Geometry of moduli spaces

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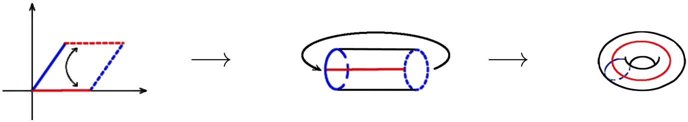
The moduli problem for algebraic curves (1)

C: compact Riemann surface

Examples:

- $C = \mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ (Riemann sphere)

- $E = \mathbb{C} \,/\, \mathbb{Z} + \mathbb{Z}\tau$ (torus, elliptic curve)



The moduli problem for algebraic curves (2)

Theorem (Riemann existence theorem, Chow's theorem): Every compact Riemann surface C is an algebraic curve, i.e. admits a holomorphic embedding $i\colon C \hookrightarrow \mathbb{P}^n = \mathbb{P}^n(\mathbb{C})$ and $i(C)$ is the set of solutions of finitely many homogeneous polynomial equations.

Conclusion: It makes no difference wether one considers the classification problem for compact Riemann surfaces or for smooth projective algebraic curves (over $\mathbb{C}$).

The moduli problem for algebraic curves (3)

Question: How can one classify algebraic curves?

The *topological* classification of orientable compact connected 2-dimensional real surfaces is given by their *genus* (= # of holes)

$g = 0$ $g = 1$ $\dots$ genus g

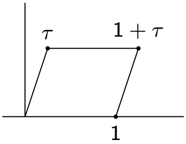
The moduli problem for algebraic curves (4)

Question: How many algebraic (= holomorphic = conformal) structures exist on a topological surface of genus g?

$g = 0$: The algebraic structure is unique, i.e. every algebraic curve of genus 0 is isomorphic to $\mathbb{P}^1$.

$g = 1$: Every curve of genus 1 arises as

$$E_\tau = \mathbb{C} \;/\; \mathbb{Z} + \tau\mathbb{Z}, \quad \operatorname{Im}\tau > 0.$$



$$\tau \in \mathbb{H}_1 = \{z \in \mathbb{C};\ \operatorname{Im} z > 0\}.$$

Question: When is $E_\tau \cong E_{\tau'}$?

The moduli problem for algebraic curves (5)

The group $\mathrm{SL}(2,\mathbb{Z})$ acts on $\mathbb{H}_1$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \quad \tau \mapsto \frac{a\tau + b}{c\tau + d} \, .$$

One shows that

$$E_\tau \cong E_{\tau'} \Longleftrightarrow \tau \sim \tau' \ \text{modulo} \ \mathrm{SL}(2,\mathbb{Z}).$$

I.e. we consider the quotient of $\mathbb{H}_1$ by $\mathrm{SL}(2,\mathbb{Z})$ and obtain

$$\mathcal{M}_1 = \mathrm{SL}(2,\mathbb{Z}) \setminus \mathbb{H}_1 = \{\text{elliptic curves}\}/\cong$$

Using the j-function $j \colon \mathbb{H}_1 \to \mathbb{C}$ one shows that

$$\bar{j} \colon \mathcal{M}_1 = \mathrm{SL}(2,\mathbb{Z}) \setminus \mathbb{H}_1 \cong \mathbb{C}.$$

The moduli problem for algebraic curves (6)

Curves of genus $g \geq 2$

Riemann (1857): Curves of genus $g \geq 2$ depend on $3g - 3$ "moduli".

$$\mathcal{M}_g = \{\text{algebraic curves of genus } g\}/\cong$$

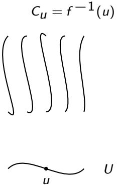
Then the following holds:

- $\mathcal{M}_g$ carries the structure of a quasi-projective variety
- $\dim \mathcal{M}_g = 3g - 3$
- $\mathcal{M}_g$ is irreducible
- $\mathcal{M}_g$ has at most finite quotient singularities.

The moduli problem for algebraic curves (7)

An important property of $\mathcal{M}_g$ is the following:

Let $f\colon \mathcal{X} \to U$ be a family of smooth projective curves, i.e.



for every $u \in U$ the fibre $C_u = f^{-1}(u)$ is a smooth projective curve of genus g. Then the map

$$\begin{aligned} \varphi_U\colon U &\to \mathcal{M}_g \\ u &\mapsto [C_u] \end{aligned}$$

is a morphism (and $\mathcal{M}_g$ is minimal with this property).

Formally one describes this in the language of representations of *functors* and (Deligne–Mumford) *stacks*.

The moduli problem for algebraic curves (8)

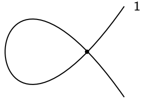
Compactifications of $\mathcal{M}_g$

$\mathcal{M}_g$ is a quasi-projective variety.

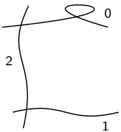
Question: Is there a geometrically meaningful compactification of $\mathcal{M}_g$?

$$\overline{\mathcal{M}}_g := \text{moduli space of stable curves of genus } g$$

Definition: A projective algebraic curve is *stable* if it has at most nodes as singularities (but it need not be irreducible) and $|\mathrm{Aut}(C)| < \infty$.



$g = 2$



$g = 4$

The geometry of $\overline{\mathcal{M}}_g$ (1)

Fact: $\overline{\mathcal{M}}_g$ carries the structure of a projective variety containing $\mathcal{M}_g$ as a (Zariski-)open set.

$g = 0$

$$\overline{\mathcal{M}}_0 = \{\mathrm{pt}\}$$

$g = 1$

$$\overline{\mathcal{M}}_1 = \mathcal{M}_1 \cup \{\mathrm{pt}\} = \mathbb{C} \cup \{\mathrm{pt}\} = \mathbb{P}^1$$

$g \geq 2$

Question: What can we say about the geometry of the variety $\overline{\mathcal{M}}_g$?

The Kodaira dimension (1)

$$X = \text{projective manifold}, \ \dim_{\mathbb{C}} X = n.$$

$$T_X = \text{tangent bundle of } X$$
$$\Omega^1_X = T_X^\vee = \text{cotangent bundle of } X.$$

The sections of Ω^1_X are 1-forms, i.e. locally of the form

$$\Omega = \sum_{i=1}^{n} f_i(z_1, \ldots, z_n)\, \mathrm{d}z_i.$$

The Kodaira dimension (2)

$$\omega_X := \Lambda^n \Omega_X^1 = \det \Omega_X^1 = \text{canonical bundle}$$

The sections of ω_X are n-forms, i.e. locally of the form

$$\omega = f(z_1, \ldots, z_n)\, dz_1 \wedge \ldots \wedge dz_n.$$

$$\omega_X^{\otimes k} = \omega_X \otimes \ldots \otimes \omega_X$$

(k-th power of the canonical bundle, $k \geq 1$)

The sections are k-fold pluricanonical forms and locally of the form

$$\omega = f(z_1, \ldots, z_n)\, (dz_1 \wedge \ldots \wedge dz_n)^k.$$

The Kodaira dimension (3)

$$X = \text{projective manifold},\ \dim_{\mathbb{C}} X = n.$$

Definition: The k-th *plurigenus* of X is defined by

$$P_k(X) := \dim_{\mathbb{C}} H^0(X, \omega_X^{\otimes k})$$

i.e. the number of independent global k-fold pluricanonical forms.

The Kodaira dimension (4)

Definition: The *Kodaira dimension* of X is defined as

$$\kappa(X) = \begin{cases} -\infty & \text{if } P_k(X) = 0 \text{ for all } k \geq 1 \\ 0 & \text{if } P_k(X) = 0 \text{ or } 1 \text{ for all } k \geq 1 \text{ and there} \\ & \text{is at least one } k_0 \geq 1 \text{ with } P_{k_0}(X) = 1 \\ \kappa & \text{if } P_k(X) \sim c \cdot k^\kappa + \text{l.o.t. } (c > 0). \end{cases}$$

Remark: $\kappa(X) \in \{-\infty, 0, 1, \dots, \dim X\}$.

Definition: X is of *general type* if $\kappa(X) = \dim X$.

The Kodaira dimension (5)

In the case of curves the situation is as follows:

- $\kappa(C) = -\infty \Leftrightarrow g(C) = 0 \Leftrightarrow C = \mathbb{P}^1$
- $\kappa(C) = 0 \Leftrightarrow g(C) = 1 \Leftrightarrow C =$ elliptic curve
- $\kappa(C) = 1 \Leftrightarrow g(C) \geq 2$

The Kodaira dimension is a rough birational invariant for algebraic varieties.

Geometry of $\overline{\mathcal{M}}_g$ (2)

Question: What is the Kodaira dimension of (a smooth model of) $\overline{\mathcal{M}}_g$?

Theorem (Harris–Mumford, Eisenbud–Harris): $\overline{\mathcal{M}}_g$ is of general type for $g \geq 24$.

Theorem (Farkas): $\overline{\mathcal{M}}_{22}$ is of general type.

Theorem: $\kappa(\overline{\mathcal{M}}_g) = -\infty$ for $g \leq 16$.

This result is due to Severi, Sernesi, Chang, Ran, Bruno, Verra, ...

Question: $\kappa(\overline{\mathcal{M}}_g) = ?$ for $17 \leq g \leq 21,\ 23$?

Proposition (Farkas): $\kappa(\overline{\mathcal{M}}_{23}) \geq 2$.

Moduli of abelian varieties (1)

Abelian varieties are higher-dimensional generalisations of elliptic curves

$$E = \mathbb{C} \;/\; \mathbb{Z}\tau + \mathbb{Z} = \mathbb{C} \;/\; \Lambda_\tau \quad (\Lambda_\tau = \mathbb{Z}\tau + \mathbb{Z}).$$

A *g-dimensional torus* is given by

$$A = \mathbb{C}^g / \;\Lambda \qquad (\Lambda = \text{ lattice of rank } 2g \text{ in } \mathbb{C}^g).$$

Clearly A is a compact complex abelian Lie group (and every such Lie group is of this form).

Definition: A torus A is called an *abelian variety* if A is projective, i.e. an embedding $A \hookrightarrow \mathbb{P}^N$ exists (for some N).

Remark: This is automatic for $g = 1$, but does not hold in general.

Moduli of abelian varieties (2)

Many aspects of the theory of elliptic curves carry over to abelian varieties. The *Siegel upper half plane* of genus g is defined as

$$\mathbb{H}_g = \{\tau \in \mathrm{Mat}(g \times g, \mathbb{C});\ \tau = {}^t\tau,\ \mathrm{Im}\,\tau > 0\}.$$

The *integral symplectic group* is defined by

$$\mathrm{Sp}(g, \mathbb{Z}) = \{M \in \mathrm{GL}(2g, \mathbb{Z});\ {}^tMJM = J\}$$

where

$$J = \left(\begin{array}{c|c} 0 & \mathbb{1}_g \\ \hline -\mathbb{1}_g & 0 \end{array}\right).$$

Remark: For $g = 1$

$$\mathrm{Sp}(1, \mathbb{Z}) = \mathrm{SL}(2, \mathbb{Z}).$$

Moduli of abelian varieties (3)

The group $\mathrm{Sp}(g, \mathbb{Z})$ acts on $\mathbb{H}_g$ by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : \quad \tau \mapsto (A\tau + B)(C\tau + D)^{-1}.$$

The quotient

$$\mathcal{A}_g = \mathrm{Sp}(g, \mathbb{Z}) \setminus \mathbb{H}_g = \text{moduli space of principally polarized abelian varieties of dimension } g.$$

A *polarized* abelian variety is a pair $(A, \mathscr{L})$ where A is an abelian variety und $\mathscr{L}$ is an ample line bundle on A.

A polarization is called *principal* if $\mathscr{L}^g = g!$.

Moduli of abelian varieties (4)

$\mathcal{A}_g$ has the following properties:

- $\mathcal{A}_g$ is quasi-projective, irreducible
- $\dim \mathcal{A}_g = \frac{1}{2}g(g+1)$
- $\mathcal{A}_g$ has only finite quotient singularities.

Moduli of abelian varieties (5)

Question: What is the Kodaira dimension of (a smooth projective model of) $\mathcal{A}_g$?

Theorem: $\kappa(\mathcal{A}_g) = -\infty$ for $g \leq 5$.

$g \leq 3$:	classical
$g = 4$:	Clemens
$g = 5$:	Donagi; Mori, Mukai; Verra

Theorem: $\mathcal{A}_g$ is of general type for $g \geq 7$.

$g \geq 9$:	Tai
$g = 8$:	Freitag
$g = 7$:	Mumford

Problem: $\kappa(\mathcal{A}_6) = ?$

Moduli of abelian varieties (6)

The Siegel domain $\mathbb{H}_g$ is the hermitian symmetric space

$$\mathbb{H}_g = \mathrm{Sp}(g, \mathbb{R})/U(g)$$

where $U(g)$ is the (up to conjugation) unique maximal compact subgroup.

It can also be realised as a bounded complex domain via the *Cayley transformation*

$$\begin{aligned} \mathbb{H}_g &\longrightarrow D_g = \{Z \in \mathrm{Mat}(g \times g, \mathbb{C});\ Z = {}^tZ,\ {}^tZ\overline{Z} < \mathbb{1}_g\} \\ \tau &\longmapsto (\tau - i \cdot \mathbb{1}_g)(\tau + i \cdot \mathbb{1}_g)^{-1}. \end{aligned}$$

Example:

$$\mathbb{H}_1 \xrightarrow{\cong} D_1 = \{z \in \mathbb{C};\ |z| < 1\}.$$

Moduli of abelian varieties (7)

Modular forms

Definition: A *modular form* of *weight k* with respect to $\mathrm{Sp}(g, \mathbb{Z})$ is a holomorphic function

$$F \colon \mathbb{H}_g \longrightarrow \mathbb{C}$$

such that

$$F(M(\tau)) = (\det(C\tau + D))^k F(\tau)$$

for all $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(g, \mathbb{Z})$.

Remark: For $g = 1$ one needs to add a condition of holomorphicity at infinity (the cusp).

Moduli of abelian varieties (8)

Definition: A modular form is a *cusp form* if it vanishes at ∞.

$$S_k(\mathrm{Sp}(g,\mathbb{Z})) = \{F;\ F \text{ is a cusp form of weight } k\}.$$

$$\dim_{\mathbb{C}} S_k(\mathrm{Sp}(g,\mathbb{Z})) < \infty.$$

Moduli of abelian varieties (9)

Modular forms and differential forms

Let F be a modular form of weight $k(g+1)$ with respect to $\mathrm{Sp}(g,\mathbb{Z})$. Define

$$\omega_F = F \cdot (\mathrm{d}\tau_{11} \wedge \mathrm{d}\tau_{12} \wedge \ldots \wedge \mathrm{d}\tau_{gg})^k.$$

Fact: ω_F is invariant w.r.t. $\mathrm{Sp}(g,\mathbb{Z})$.

Hence we can view ω_F as a k-fold *pluricanonical form* on (an open part of) $\mathcal{A}_g$.

More precisely

$$\mathcal{A}_g^\circ = \mathcal{A}_g \setminus \{\text{fixed points of } \mathrm{Sp}(g,\mathbb{Z})\}$$

Then

$$\omega_F \in H^0(\mathcal{A}_g^\circ,\ \omega_{\mathcal{A}_g^\circ}^{\otimes k}).$$

Enriques classification of surfaces

S: smooth projective surface, minimal

The Kodaira classification classifies surfaces according to Kodaira dimension.

$\kappa(S) = -\infty$ S is rational or a geometrically ruled surface

$\kappa(S) = 0$ S is one of the following:

- abelian surface
- $K3$ surface
- Enriques surface
- bielliptic surface

$\kappa(S) = 1$ S is elliptic fibration

$\kappa(S) = 2$ S is of general type.

$K3$ surfaces (1)

Definition: A $K3$ surface is a compact complex surface with the following properties

- $\omega_S = \mathcal{O}_S$
- $\pi_1(S) = \{0\}$.

Examples: (1) Quartic surfaces in $\mathbb{P}^3$:

$$S = \{(x_0 : x_1 : x_2 : x_3) \in \mathbb{P}^3;\ f_4(x_0, \dots, x_3) = 0\}$$

where f_4 is homogeneous of degree 4, general. E.g.

$$S = \{x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0\}.$$

(2) Complete intersections of degree $(2,3)$ in $\mathbb{P}^4$:

$$S = \{f_2 = f_3 = 0\} \subset \mathbb{P}^4.$$

Remark: There exist non-algebraic $K3$ surfaces.

$K3$ surfaces (2)

The second cohomology group $H^2(S,\mathbb{Z})$ of a $K3$ surface has the structure of a *lattice*, i.e. it carries a non-degenerate symmetric bilinear form (given by the intersection form). More precisely

$$H^2(S,\mathbb{Z}) = 3U \oplus 2E_8(-1)$$

where

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{(hyperbolic plane)}$$

and

$$E_8(-1) = \text{ non-degenerate, even, unimodular lattice of rank 8.}$$

The second cohomology group with complex coefficients has a Hodge decomposition

$$H^2(S,\mathbb{C}) = \underbrace{H^{20}}_{\text{(2,0)-forms}} \oplus \underbrace{H^{11}}_{\text{(1,1)-forms}} \oplus \underbrace{H^{02}}_{\text{(0,2)-forms}} \qquad (H^{20} = \overline{H^{02}}).$$

$K3$ surfaces (3)

Since $\omega_S = \mathcal{O}_S$ it follows that

$$\dim_{\mathbb{C}} H^{20} = \dim_{\mathbb{C}} H^0(S, \omega_S) = \dim_{\mathbb{C}} H^0(S, \mathcal{O}_S) = 1.$$

I.e. we have a 1-dimensional subspace

$$\mathbb{C} \cong H^{20} \subset H^2(S, \mathbb{C}) \cong \mathbb{C}^{22}.$$

This inclusion carries all information on S (Torelli theorem).

$$L_{K3} = 3U + 2E_8(-1) \qquad (\mathrm{sign}(L_{K3}) = (3, 19)).$$

A *marking* of a $K3$ surface is an isomorphism

$$\varphi\colon H^2(S, \mathbb{Z}) \cong L_{K3}.$$

This defines a 1-dimensional subspace

$$\varphi(\mathbb{C}\cdot\omega_S) = \varphi(H^{20}) \subset L_{K3} \otimes \mathbb{C}.$$

$K3$ surfaces (4)

With respect to the intersection form on $H^2(S, \mathbb{C})$ one has

$$(\omega_S, \omega_S) = 0, \quad (\omega_S, \overline{\omega}_S) > 0.$$

This leads one to define

$$\Omega_{K3} = \{[x] \in \mathbb{P}(L_{K3} \otimes \mathbb{C});\ (x, x) = 0,\ (x, \overline{x}) > 0\}.$$

$$\dim_{\mathbb{C}} \Omega_{K3} = 20$$

Given a marking $\varphi\colon H^2(S, \mathbb{Z}) \cong L_{K3}$ of a $K3$ surface one defines the *period point*

$$\varphi(\omega_S) = [\mathbb{C}\, \omega_S] \in \Omega_{K3}.$$

$K3$ surfaces (5)

$$\mathrm{O}(L_{K3}) = \{g;\ g \text{ is an orthogonal transformation of } L_{K3}\}.$$

This group acts on Ω_{K3}.

Theorem (Torelli): The $K3$ surface S can be reconstructed from its period point. I.e. the quotient

$$\mathcal{M}_{K3} = \mathrm{O}(L_{K3}) \setminus \Omega_{K3}$$

is the moduli space of $K3$ surfaces.

Remark: The group action is badly behaved, e.g. the quotient $\mathcal{M}_{K3}$ is not hausdorff.

$K3$ surfaces (6)

The situation improves when one restricts to the algebraic case, i.e. to *polarized* $K3$ surfaces.

A polarized $K3$ surface is a pair $(S, \mathscr{L})$ where $\mathscr{L}$ is an ample line bundle.

Instead of $\mathscr{L}$ it suffices to consider

$$h = c_1(\mathscr{L}) \in H^2(S, \mathbb{Z}).$$

The degree of $\mathscr{L}$ (resp. h) is

$$\deg \mathscr{L} = c_1(\mathscr{L})^2 = h^2 = 2d > 0.$$

Question: How can we describe moduli of polarized $K3$ surfaces (of given degree $2d$)?

$K3$ surfaces (7)

If $h \in L_{K3}$ is a primitive element with $h^2 = 2d$, then

$$h^{\perp}_{L_{K3}} \cong 2U \oplus 2E_8(-1) \oplus \langle -2d \rangle =: L_{2d}.$$

As before we consider

$$\Omega_{L_{2d}} = \{[x] \in \mathbb{P}(L_{2d} \otimes \mathbb{C});\ (x,x) = 0,\ (x,\overline{x}) > 0\}.$$

Then

- $\dim \Omega_{L_{2d}} = 19$
- $\Omega_{L_{2d}} = \mathcal{D}_{L_{2d}} \cup \mathcal{D}'_{L_{2d}}$

where $\mathcal{D}_{L_{2d}}$ is a homogeneous symmetric domain of type IV.

$K3$ surfaces (8)

Let

$$\widetilde{\mathrm{O}}(L_{2d}) = \{g \in \mathrm{O}(L_{K3});\ g(h) = h\}$$

and set

$$\mathcal{F}_{2d} = \widetilde{\mathrm{O}}(L_{2d}) \setminus \Omega_{L_{2d}}.$$

Theorem (Torelli): The quotient $\mathcal{F}_{2d}$ is the moduli space of pseudo-polarized $K3$ surfaces of degree $2d$.

$K3$ surfaces (9)

The action of $\widetilde{\mathrm{O}}(L_{2d})$ on $\Omega_{L_{2d}}$ is properly discontinuous. The following holds:

- $\dim \mathcal{F}_{2d} = 19$
- $\mathcal{F}_{2d}$ has only finite quotient singularities
- $\mathcal{F}_{2d}$ is quasi-projective (Baily–Borel).

There exist different compactifications of $\mathcal{F}_{2d}$:

- $\mathcal{F}_{2d}^{BB}$: Baily–Borel compactification
- $\mathcal{F}_{2d}^{tor}$: toroidal compactifications

$K3$ surfaces (10)

Question: What is the Kodaira dimension of $\mathcal{F}_{2d}$?

Theorem (Mukai): $\mathcal{F}_{2d}$ is unirational (and hence $\kappa(\mathcal{F}_{2d}) = -\infty$) if $1 \leq d \leq 10$, $d = 12, 16, 17$ and 19.

Theorem (Gritsenko, H., Sankaran, 2007): $\mathcal{F}_{2d}$ is of general type for $d > 61$ and $d = 46, 50, 54, 57, 58, 60$.

Remark (A. Peterson): $\mathcal{F}_{2d}$ is also of general type for $d = 52$.

Automorphic forms (1)

Recall that

$$\Omega_{L_{2d}} = \{[x] \in \mathbb{P}(L_{2d} \otimes \mathbb{C});\ (x,x) = 0,\ (x,\overline{x}) > 0\} = \mathcal{D}_{L_{2d}} \cup \mathcal{D}'_{L_{2d}}.$$

Let

$$(L_{2d} \otimes \mathbb{C}) \setminus \{0\} \supset \mathcal{D}^{\bullet}_{L_{2d}} = \text{affine cone over } \mathcal{D}_{L_{2d}} \subset \mathbb{P}(L_{2d} \otimes \mathbb{C})$$

and

$$\mathrm{O}^{+}(L_{2d}) = \{g \in \mathrm{O}(L_{2d});\ g(\mathcal{D}_{L_{2d}}) = \mathcal{D}_{L_{2d}}\}.$$

Automorphic forms (2)

Definition: Let $\Gamma \subset \mathrm{O}^+(L_{2d})$ be a group of finite index. A *modular form* (*automorphic form*) of *weight* k with respect to Γ and with a character χ is a holomorphic function

$$F\colon \mathcal{D}_{L_{2d}}^{\bullet} \longrightarrow \mathbb{C}$$

such that

(1) $F(tZ) = t^{-k}F(Z) \quad (t \in \mathbb{C}^*)$
(2) $F(\gamma Z) = \chi(\gamma)F(Z) \quad (\gamma \in \Gamma)$.

Automorphic forms (3)

Fact: $S_{19k}(\widetilde{\mathrm{O}}^+(L_{2d}), \det^k) \ni F \rightsquigarrow F\cdot \underset{\substack{\uparrow \\ \text{suitable volume form}}}{(\mathrm{d}z)^k} \in H^0(\mathcal{F}_{2d}^\circ, \omega_{\mathcal{F}_{2d}^\circ}^{\otimes k})$

where

$$\mathcal{F}_{2d}^\circ = \mathcal{F}_{2d} \setminus \{\text{fixed points of } \widetilde{\mathrm{O}}^+(L_{2d})\}.$$

Question: When can one extend these forms to pluricanonical forms on a smooth projective model $\overline{\mathcal{F}}_{2d}$ of $\mathcal{F}_{2d}$?

Obstructions:

(1) *elliptic* obstructions (from singularities of $\overline{\mathcal{F}}_{2d}$)

(2) *reflective* obstructions (from quasi-reflections in $\widetilde{\mathrm{O}}^+(L_{2d})$)

(3) *parabolic* obstructions ($\mathrm{d}z$ picks up poles along the boundary).

We will concentrate on (3) in the sequel.

Automorphic forms (4)

Low weight cusp form trick

Assume we have a cusp form F_a of *low* weight a (i.e. $a < 19 = \dim \Omega_{L_{2d}}$), i.e.

$$F_a \in S_a(\widetilde{\mathrm{O}}^+(L_{2d}), \det^{\varepsilon}); \quad \varepsilon = 0 \text{ or } 1.$$

Let k be even and consider

$$F \in F_a^k \cdot S_{k\underbrace{(19-a)}_{>0}}(\widetilde{\mathrm{O}}^+(L_{2d})) \subset S_{19k}(\widetilde{\mathrm{O}}^+(L_{2d})) = S_{19k}(\widetilde{\mathrm{O}}^+(L_{2d}), \det^k).$$

Then

$$\omega_F = F \cdot (\mathrm{d}z)^k$$

has no poles along the boundary.

Question: Construction of low weight cusp forms?

The Borcherds form (1)

Let

$$L_{2,26} = 2U \oplus 3E_8(-1) = 2U \oplus \Lambda \quad (\Lambda = \text{ Leech lattice}).$$

Then Borcherds has constructed a modular form

$$\Phi_{12} \colon \mathcal{D}^{\bullet}_{L_{2,26}} \longrightarrow \mathbb{C}$$

of weight 12.

Idea: "Restrict" this form to $\mathcal{D}^{\bullet}_{L_{2d}}$.

Choose $l \in E_8(-1), l^2 = -2d < 0$. This defines embeddings

$$L_{2d} = 2U \oplus E_8(-1) \oplus \langle -2d \rangle \hookrightarrow L_{2,26}$$

resp.

$$\Omega_{L_{2d}} \subset \Omega_{L_{2,26}}.$$

The Borcherds form (2)

Let

$$R_l := \{r \in E_8(-1);\ r^2 = -2,\ (r,l) = 0\}$$

$$N_l := |R_l|.$$

We define the "quasi-pullback" of Φ_{12} to $\mathcal{D}_{L_{2d}}^{\bullet}$ by

$$F_l := \left. \frac{\Phi_{12}(z)}{\prod\limits_{\pm r \in R_l} (r,z)} \right|_{\mathcal{D}_{L_{2d}}^{\bullet}}.$$

Proposition: If $N_l > 0$ then

$$0 \neq F_l \in S_{12+\frac{N_l}{2}}(\widetilde{\mathrm{O}}^+(L_{2d}), \det).$$

This gives us low weight cusp forms provided

$$2 \leq N_l \leq 12.$$

The Borchards form (3)

Question: When can we find $l \in E_8(-1)$ with $l^2 = -2d$ which is orthogonal to at least 2 and at most 12 roots?

Proposition: Such l exists if

$$4N_{E_7}(2d) > 28N_{E_6}(2d) + 63N_{D_6}(2d) \qquad (*)$$

where $N_L(2d)$ is the number of representations of the integer $2d$ in the lattice L.

- One can show that $(*)$ holds for $d > 143$.
- The remaining small d in the theorem can be done by a computer search.

Further outlook (1)

This technique can also be applied to other moduli problems.

Definition: An *irreducible symplectic manifold* is a compact complex manifold X with the following properties:

(1) X is Kähler

(2) There exists a (up to scalar) unique non-degenerate 2-form $\omega_X \in H^0(X, \Omega_X^2) \quad (\Rightarrow \omega_X = \mathcal{O}_X)$

(3) X is simply connected.

Examples

(1) $\dim X = 2$: $X = S = K3$ surface

(2) X is a deformation of $\mathrm{Hilb}^n S$

(3) X is a deformation of a generalized Kummer variety

(4) O'Grady's sporadic examples in dimension 6 and 10.

Further outlook (2)

In this case Torelli does not hold (in the strict form). However one still has a finite dominant map

$$\begin{array}{ccc} \mathcal{M} & \longrightarrow & \Gamma \setminus \mathcal{D}_L \\ \uparrow & & \end{array}$$

component of a moduli space
of polarized irreducible
symplectic manifolds

where L is a lattice of signature $(2, n)$ and Γ is a suitable group.

General type results have been obtained in the case

$X \sim_{\text{def}} \text{Hilb}^2 S$, split polarization (Gritsenko, H., Sankaran).

Work in progress

- $X \sim_{\text{def}} \text{Hilb}^n S$, general polarization
- O'Grady's 10-dimensional sporadic case.