Measuring Scale Before Simplification

Herbert Edelsbrunner

BMS

Measuring Scale Before Simplification

I. Functions

II. Curves

III. Somites

IV. Motion

I. Functions

I.1 PERSISTENCE

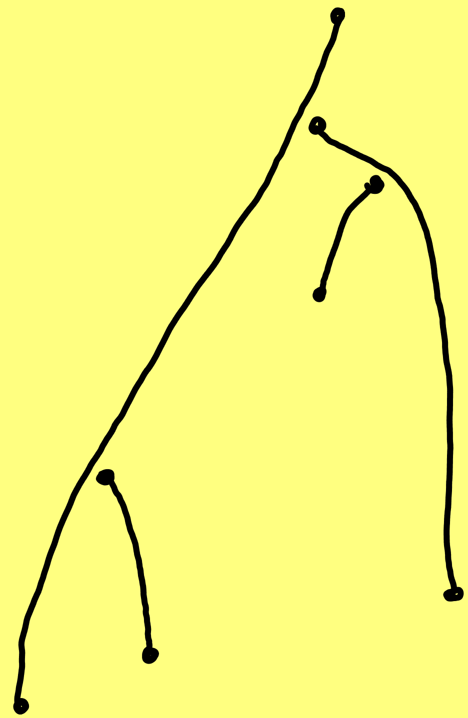
function

$f : \mathbb{S}^1 \to \mathbb{R}$

I.1 Persistence

function

$f : \mathbb{S}^1 \to \mathbb{R}$



merge tree

I.1 Persistence

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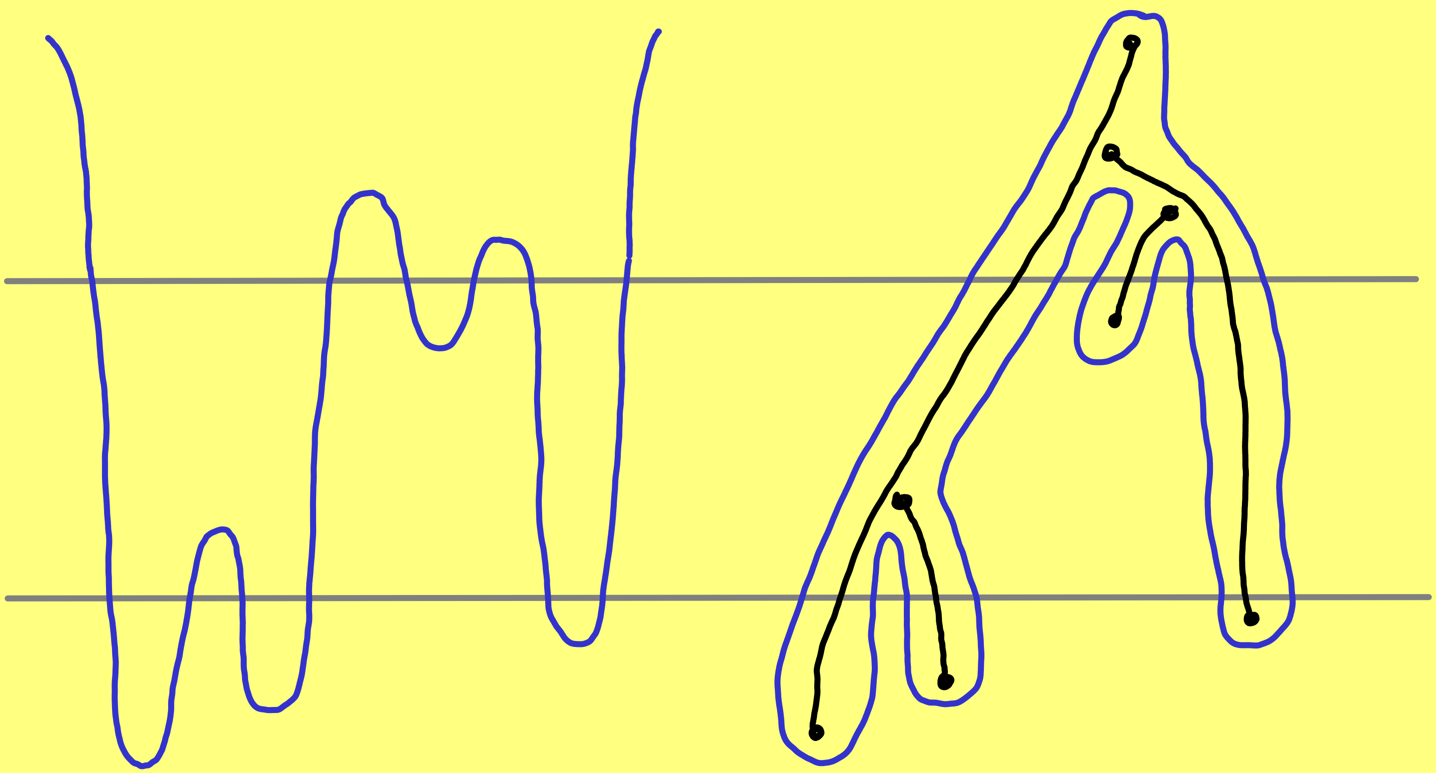

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merge tree

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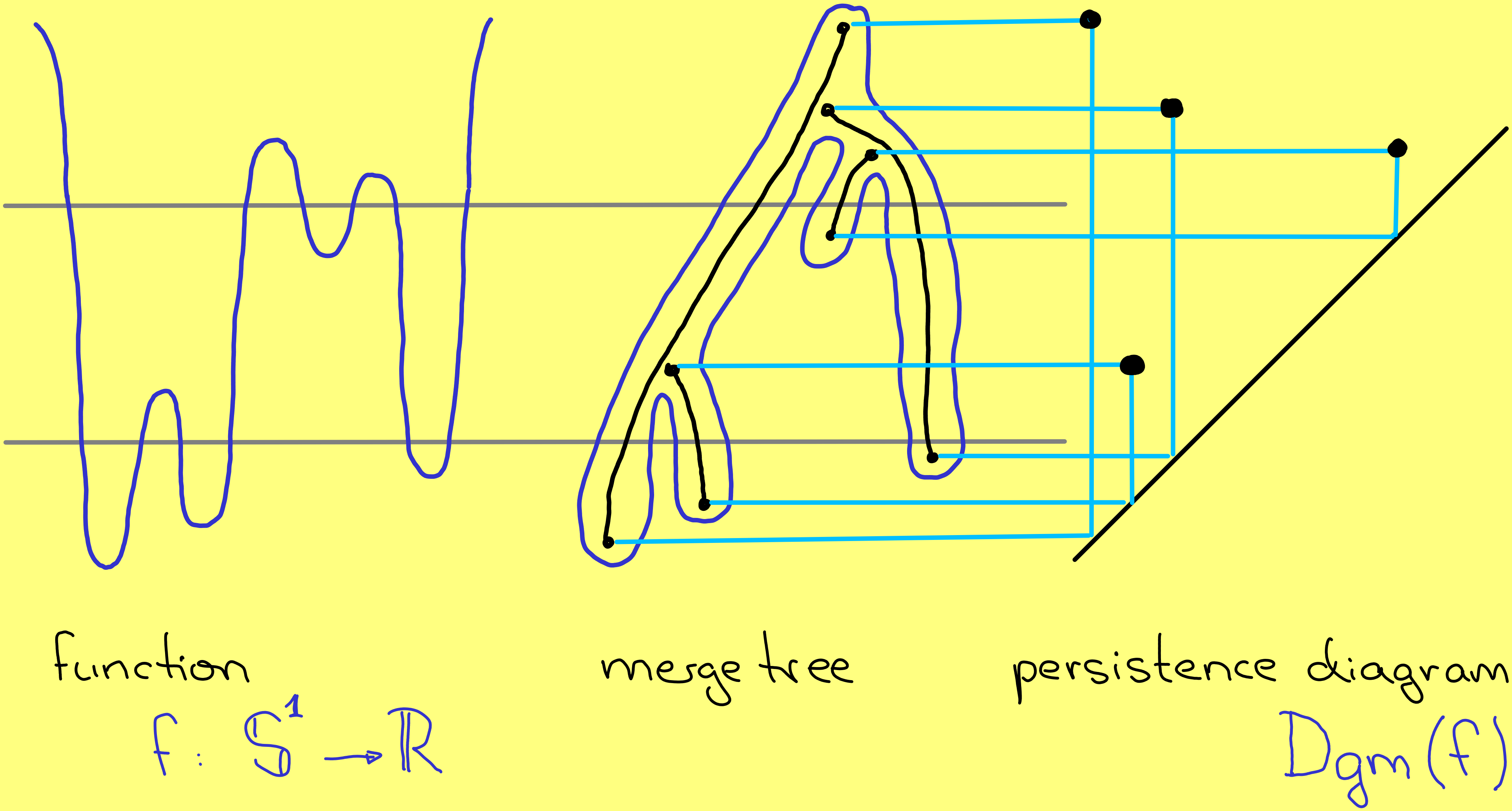

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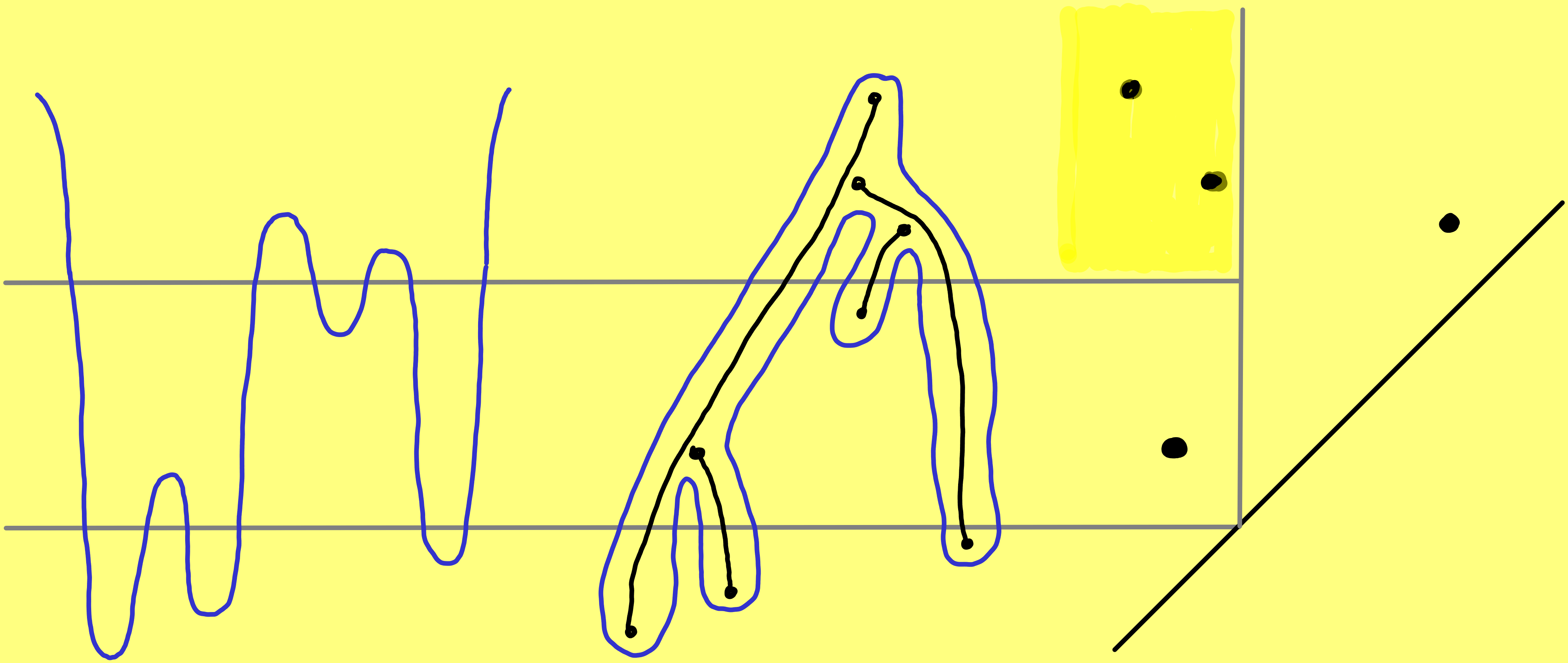

function $f: \mathbb{S}^1 \to \mathbb{R}$

merge tree

persistence diagram $\mathrm{Dgm}(f)$

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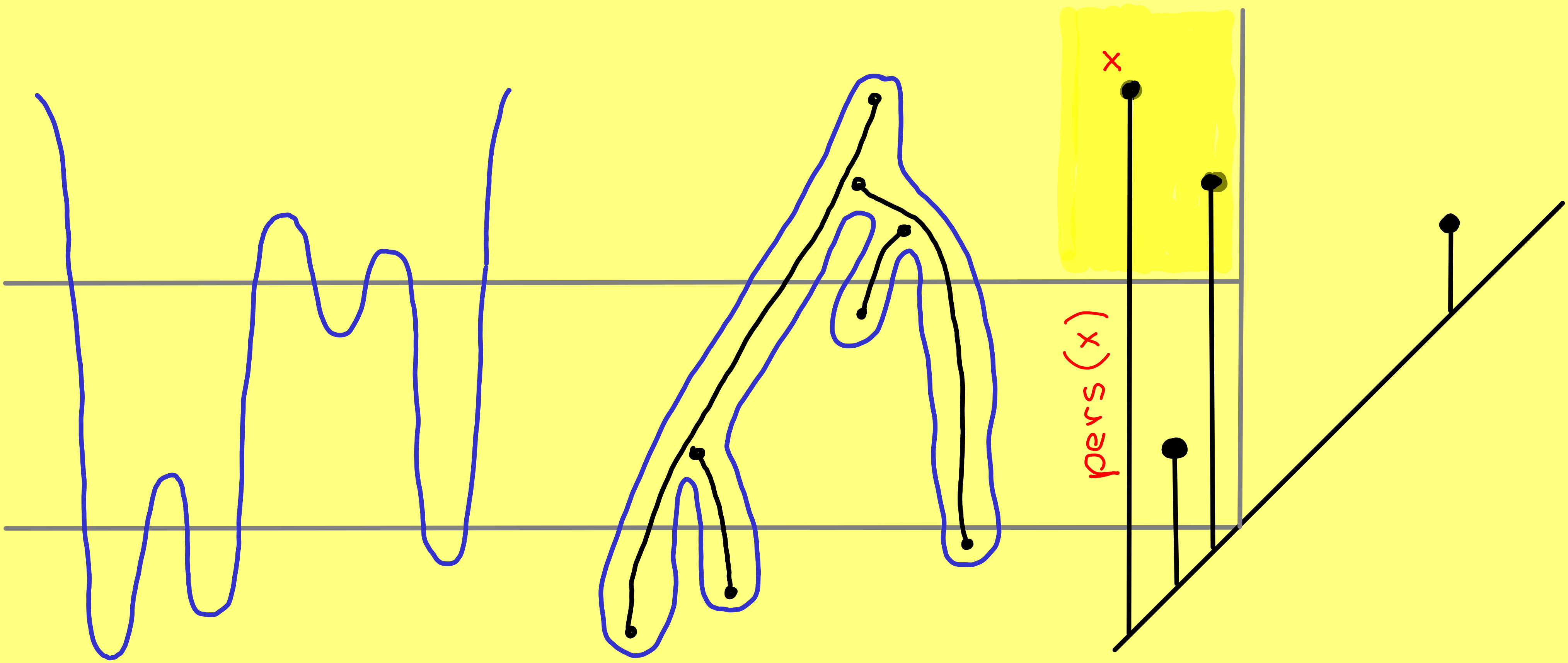


function

$f : \mathbb{S}^1 \to \mathbb{R}$

merge tree

persistence diagram

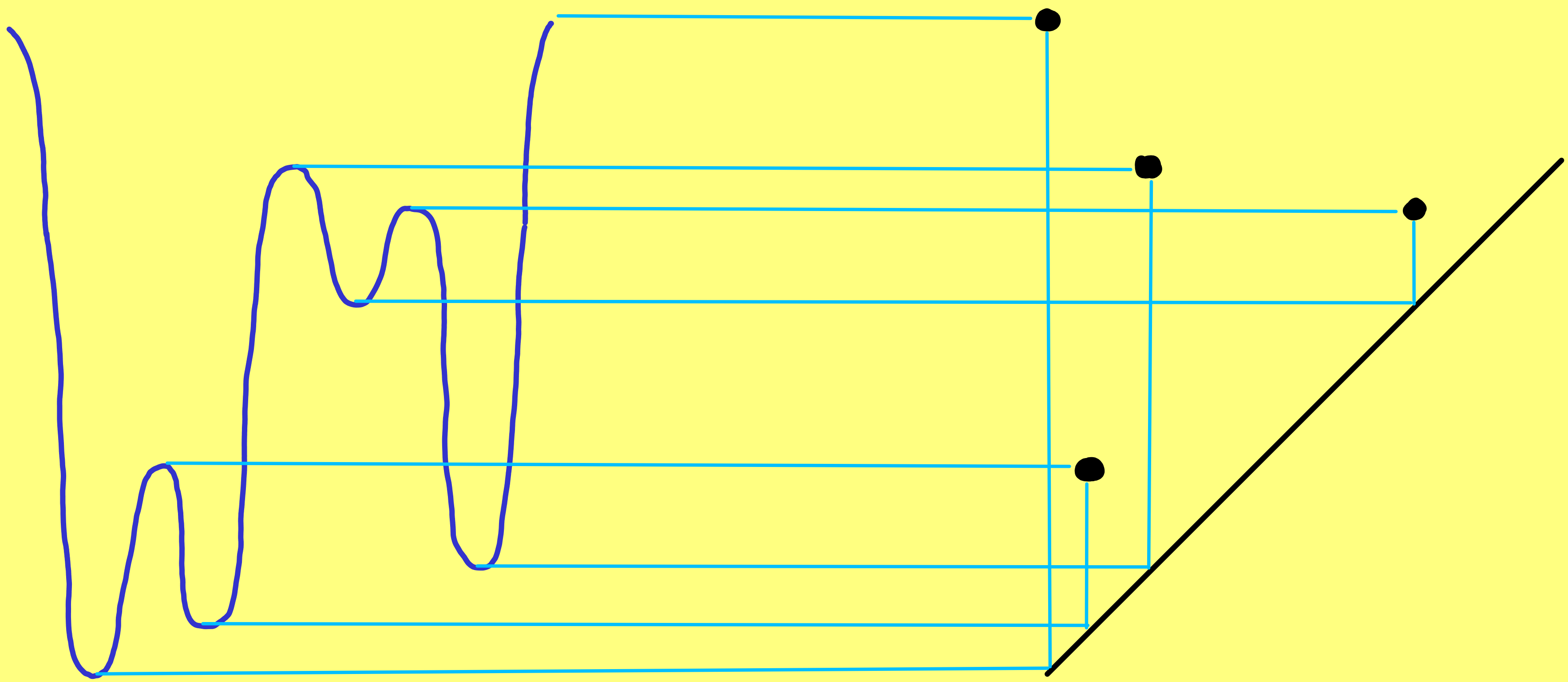
$\mathrm{Dgm}(f)$

I.2 L_∞ - STABILITY

THM. For tame functions $f, g : \mathbb{X} \to \mathbb{R}$ the bottleneck distance between their diagrams is bounded by the max-difference between the functions,

$$d_B(\mathrm{Dgm}(f), \mathrm{Dgm}(g)) \leq \|f - g\|_\infty .$$

I.2 L_∞-STABILITY



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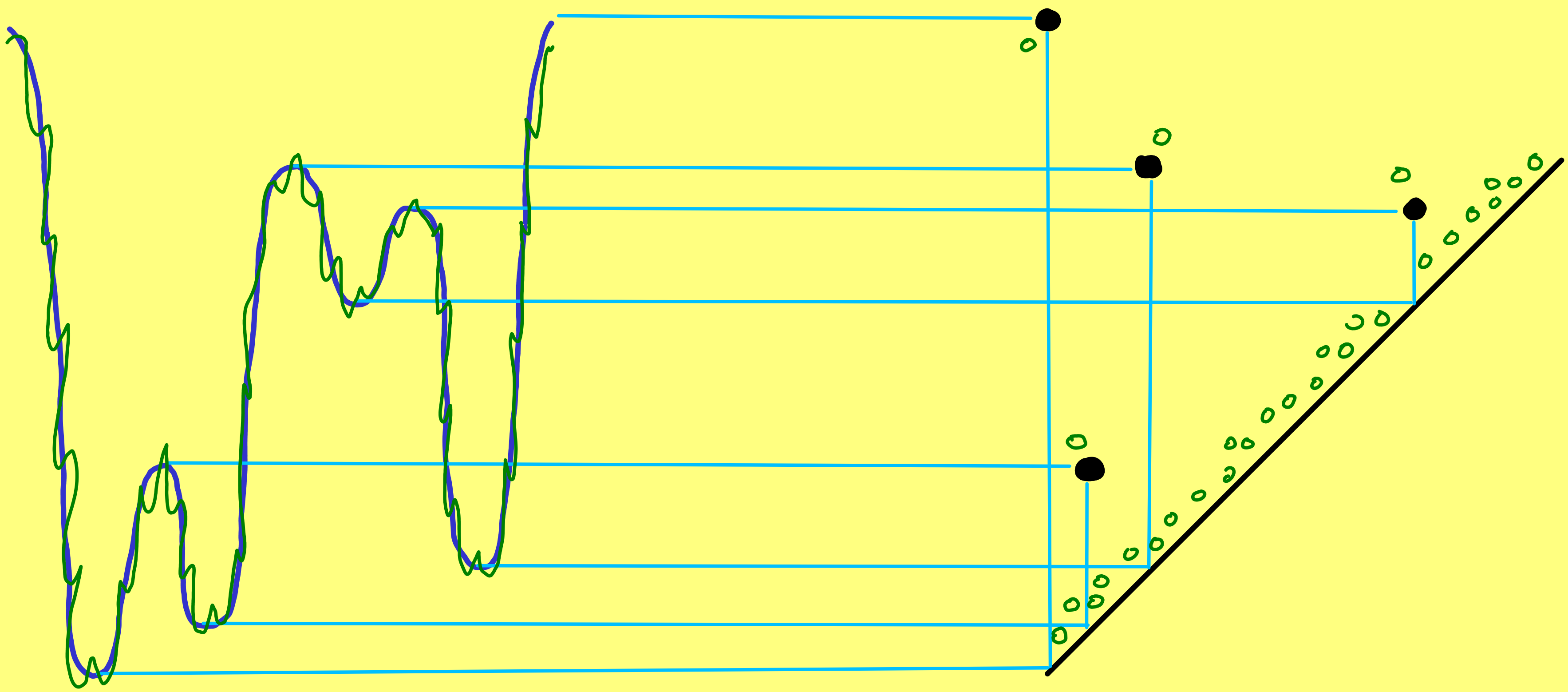

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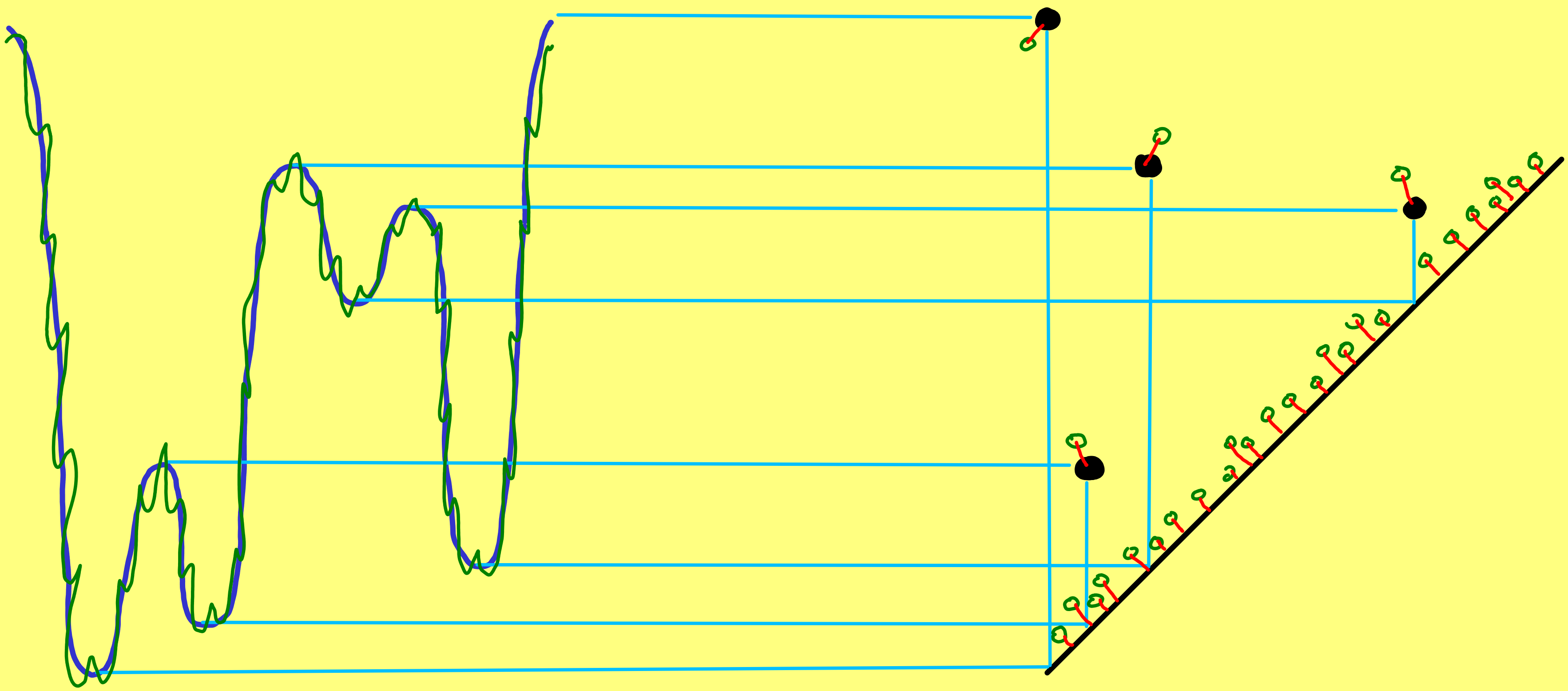


$$d_B(\mathrm{Dgm}(f), \mathrm{Dgm}(g)) \leq \|f-g\|_\infty.$$

I.3 L_p - STABILITY

DEF. $f : \mathbb{X} \to \mathbb{R}$ has Lipschitz constant C if $f(x) - f(y) \leq C \|x-y\|$.

I.3 L_p-STABILITY

DEF. $f : \mathbb{X} \to \mathbb{R}$ has Lipschitz constant C if $f(x) - f(y) \leq C \|x-y\|$.

DEF. The p-th Wasserstein distance between the diagrams of f and g is

$$W_p(f,g) = \inf_{\gamma : Dgm(f) \to Dgm(g)} \sum_{x \in Dgm(f)} \|x - \gamma(x)\|_1^p .$$

bijection

I.3 L_p - STABILITY

THM. Let $\mathbb{X}$ be a compact metric space and $f, g : \mathbb{X} \to \mathbb{R}$ two Lipschitz functions. Then there exist constants k and C that depend on $\mathbb{X}$ and the Lipschitz constants of f and g s.t.

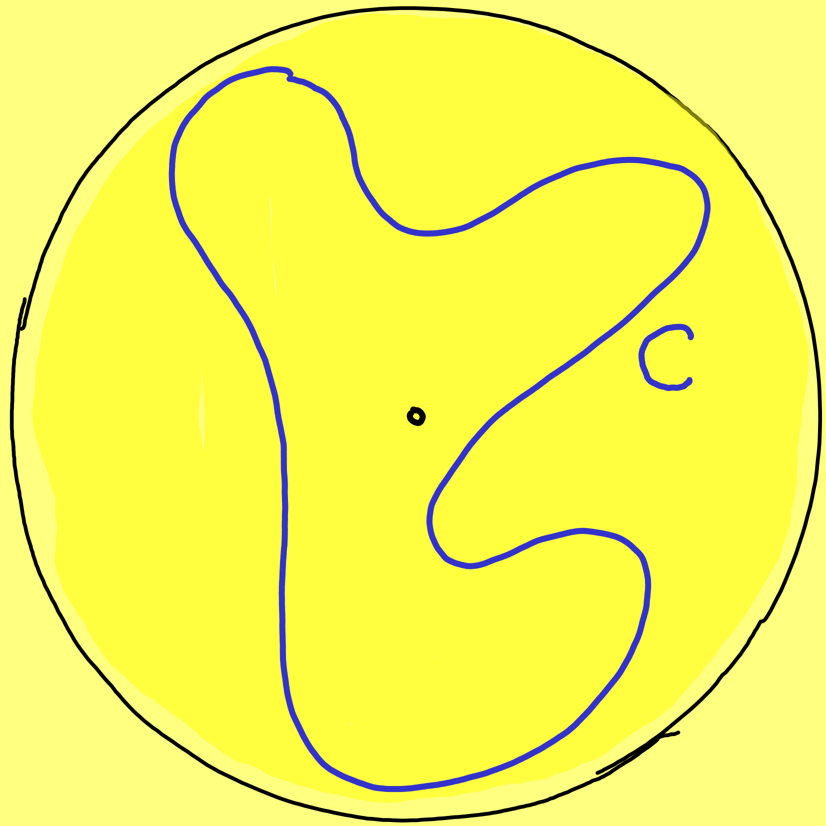
$$W_p(f, g) \leq C \|f - g\|_\infty^{p-k}$$

for every $p \geq k$.

II. CURVES

II.1 Fary's Theorem

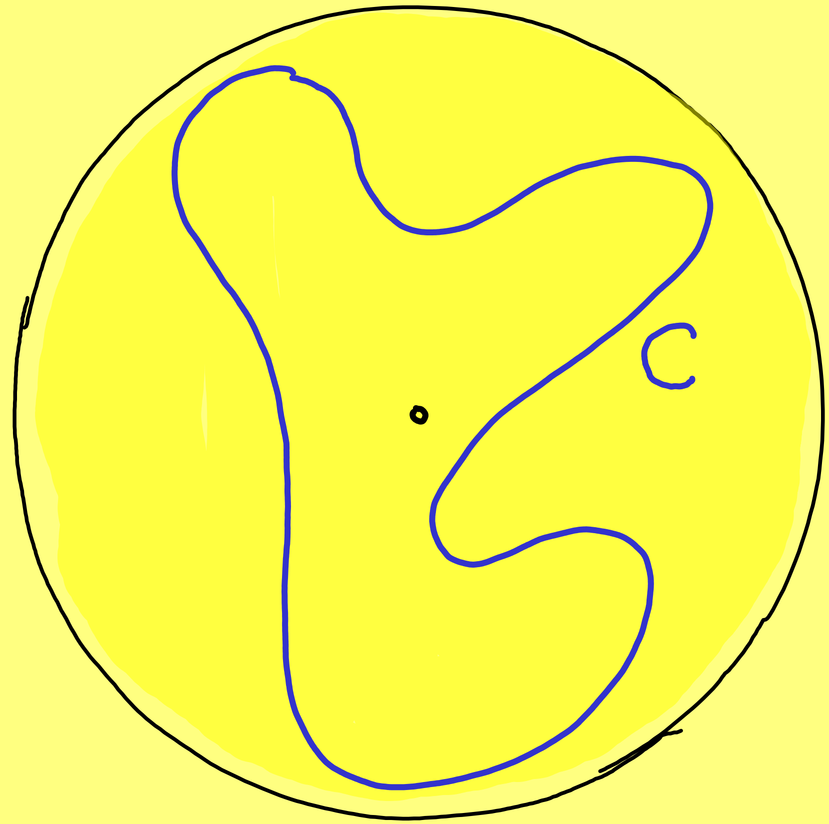
C = smooth curve inside unit disk



II.1 Fary's Theorem

C = smooth curve inside unit disk

$L(C)$ = length



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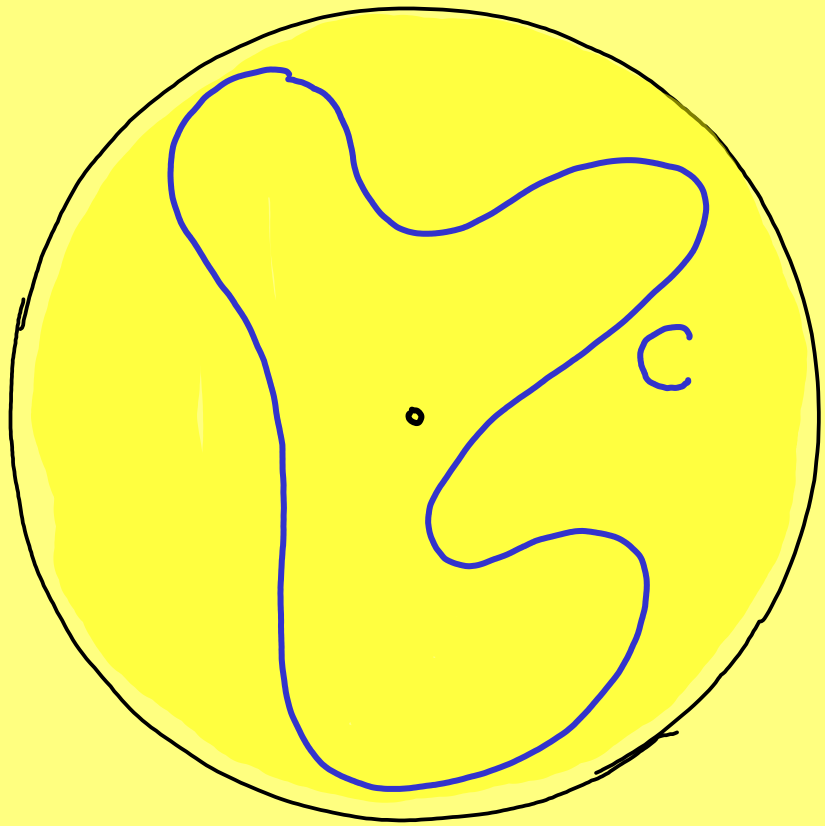

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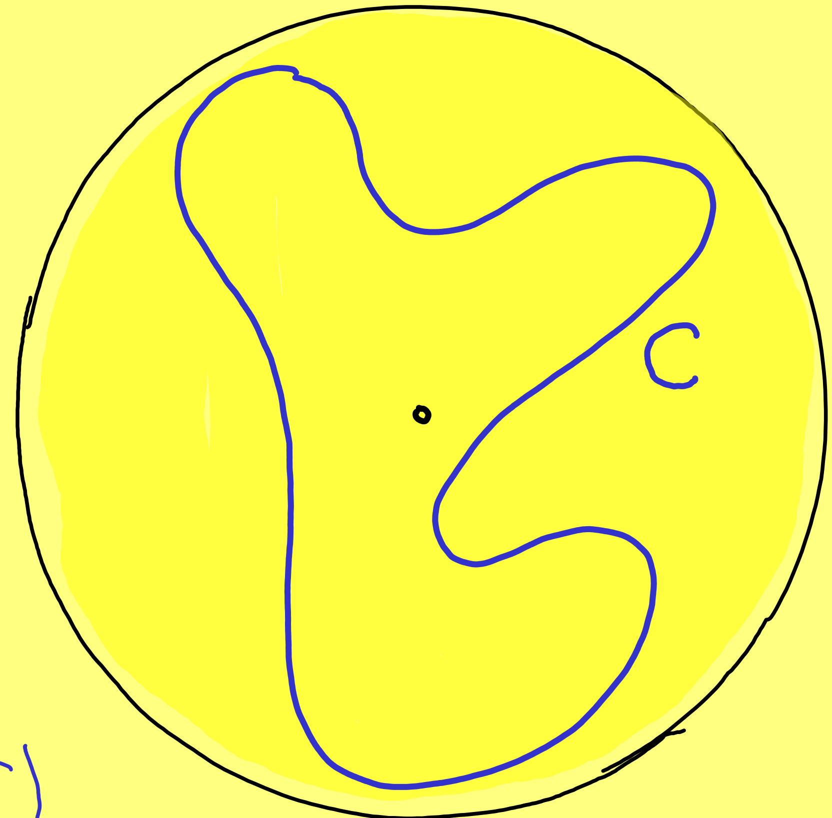
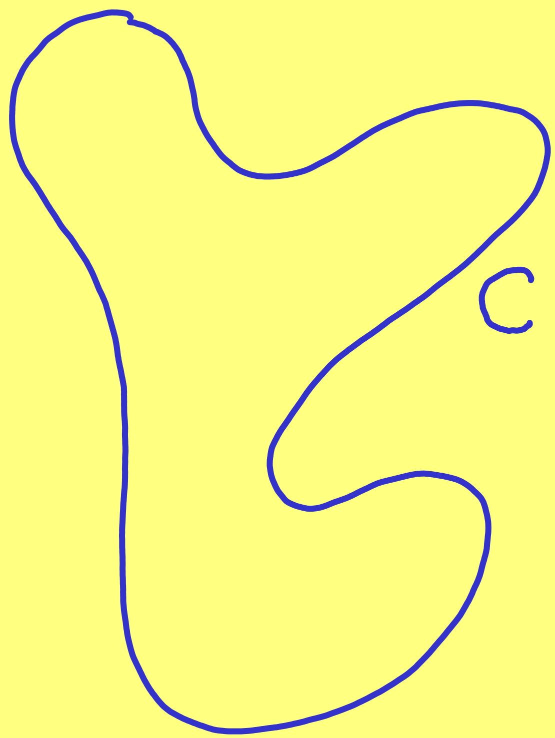


Fary: $L(C) \leq K(C)$.

II.2 Total Curvature



$$K(C) = \int_{x \in C} |\kappa(x)|\, dx$$

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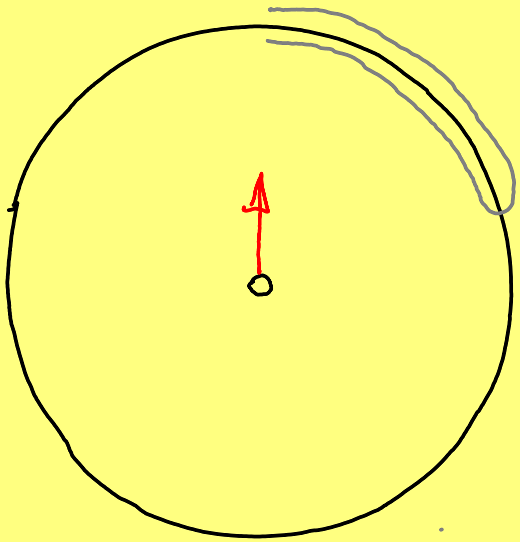

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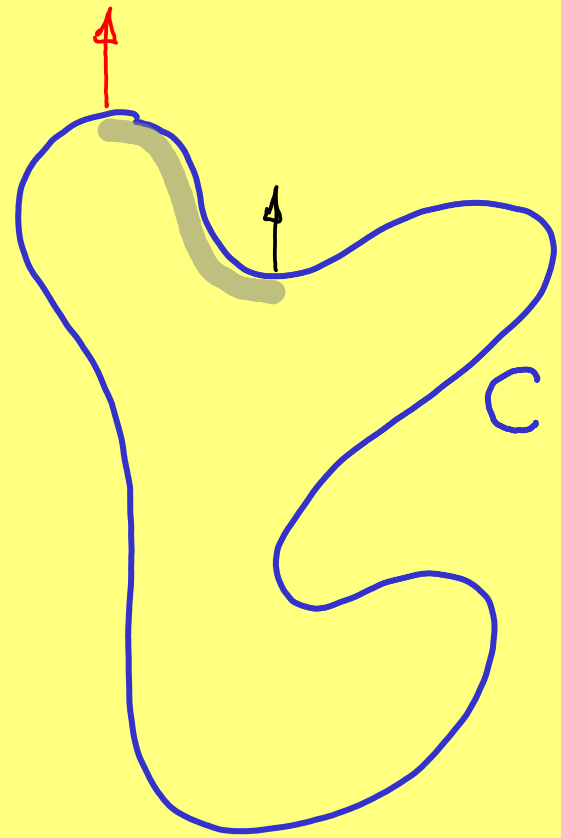

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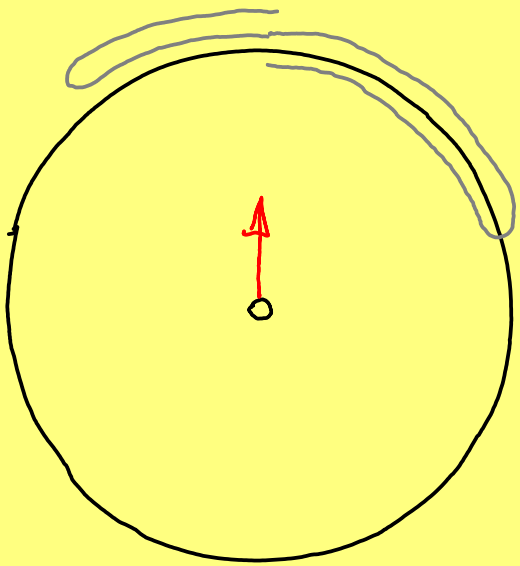

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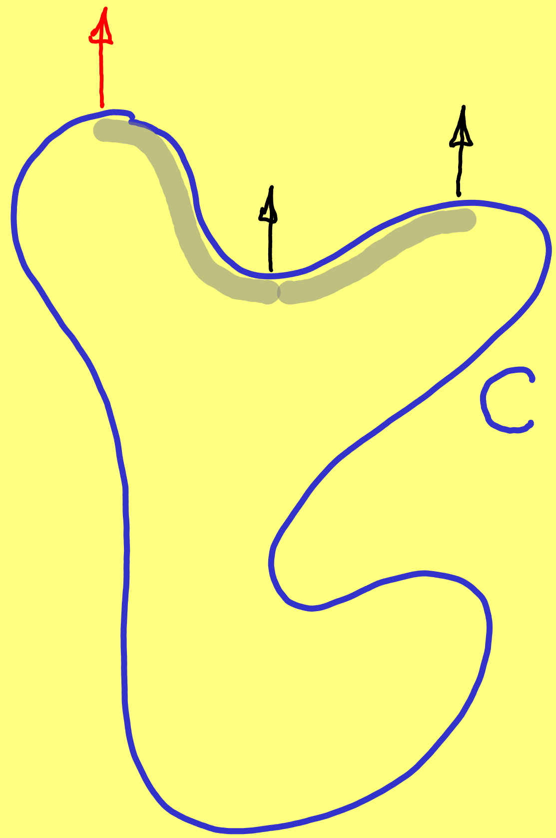

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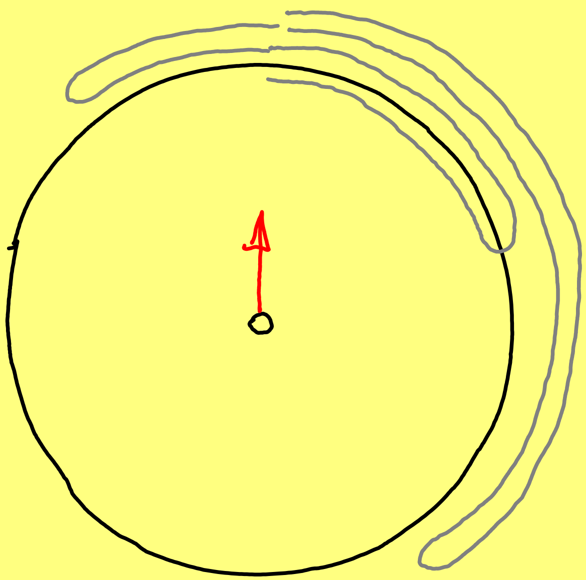

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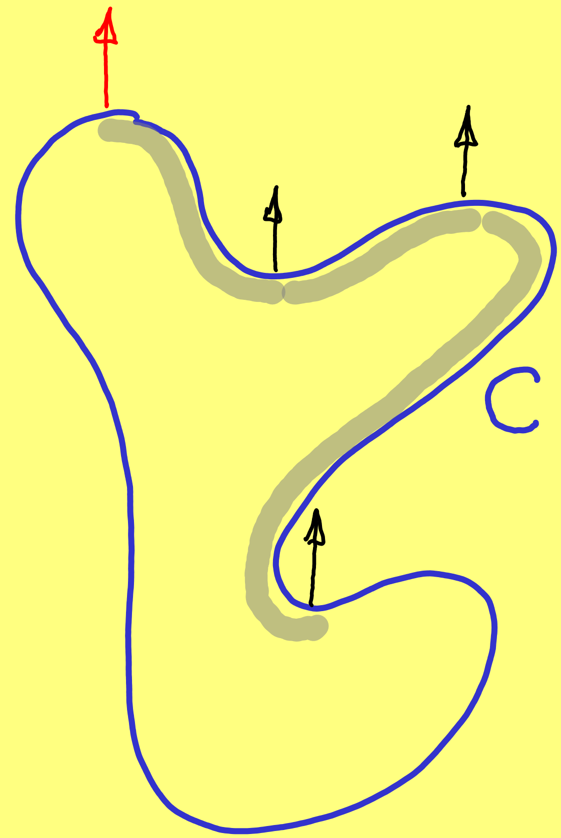

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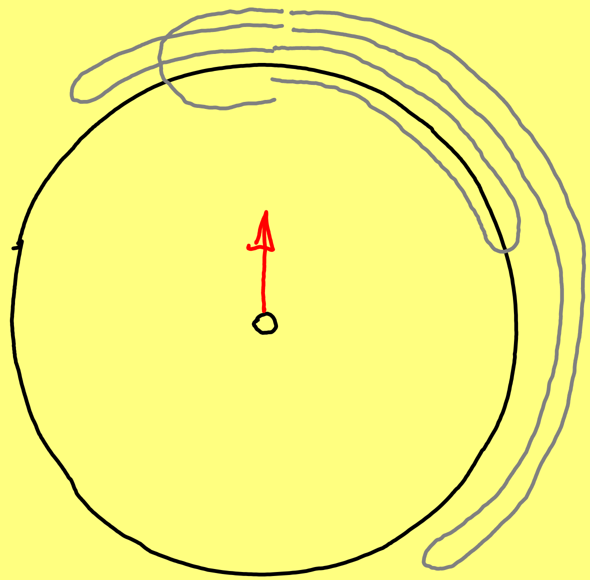

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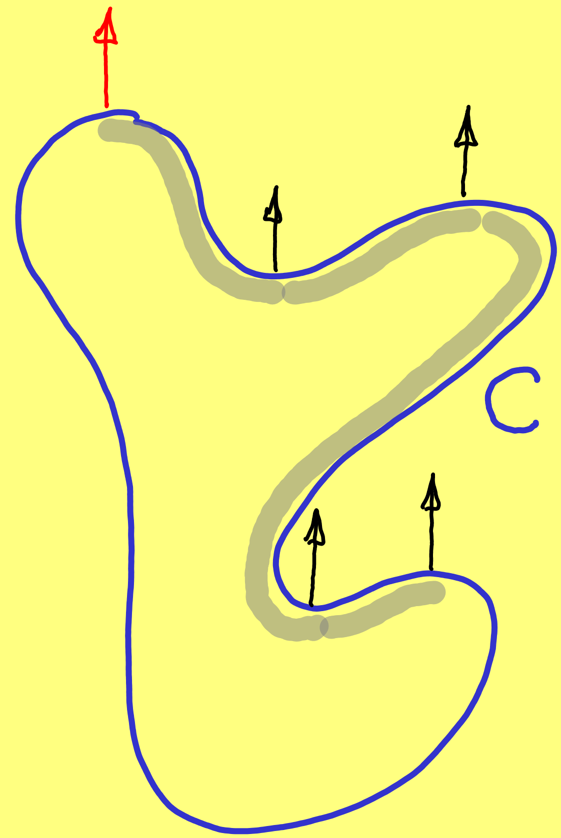

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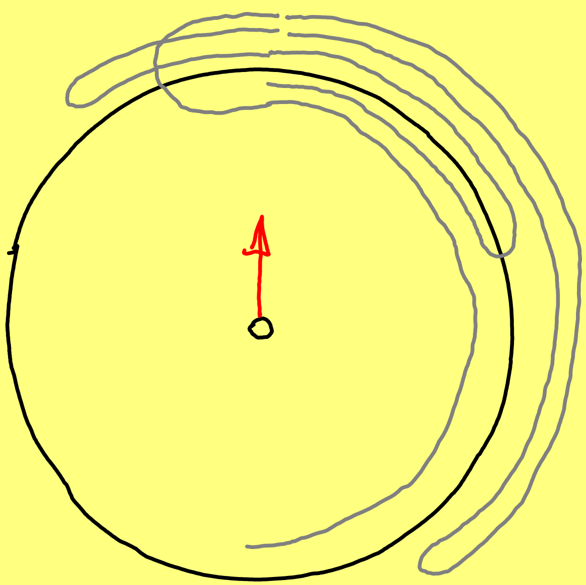

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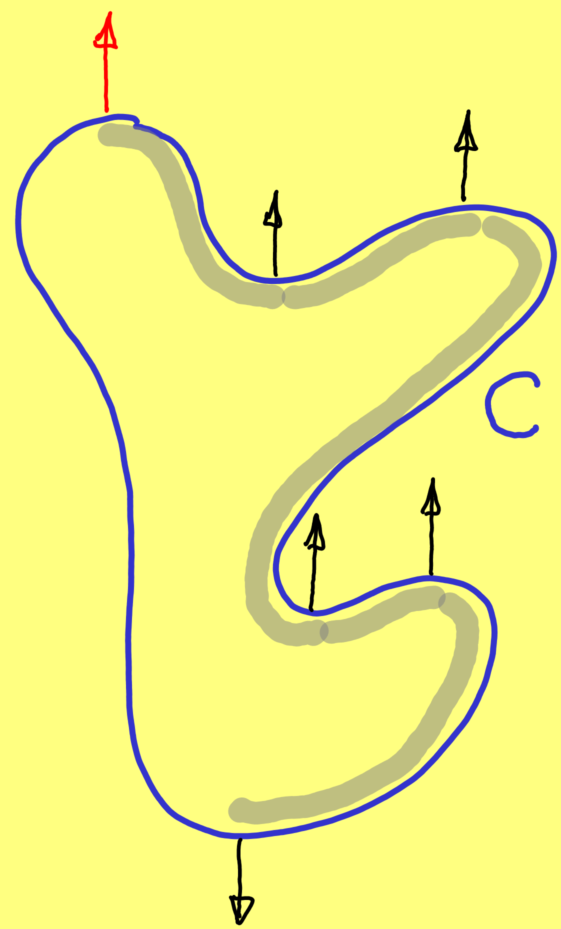

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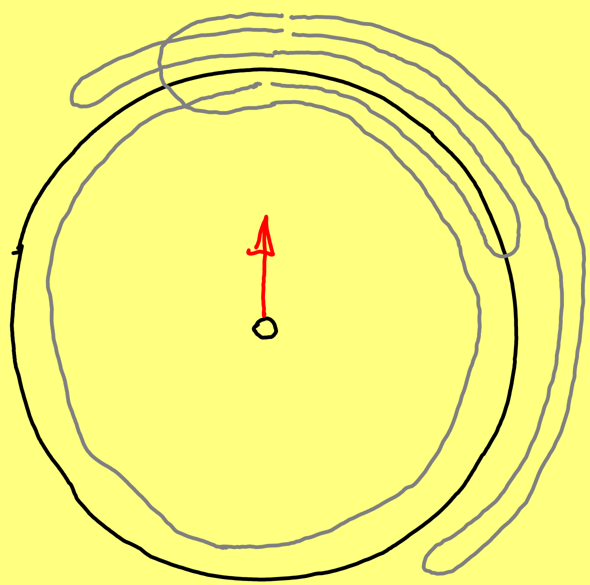

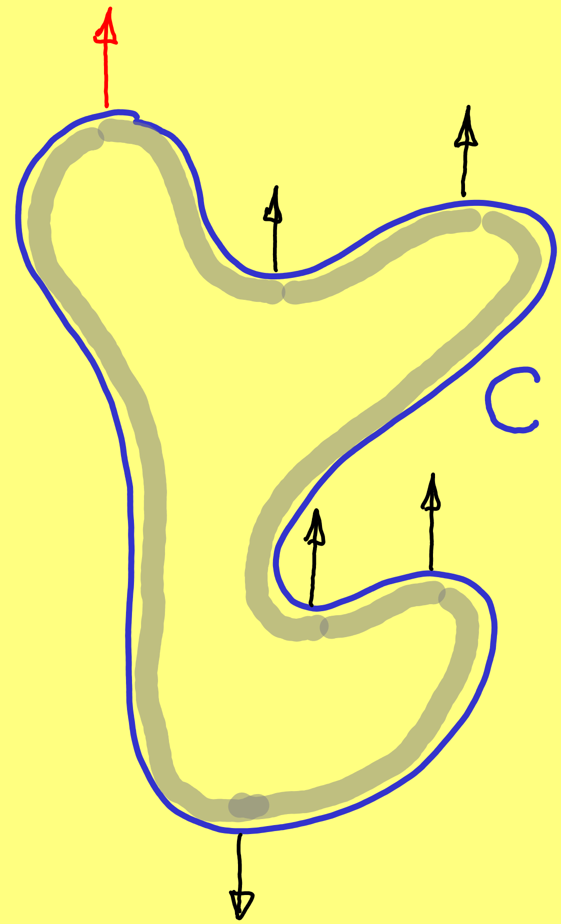

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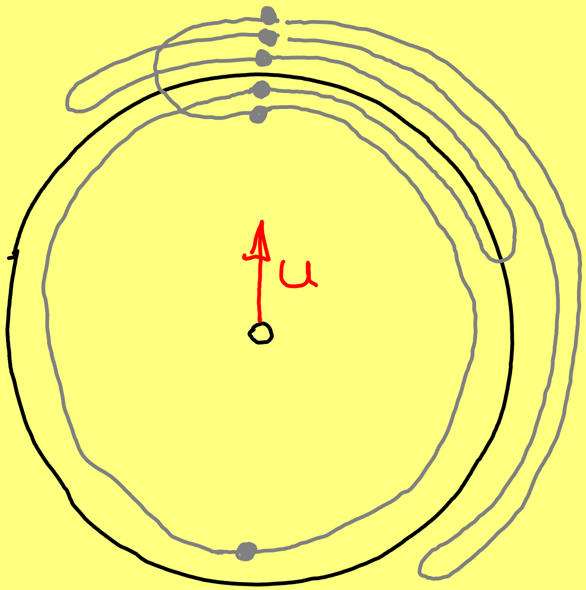
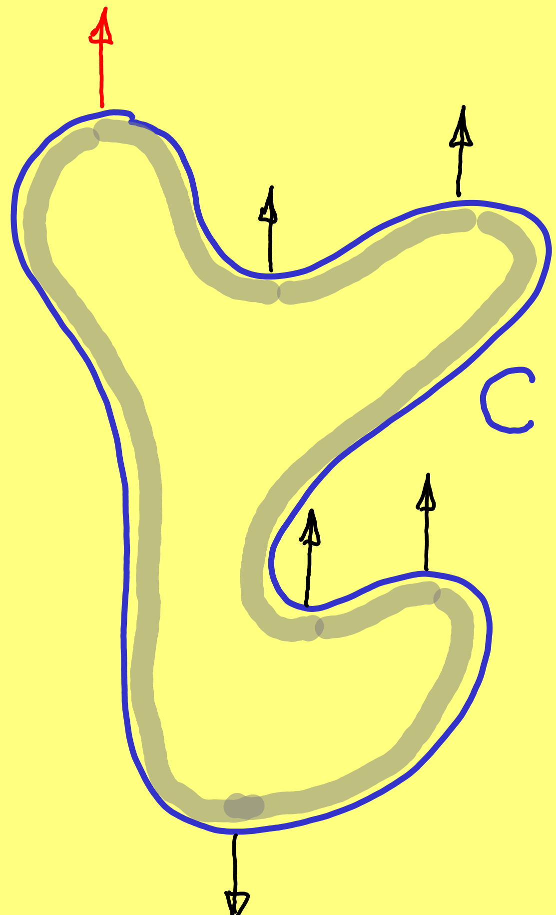


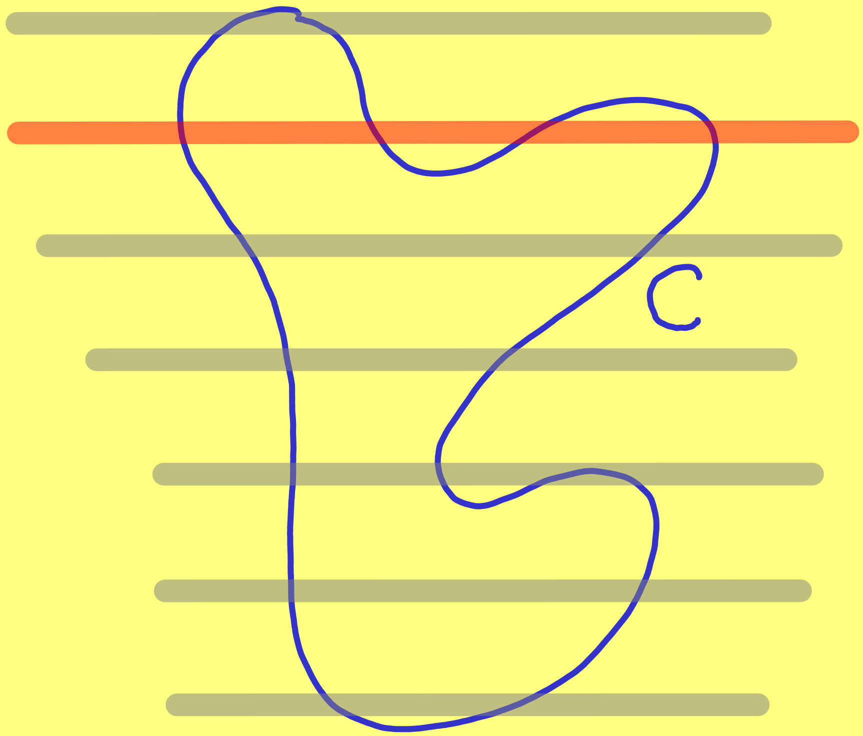


$$K(C) = \int_{x \in C} |\kappa(x)|\, dx$$

$$= \frac{1}{2} \int_{u \in S^1} (\#\,\text{crit. pts. of } f_u)\, du$$

$$= \int_{u \in S^1} |\mathrm{Dgm}(f_u)|\, du$$

II.3 LENGTH



$$L(C) = \frac{1}{2} \int_u \int_z \#\mathrm{Xing}(z)\, dz\, du$$

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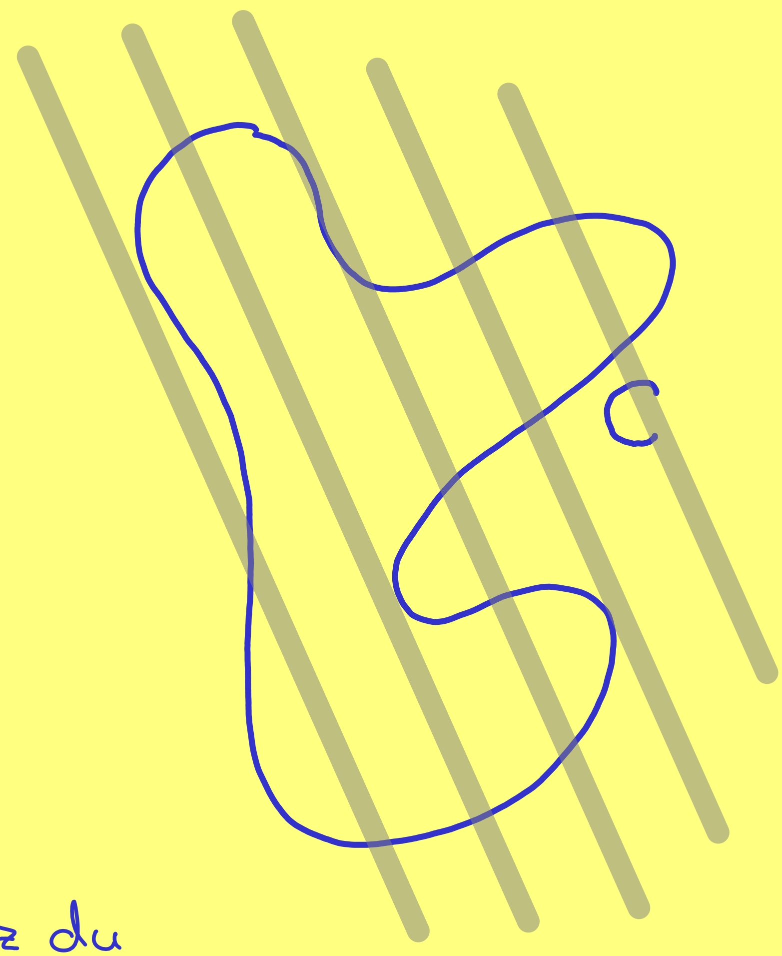

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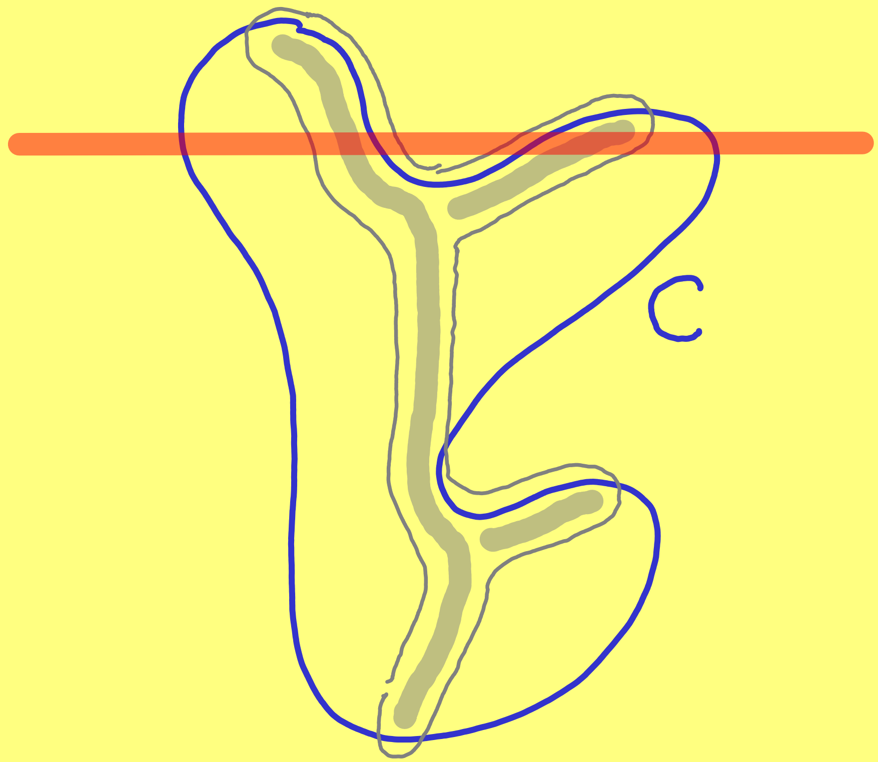

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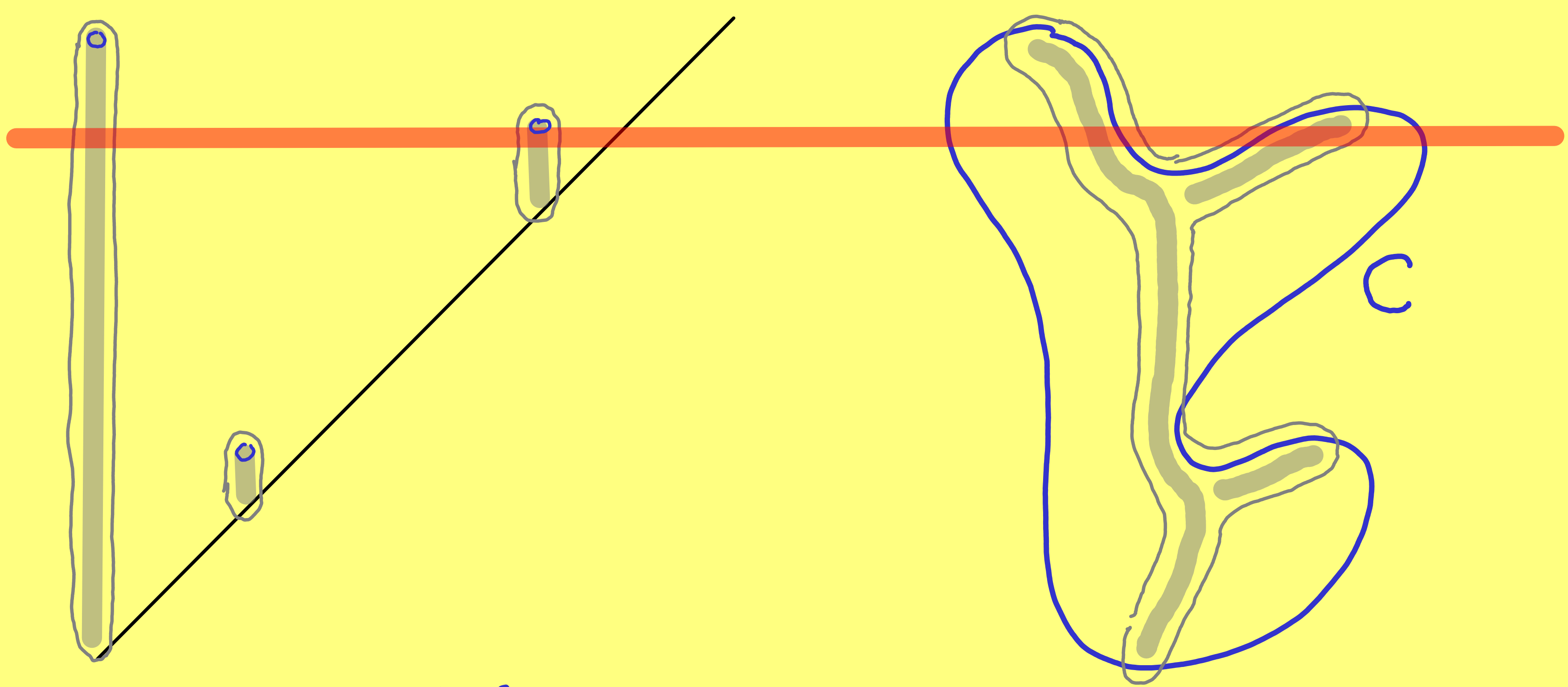

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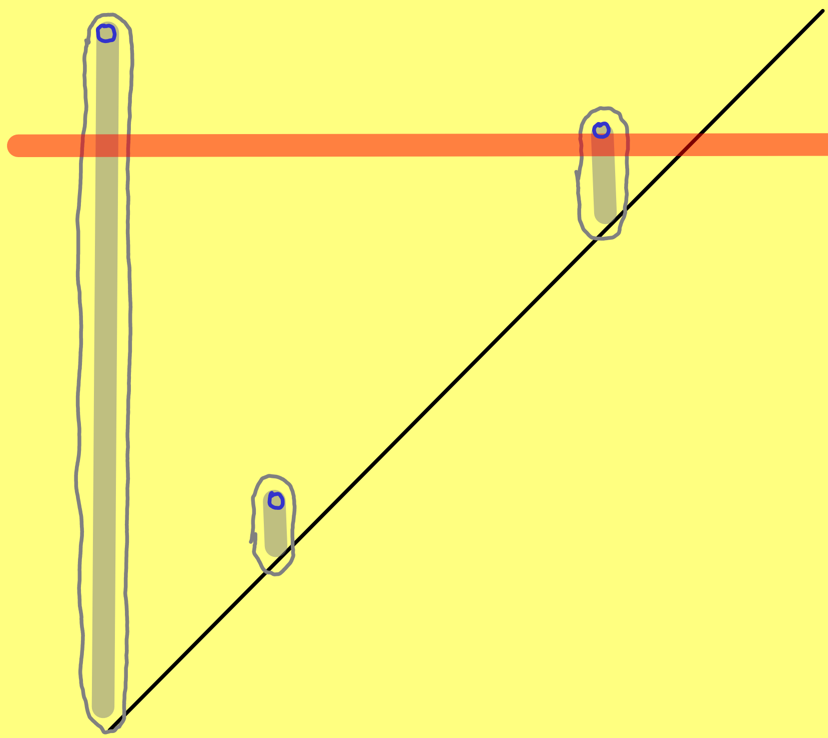
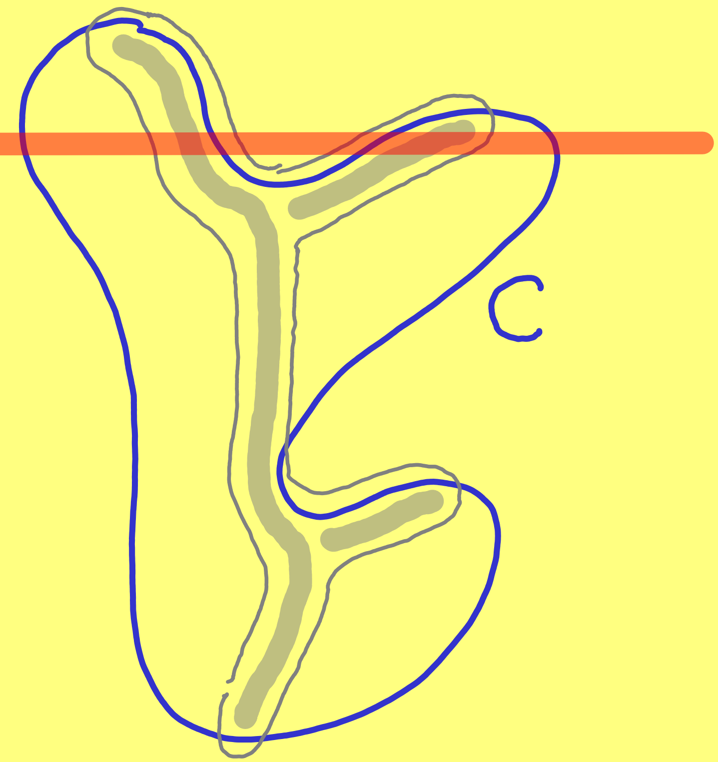
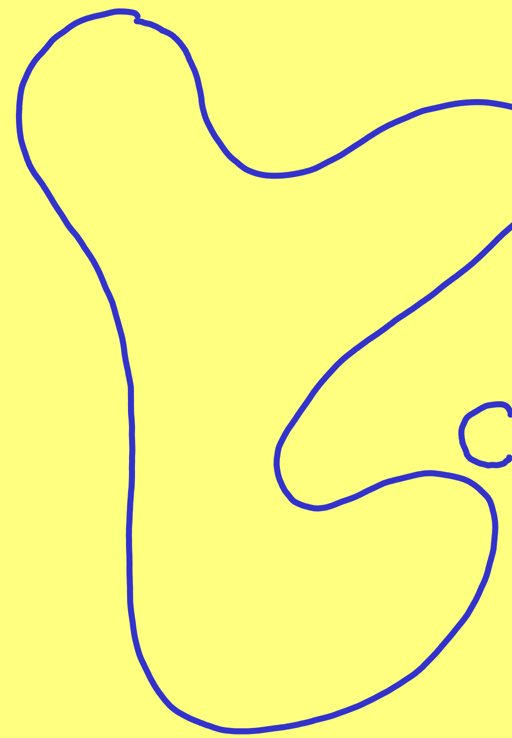




$$L(C) = \frac{1}{2} \int_u \int_z \# \mathrm{Xing}(z)\, dz\, du$$

$$= \int_u \sum_{x \in \mathrm{Dgm}(f_u)} \mathrm{pers}(x)\, du$$

II.4 Generalized Fary Theorem



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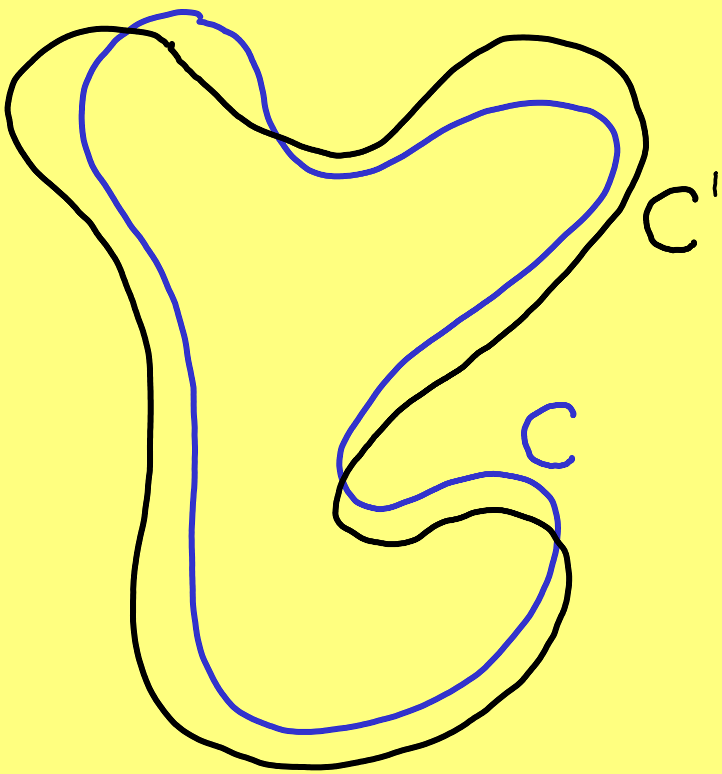

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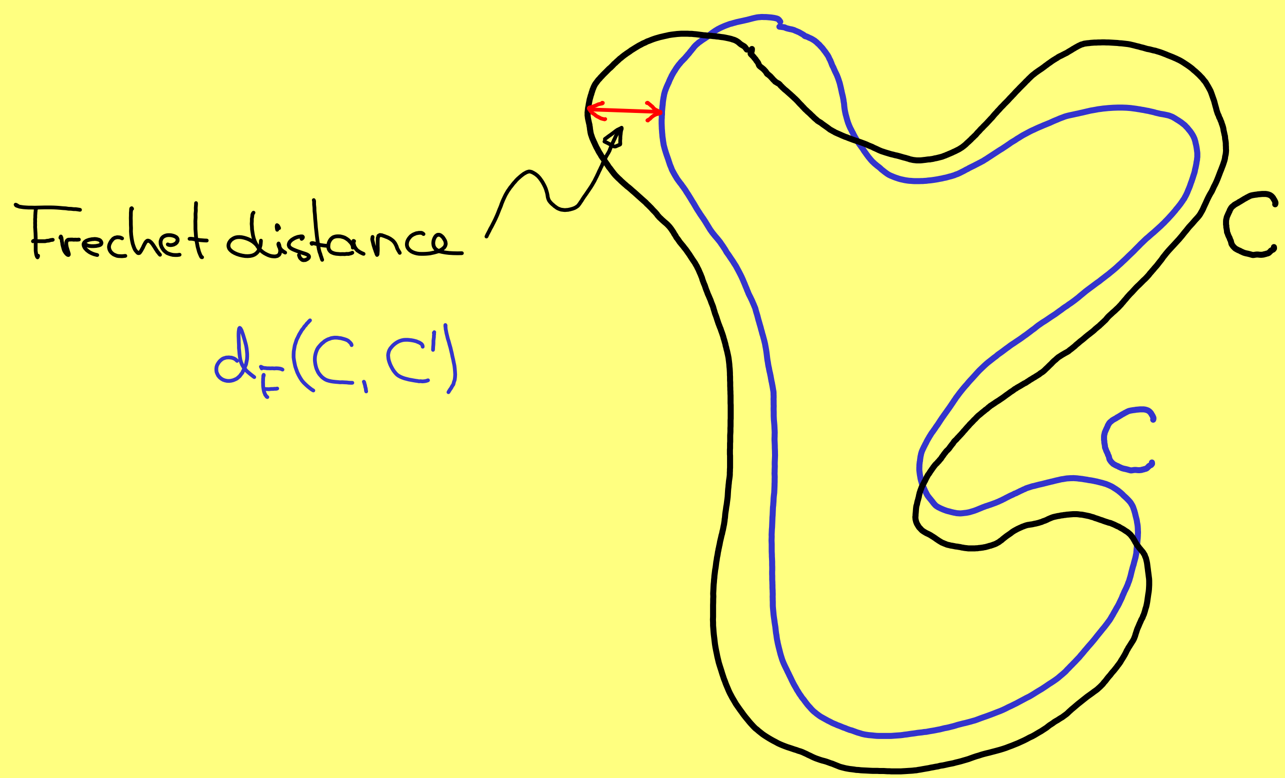

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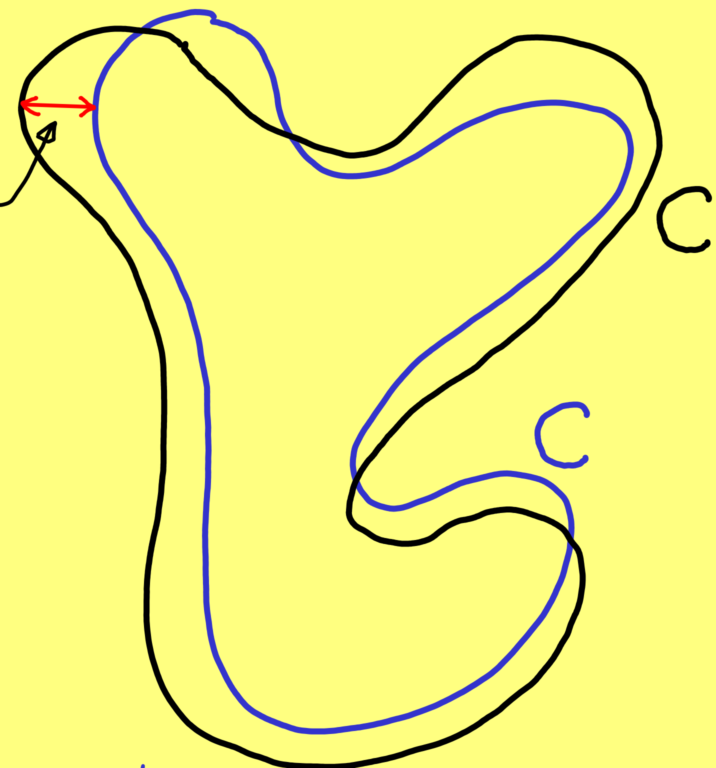
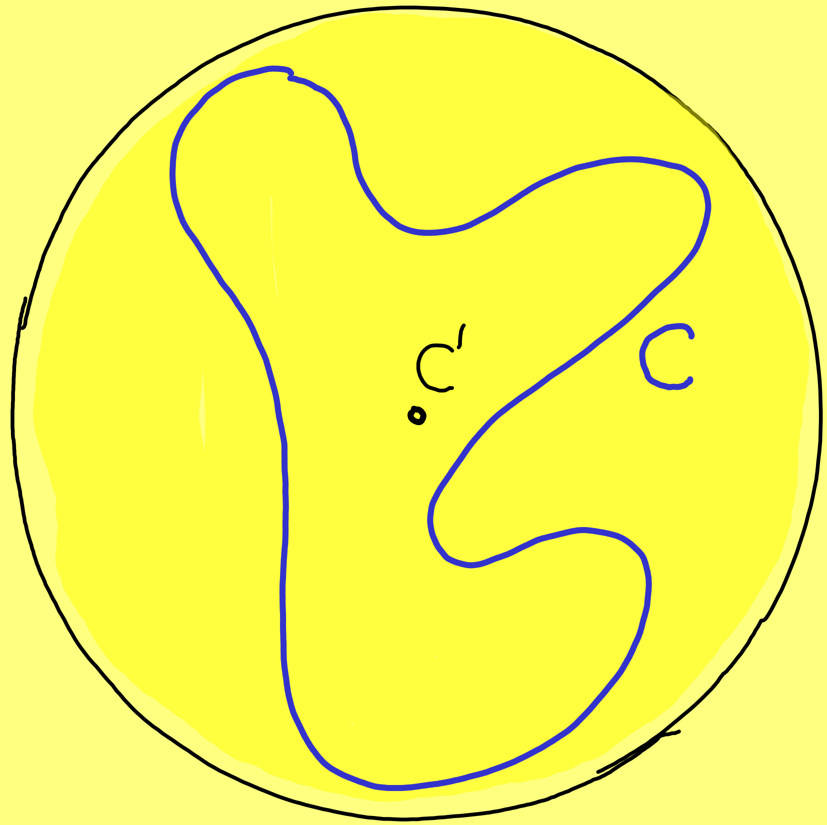


Frechet distance

$d_F(C, C')$

Thm. $|L(C) - L(C')| \leq (K(C) + K(C') - 2\pi)\, d_F(C, C').$

II.4 Generalized Fary Theorem



Thm. $|L(C) - L(C')| \leq (K(C) + K(C') - 2\pi)\, d_F(C, C')$.

III. Somites

work by Olivier Pourquié
Mary-Lee Dequéant

III.1 SEGMENTATION of vertebrate body plan

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adult mouse

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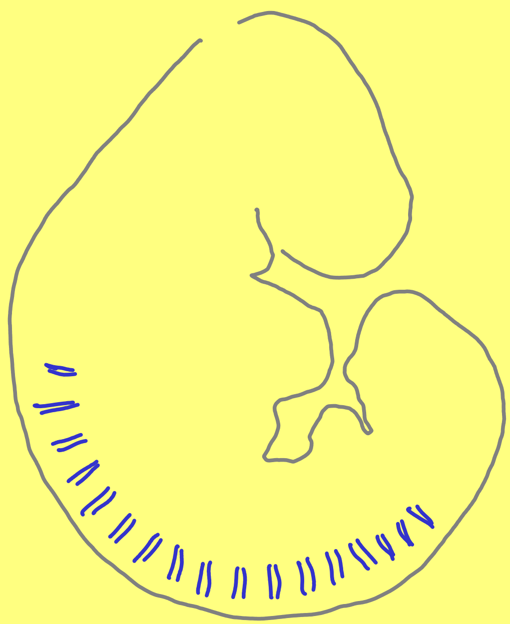
adult mouse

mouse embryo



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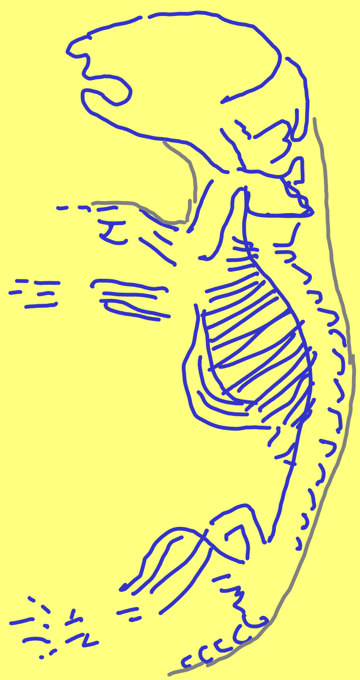
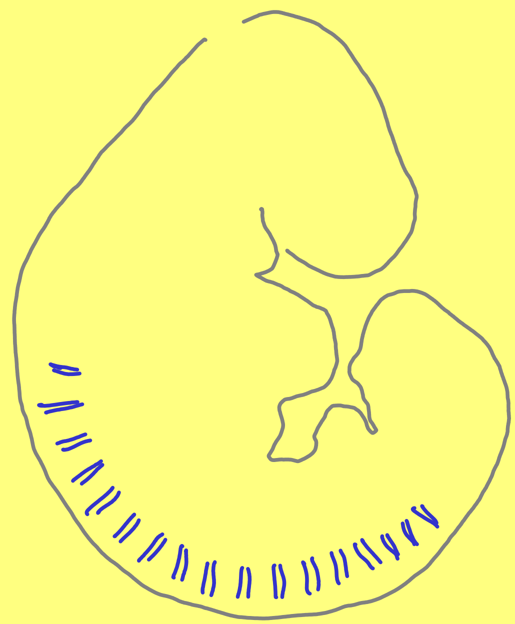
adult mouse

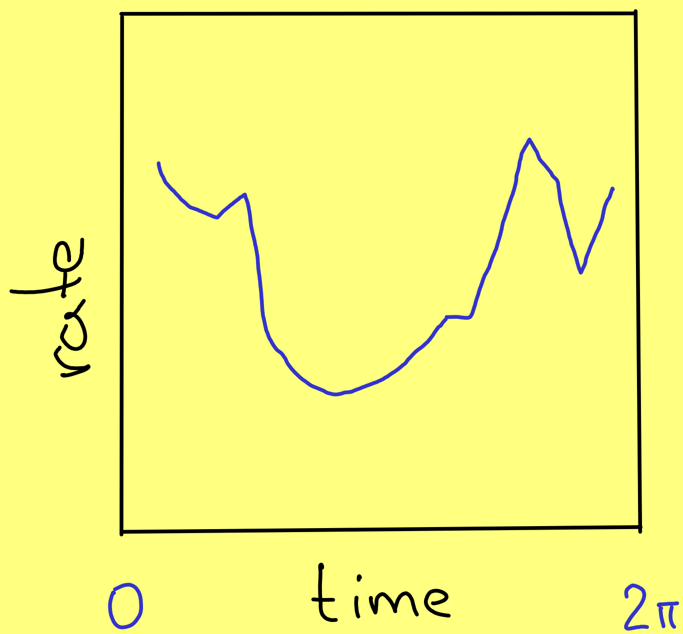


mouse embryo



somite development is a rythmic process

III.2 Gene Expression Data



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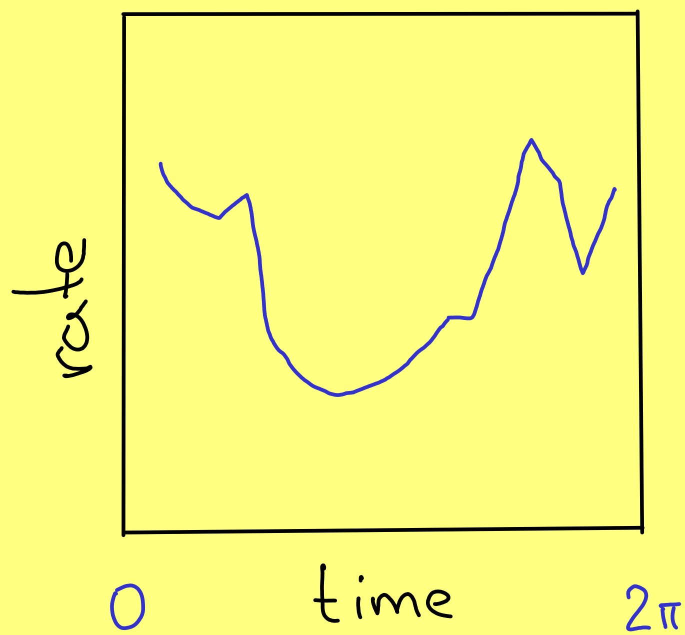


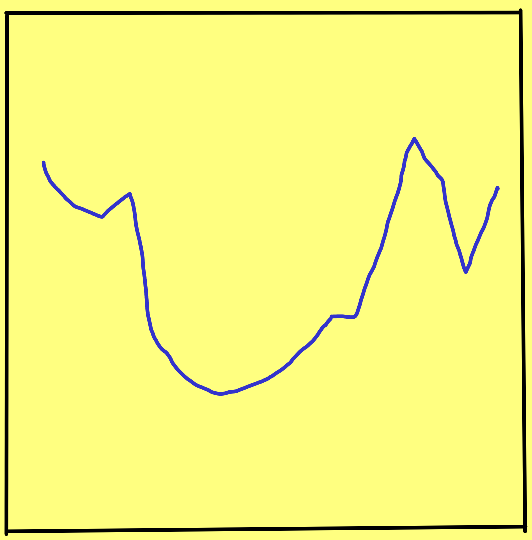
normalized to $\mathrm{amp}(f) = \max_x f(x) - \min_x f(x) = 1$

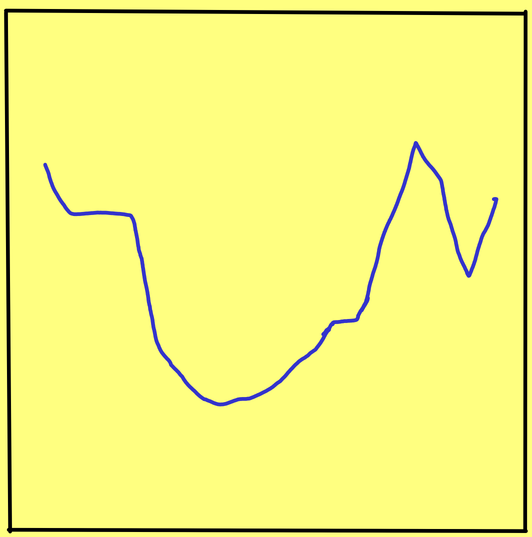
III.3 SIMPLIFICATION

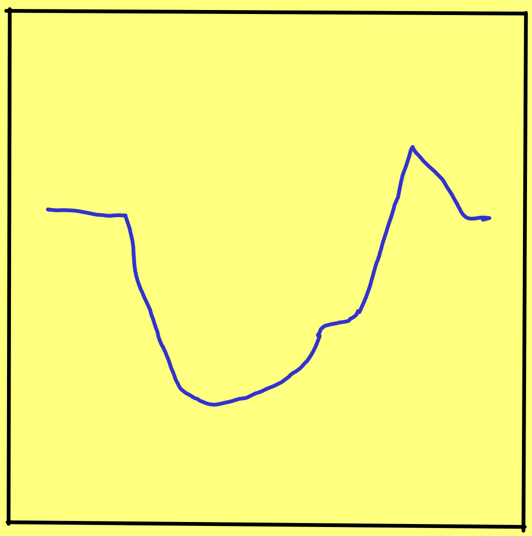
DEF. An ε-simplification of f is a function $f_\varepsilon : \mathbb{X} \to \mathbb{R}$ with $\|f - f_\varepsilon\|_\infty \le \varepsilon$ whose diagram $\mathrm{Dgm}(f_\varepsilon)$ is same as $\mathrm{Dgm}(f)$ without points of persistence at most ε.

III.3 Simplification

Def. An **ε-simplification** of f is a function $f_\varepsilon : \mathbb{X} \to \mathbb{R}$ with $\| f - f_\varepsilon \|_\infty \leq \varepsilon$ whose diagram $\mathrm{Dgm}(f_\varepsilon)$ is same as $\mathrm{Dgm}(f)$ without points of persistence at most ε.







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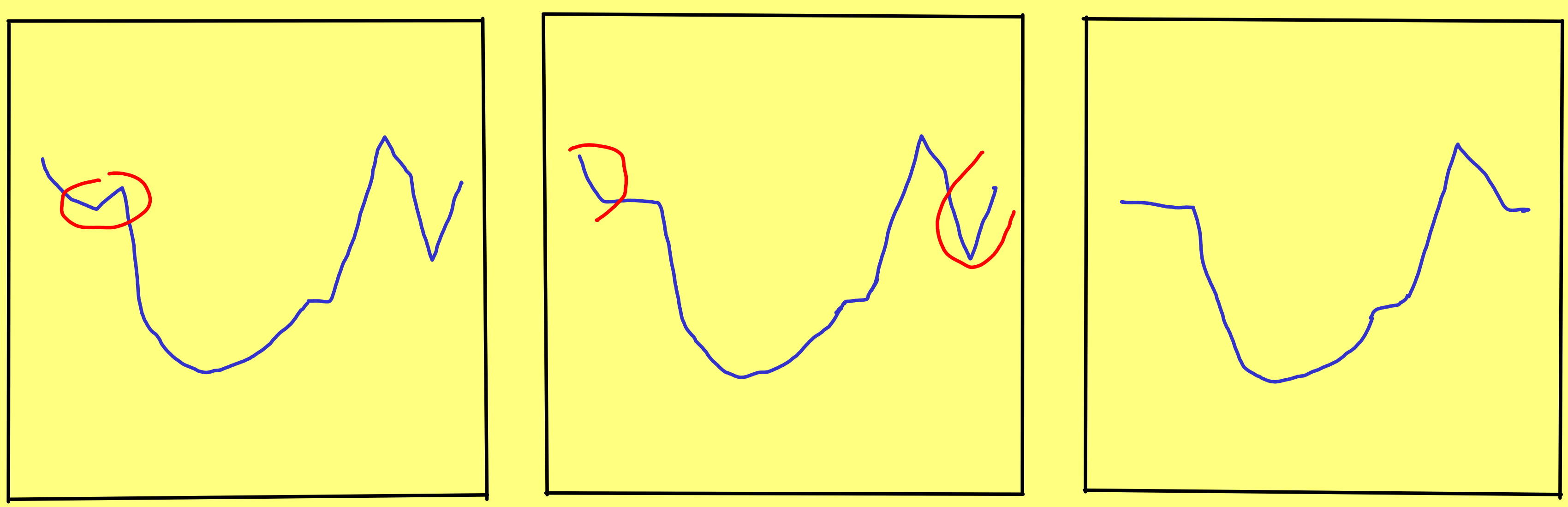

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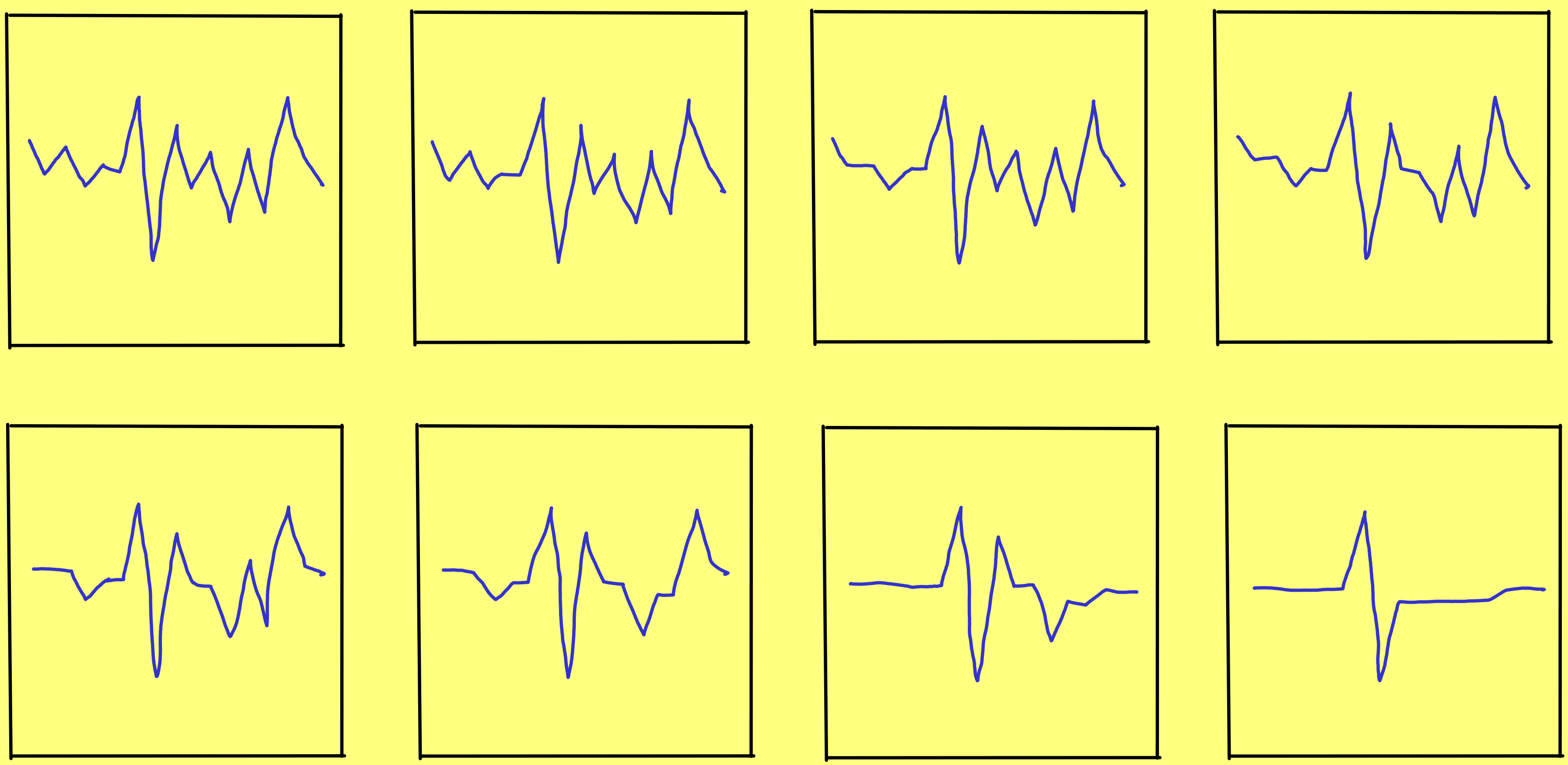

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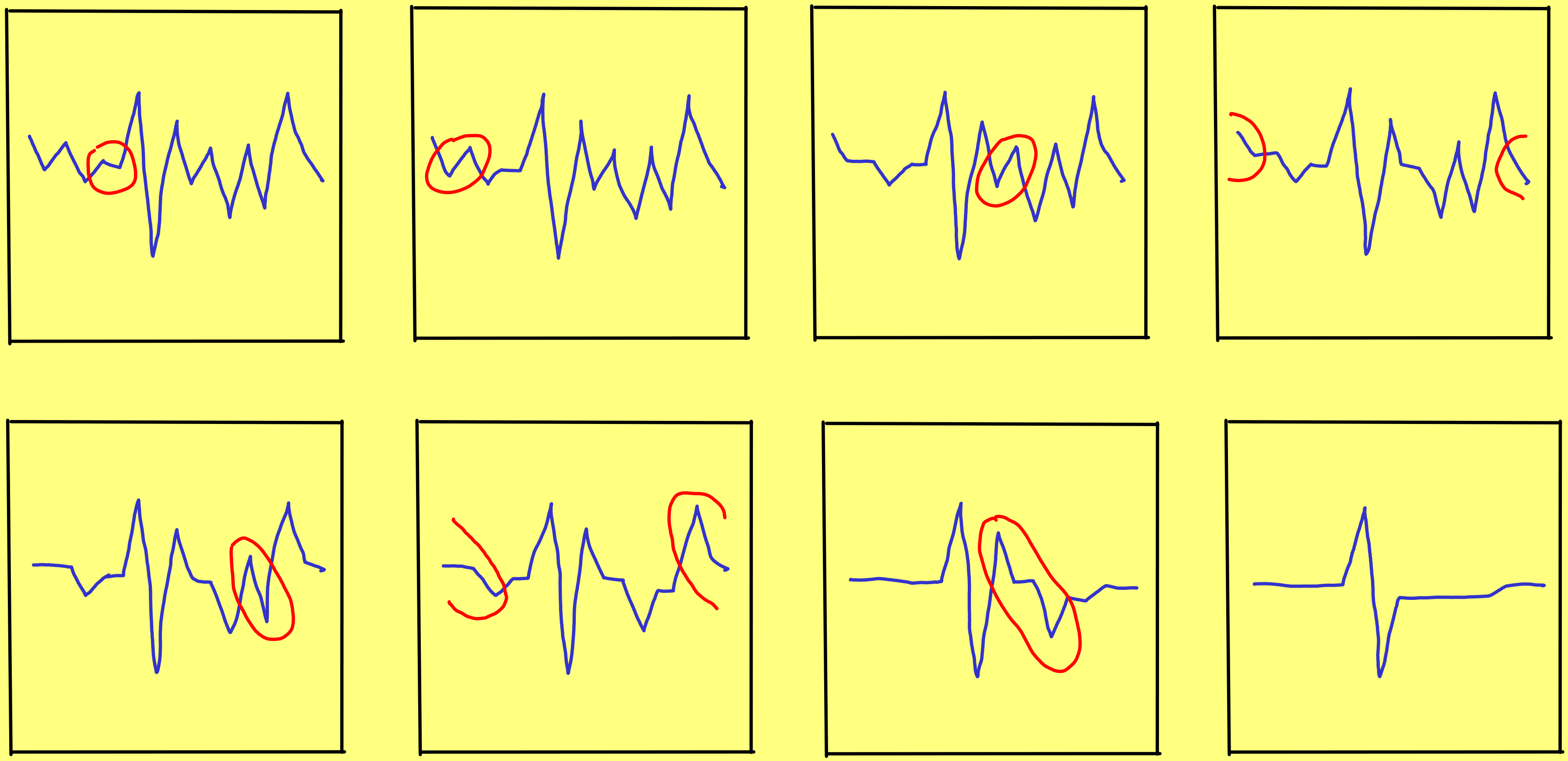

III.4 Measures

$f : S^1 \to \mathbb{R}$

$M_0(f) = \frac{1}{2}$ # critical points

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III.4 MEASURES

$$f : \mathbb{S}^1 \to \mathbb{R}$$

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$$M_2(f) = \int_{\varepsilon=0}^{1} M_1(f_\varepsilon)\, d\varepsilon = \sum_x \operatorname{pers}(x)^2$$

etc.

I.3 L_p - STABILITY

THM. Let $\mathbb{X}$ be a compact metric space and $f, g : \mathbb{X} \to \mathbb{R}$ two Lipschitz functions. Then there exist constants k and C that depend on $\mathbb{X}$ and the Lipschitz constants of f and g s.t.

$$W_p(f, g) \leq C \|f - g\|_\infty^{p-k}$$

for every $p \geq k$.

$$\boxed{\mathbb{X} = S^1 \Longrightarrow k = 1}$$

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III.4 Measures

$$f : \mathbb{S}^1 \to \mathbb{R}$$

not stable

$$M_0(f) = \frac{1}{2} \# \text{critical points}$$

$$M_1(f) = \int_{\varepsilon=0}^{1} M_0(f_\varepsilon)\, d\varepsilon = \sum_x \text{pers}(x)$$

stable

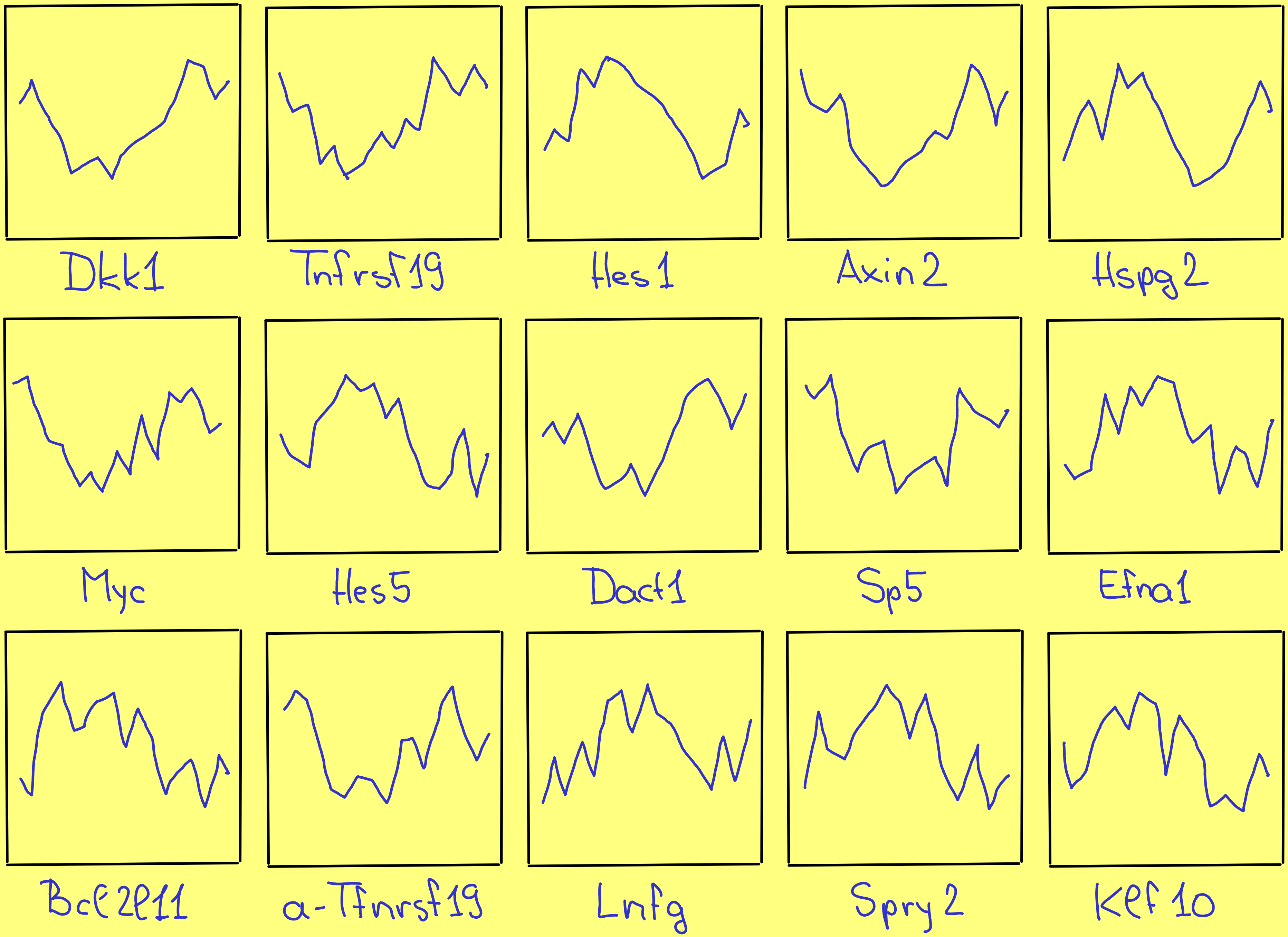
$$M_2(f) = \int_{\varepsilon=0}^{1} M_1(f_\varepsilon)\, d\varepsilon = \sum_x \text{pers}(x)^2$$

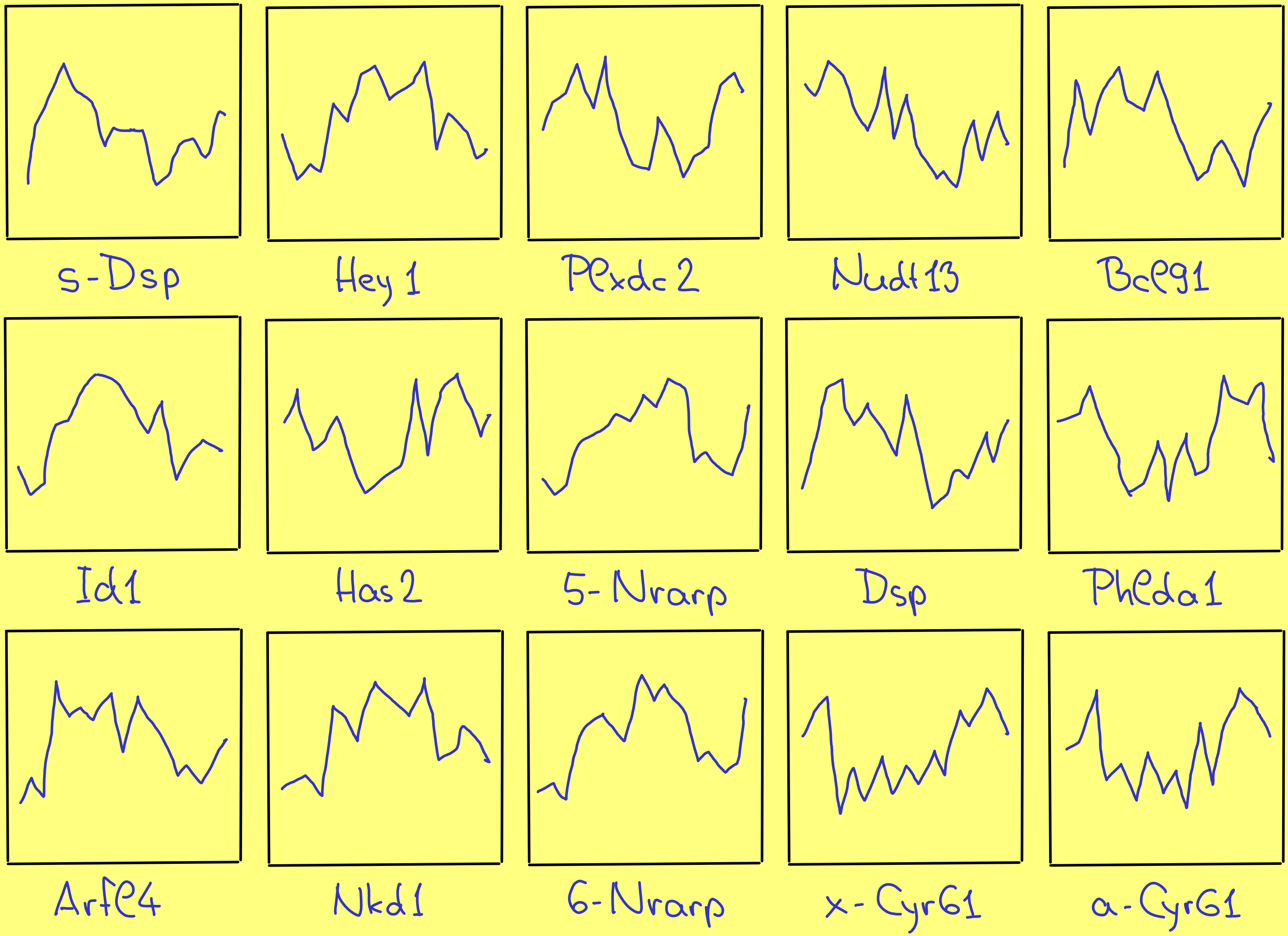
etc.

III.5 Results

using 30 genes biologically confirmed
to participate in somitogenesis

Dkk1	Tnfrsf19	Hes1	Axin2	Hspg2
Myc	Hes5	Dact1	Sp5	Efna1
Bcl2l11	a-Tnfrsf19	Lfng	Spry2	Klf10
s-Dsp	Hey1	Plxdc2	Nudt13	Bcl9l
Id1	Has2	5-Nrarp	Dsp	Phlda1
Arfl4	Nkd1	6-Nrarp	x-Cyr61	a-Cyr61


Dkk1
Tnfrsf19
Hes1
Axin2
Hspg2
Myc
Hes5
Dact1
Sp5
Efna1
Bcl2l11
a-Tfnrsf19
Lnfg
Spry2
Klf10



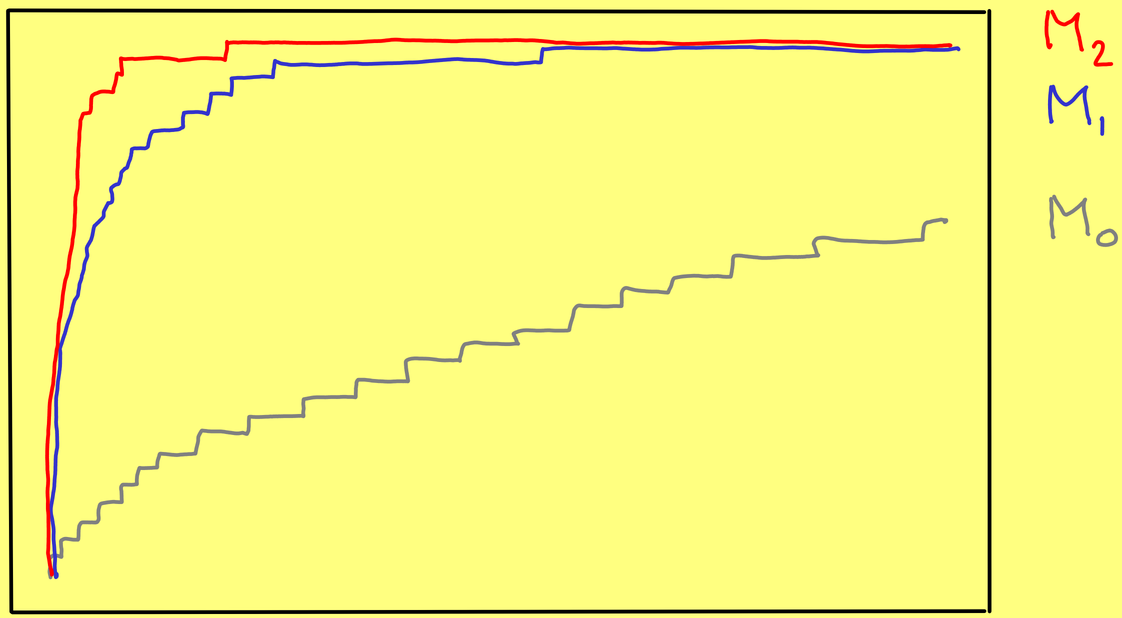
s-Dsp
Hey 1
Plxdc 2
Nudt13
Bcl9l
Id1
Has 2
5-Nrarp
Dsp
Phlda1
Arfl4
Nkd1
6-Nrarp
x-Cyr61
a-Cyr61

III.5 Results

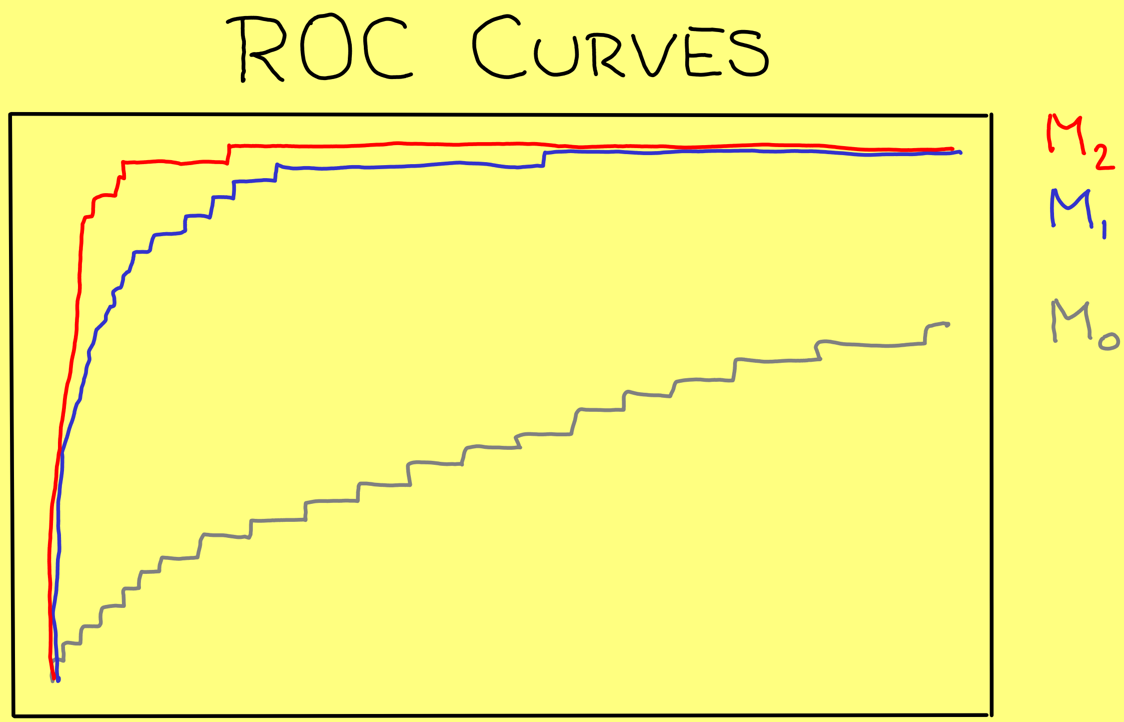
	M_1	M_2	M_3	M_4	avg
Dkk1	1	1	1	1	1
Tnfrsf19	12	2	2	2	5
Hey1	23	12	9	9	13
Nudt13	882	170	104	83	310
Arfl4	302	86	74	68	133
...	...	...	...	...	...
avg	250	109	104	117	145

III.5 Results

ROC Curves



III.5 Results



	M_0	M_1	M_2	M_3	M_4	max
Area	4.42	9.75	10.18	10.19	10.15	10.50

III.5 Results

Promising Candidates

	M_1	M_2	M_3	M_4	L
Tnfrsf9	5	4	4	6	8
Ptpn11	8	6	6	7	22
Mtm1	9	7	5	3	15
Stom	7	8	12	16	299
Star	6	9	11	13	39
...	...	...	...	...	...

IV. MOTION

IV.1 Homotopy

$F : \mathbb{M} \times [0,1] \to \mathbb{R}$,

$f_t(x) = F(x,t)$ leads to $\mathrm{Dgm}(f_t)$

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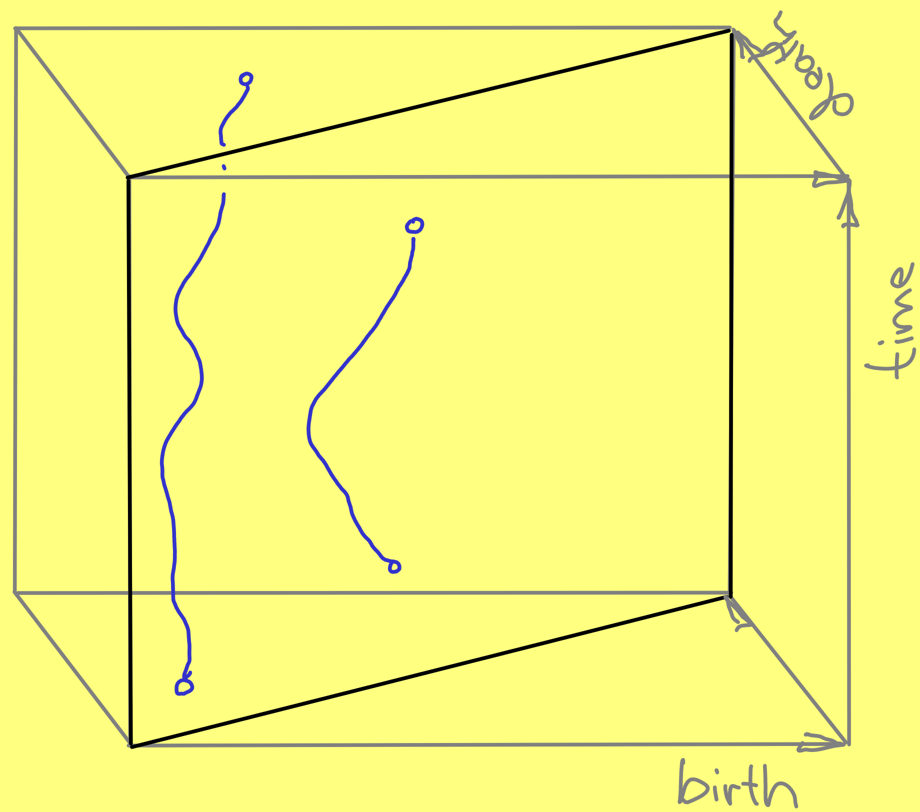
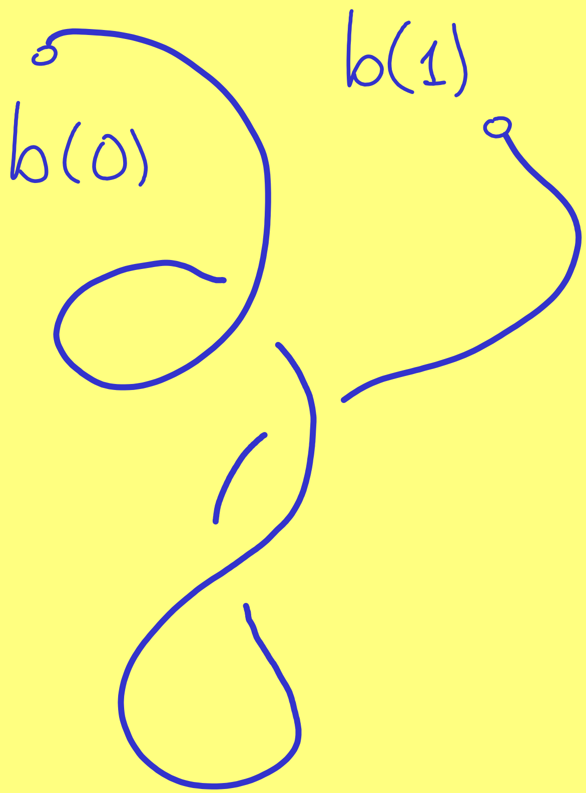
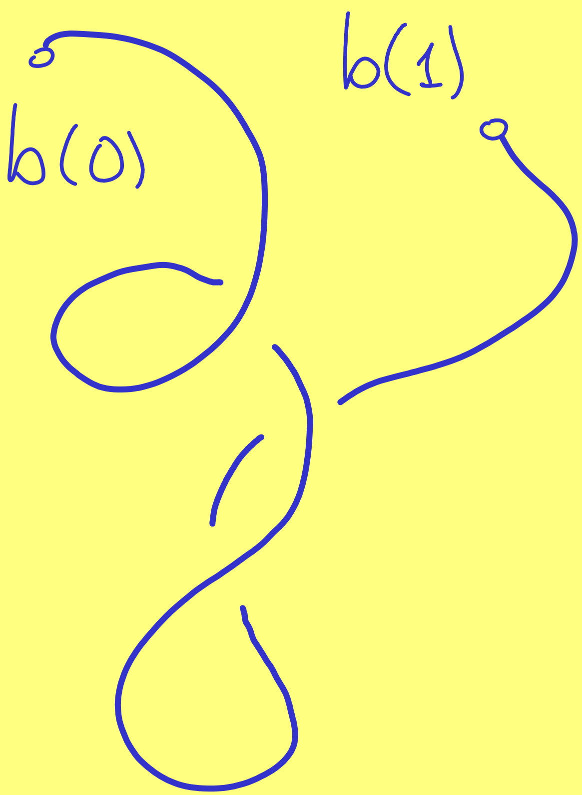


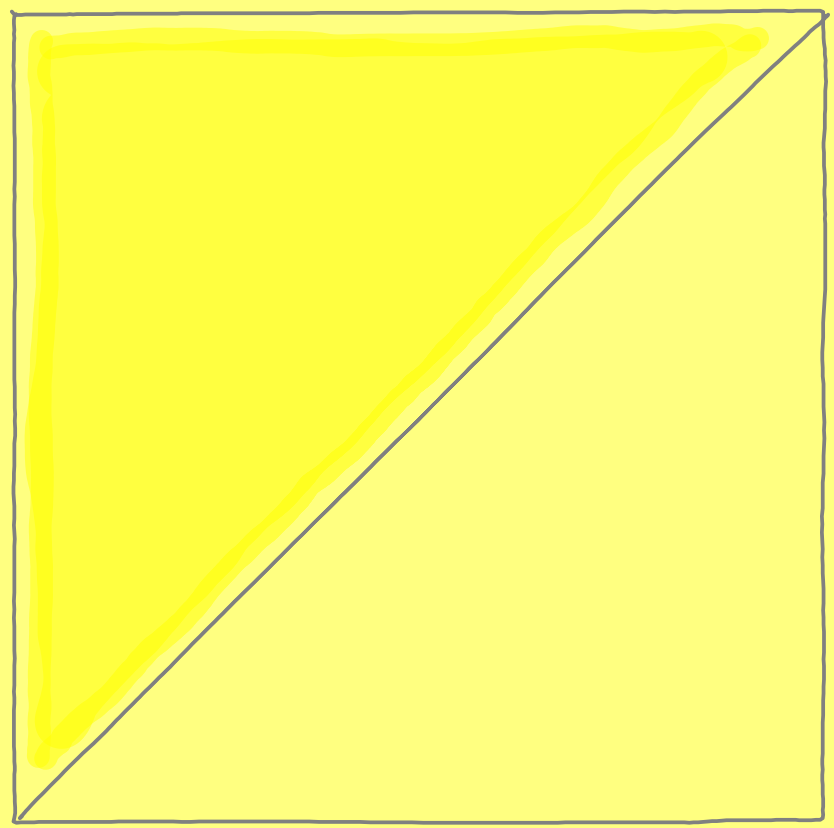
IV.2 Pairwise Distance Function



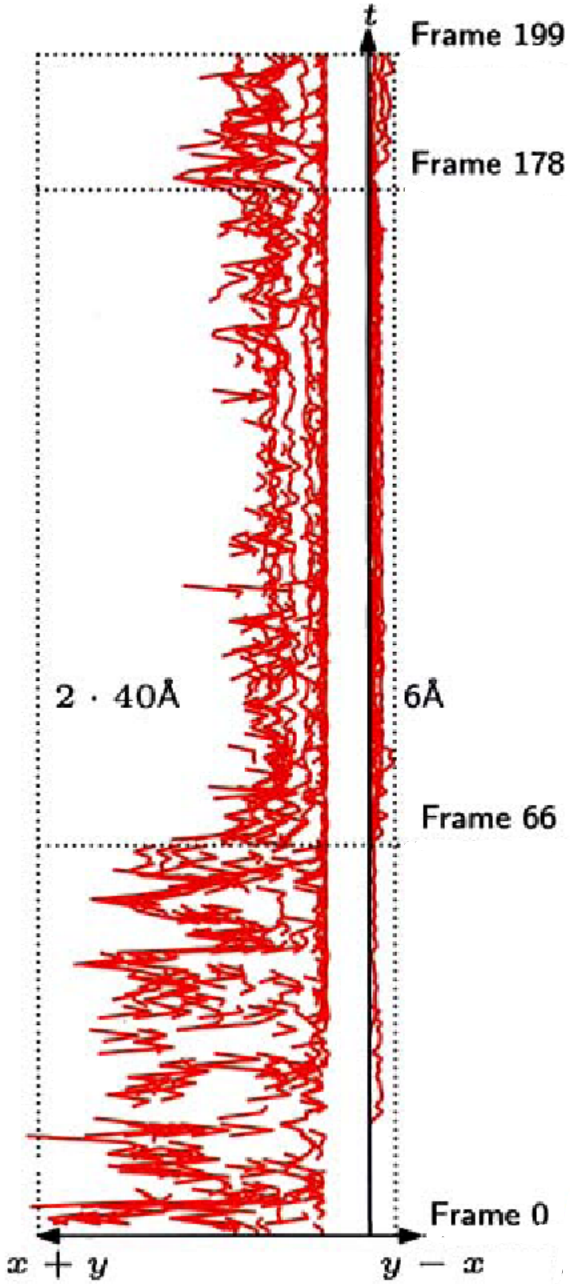
IV.2 Pairwise Distance Function

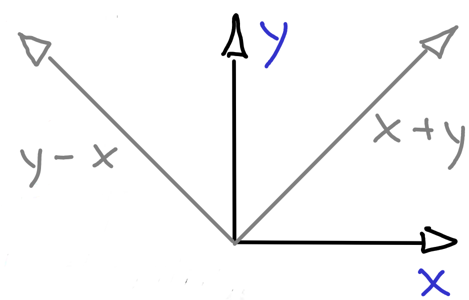
$$f(x,y) = \| b(x) - b(y) \|$$



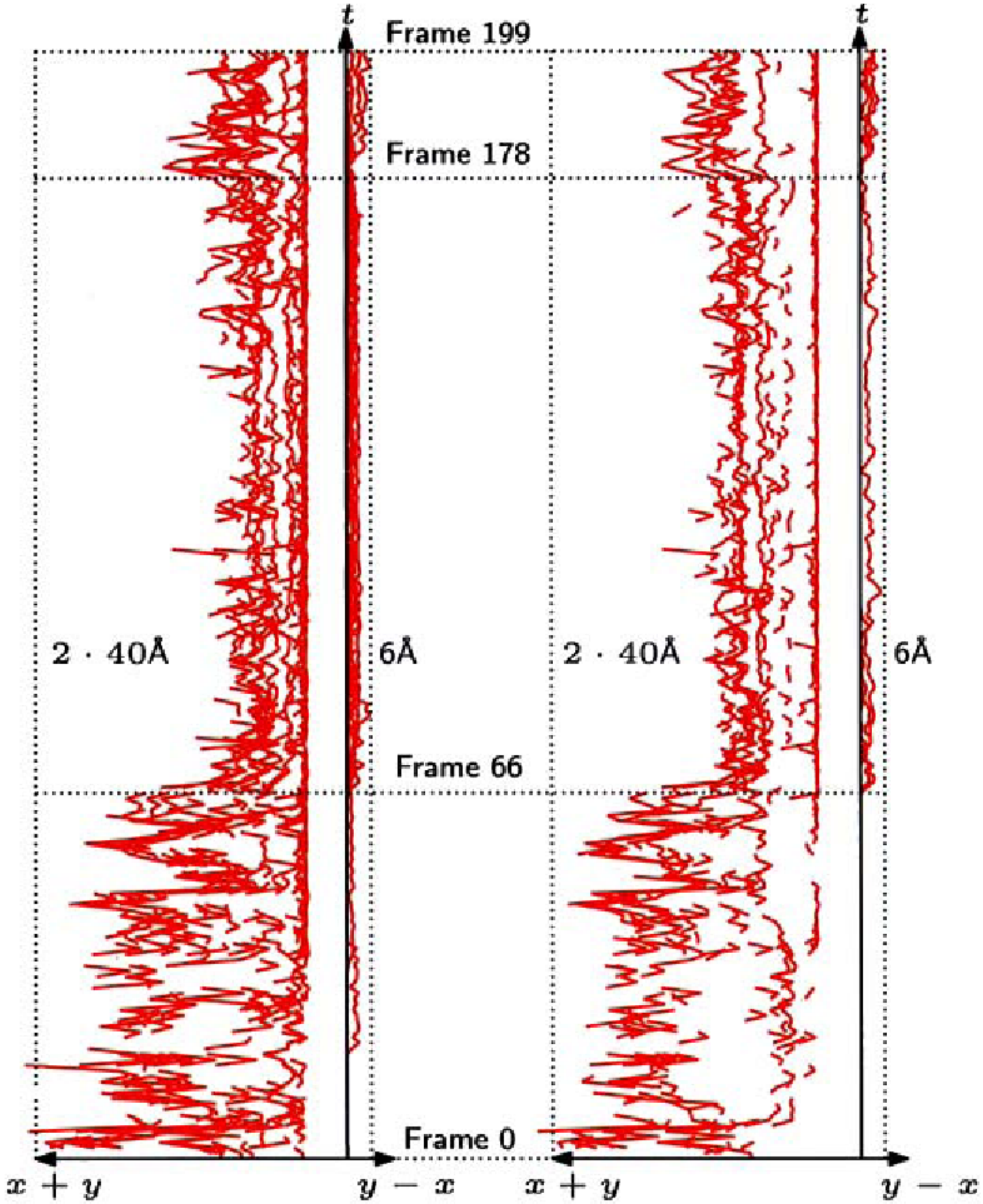


IV.3 VINEYARD





IV.3 VINEYARD



THANK YOU