The a set of words $S \subset A^*$ is re (recursively enumerable) if $\exists$ Turing machine that can generate a list of the elements of S. And S is recursive if both S and $A^* \smallsetminus S$ are r.e.

Proposition

There exist r.e. sets of integers that are not recursive.

Proposition

If $S \subset \mathbb{N}$ is r.e. not recursive,

$$G = \langle a, b, t \mid \ t\,(b^n a b^{-n}) = (b^n a b^{-n})\, t \ \forall n \in S \rangle$$

has an unsolvable word problem.

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Undecidable properties of finitely presented groups

Higman embedding gives $G \subset \Gamma$ with Γ finitely presented.

Corollary (Novikov, Boone)

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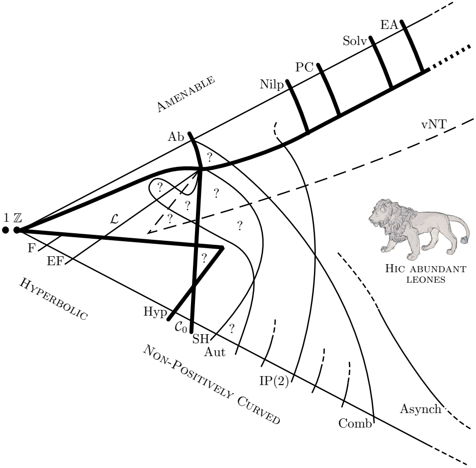

Figure 1: The universe of groups.

Nilpotent Groups: polynomial growth

$$\begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$[x, y] = xyx^{-1}y^{-1}, \quad [x_1, [x_2, [x_3, \ldots x_c]]\ldots] = 1$$

The 3-dimensional Heisenberg group

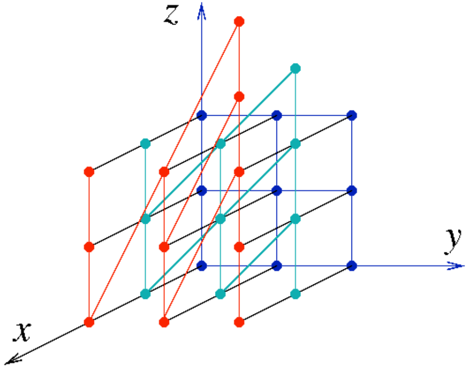

Figure: $\langle x, y, z \mid xyx^{-1}y^{-1} = z,\ xz = zx,\ yz = zy \rangle$

Groups as geometric objects (Gromov)

Study finitely generated groups as geometric objects in their own right, via their intrinsic geometry.

$$\Gamma = \langle a_1, \ldots, a_n \mid r_1,\ r_2, \ldots \rangle$$

Word Metric:

$$d(\gamma_1, \gamma_2) = \min\{|w| : w \in F(\mathcal{A}),\ w \overset{\Gamma}{=} \gamma_1^{-1}\gamma_2\}.$$

Cayley Graph (1878) $= \widetilde{K(\mathcal{P})}^{(1)}$

- Word metric and Cayley graph are independent of generating set, up to quasi-isometry.
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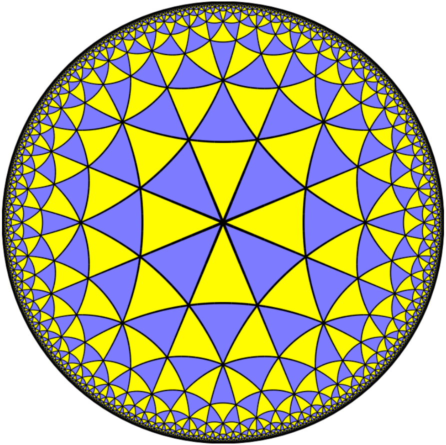

Figure: $\langle a, b, c \mid abc = 1,\ a^3 = b^3 = c^4 \rangle$

Local non-positive curvature conditions

Classical, then A.D. Alexandrov, Gromov [ref: Bridson-Haefliger]

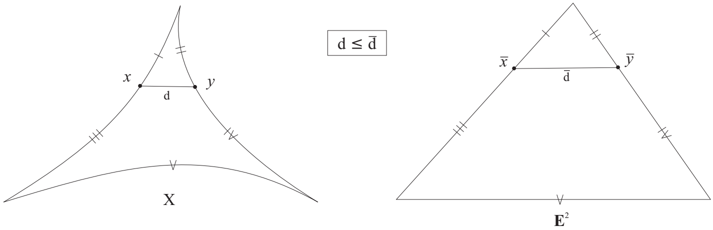

Figure: The CAT(0) inequality

Local-to-global: If X is complete and every point has a neighbourhood in which triangles satisfy this inequality, then in $\tilde{X}$ all triangles satisfy this inequality.

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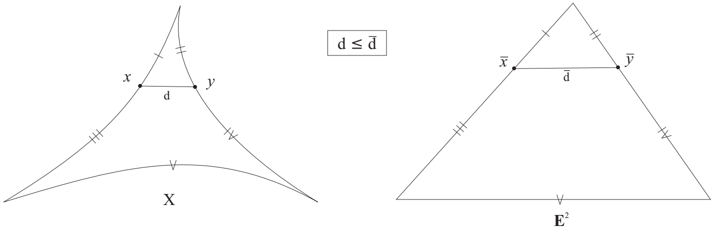

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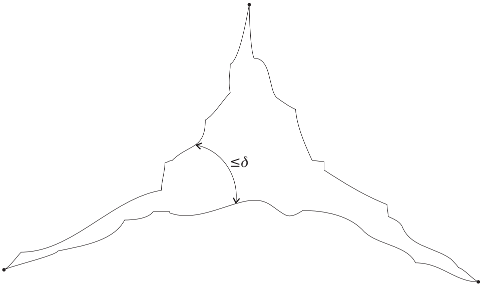

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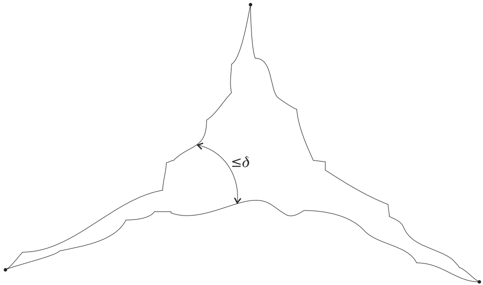

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Gromov's hyperbolic groups

If Γ is hyperbolic then it

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• has only finitely many conjugacy classes of finite subgroups and its abelian subgroups are virtually cyclic

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Subgroups of $\mathrm{SL}(n, \mathbb{Z})$

Question

How complicated are the finitely presented subgroups of $\mathrm{SL}(n, \mathbb{Z})$??

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If Γ is residually finite, what can one tell about it from it's set of finite homomorphic images, i.e. from its actions on all finite sets?

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Grothendieck's Question (1970)

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$A \neq 0$ a commutative ring, Γ a finitely generated group, $\mathrm{Rep}_A(\Gamma)$ the category of Γ-actions on fin. pres. A-modules.
Any homomorphism $u : \Gamma_1 \to \Gamma_2$ of groups induces

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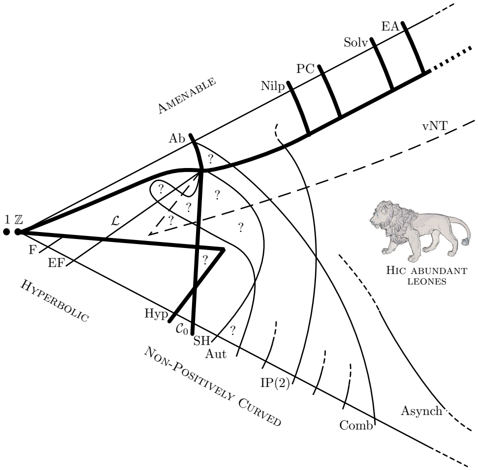

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Taming monsters: Rips and 1-2-3 Thm

$\exists$ algorithm with input a finite, aspherical presentation $\mathcal{Q}$ and output a FINITE presentation for the fibre-product

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associated to a s.e.s.

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"1-2-3 Thm" refers to fact that N, H and Q are of type $\mathcal{F}_1, \mathcal{F}_2$ and $\mathcal{F}_3$ respectively. [Baumslag, B, Miller, Short]

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Solution of Grothendieck's Problem

Question

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Grothendieck proved that the answer is yes in many cases, e.g. arithmetic groups. Platonov-Tavgen (later Bass–Lubotzky, Pyber) proved answer no for finitely generated groups in general.

Theorem (B-Grunewald, 2003)

$\exists$ *hyperbolic subgroups* $H < \mathrm{SL}(n, \mathbb{Z})$ *and finitely presented subgroups* $P \hookrightarrow \Gamma := H \times H$ *of infinite index, such that* P *is not abstractly isomorphic to* Γ, *but the inclusion* $u : P \hookrightarrow \Gamma$ *induces an isomorphism* $\hat{u} : \hat{P} \to \hat{\Gamma}$.

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