Folds, Intersections, and Inflections for Smooth and Polyhedral Surfaces: Is That a Cylinder or a Möbius Band?

BMS Friday Colloquium
June 22, 2012
Thomas Banchoff

Heinz Hopf 1894-1971

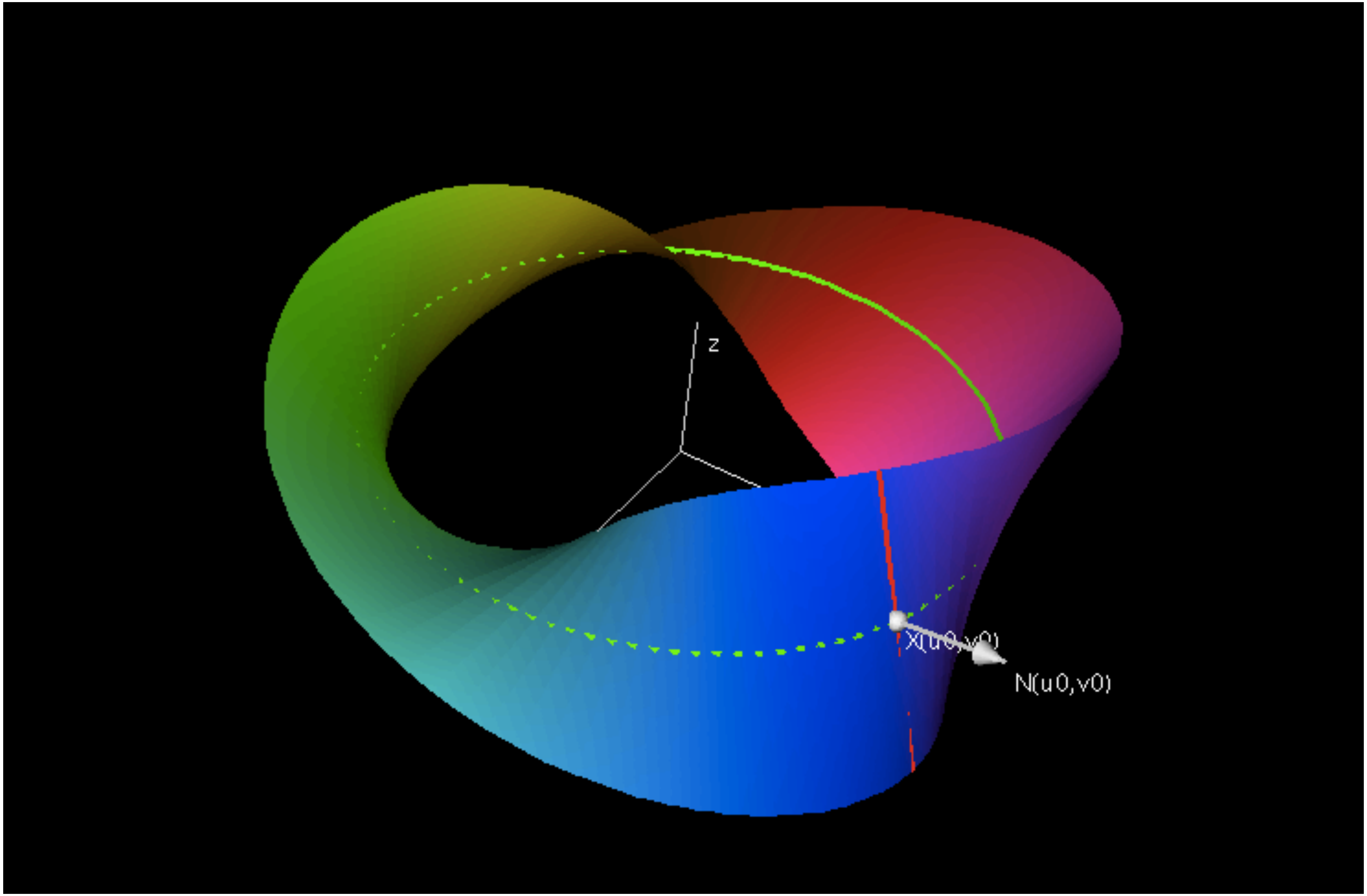
1000

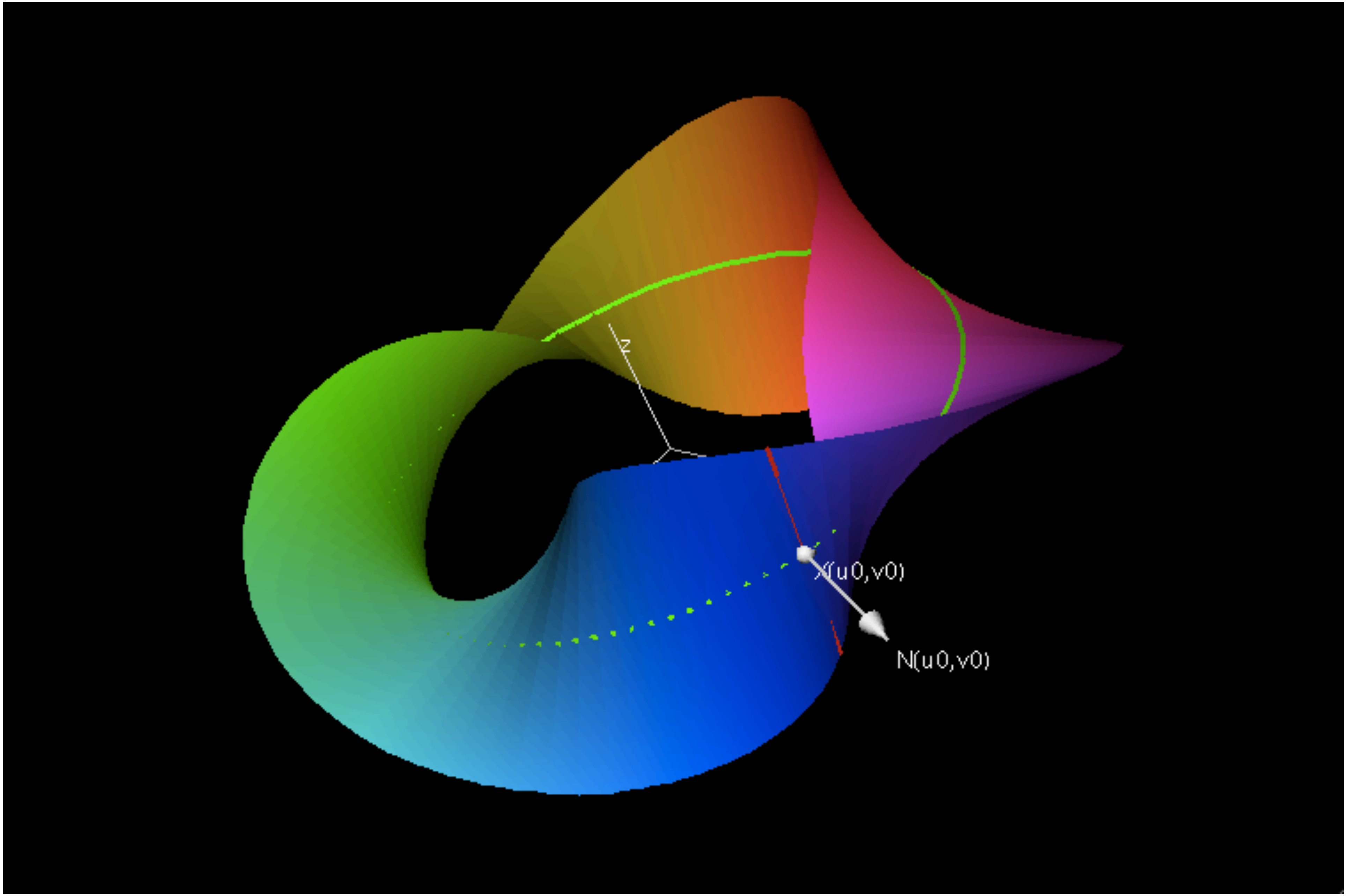
Heinz Hopf

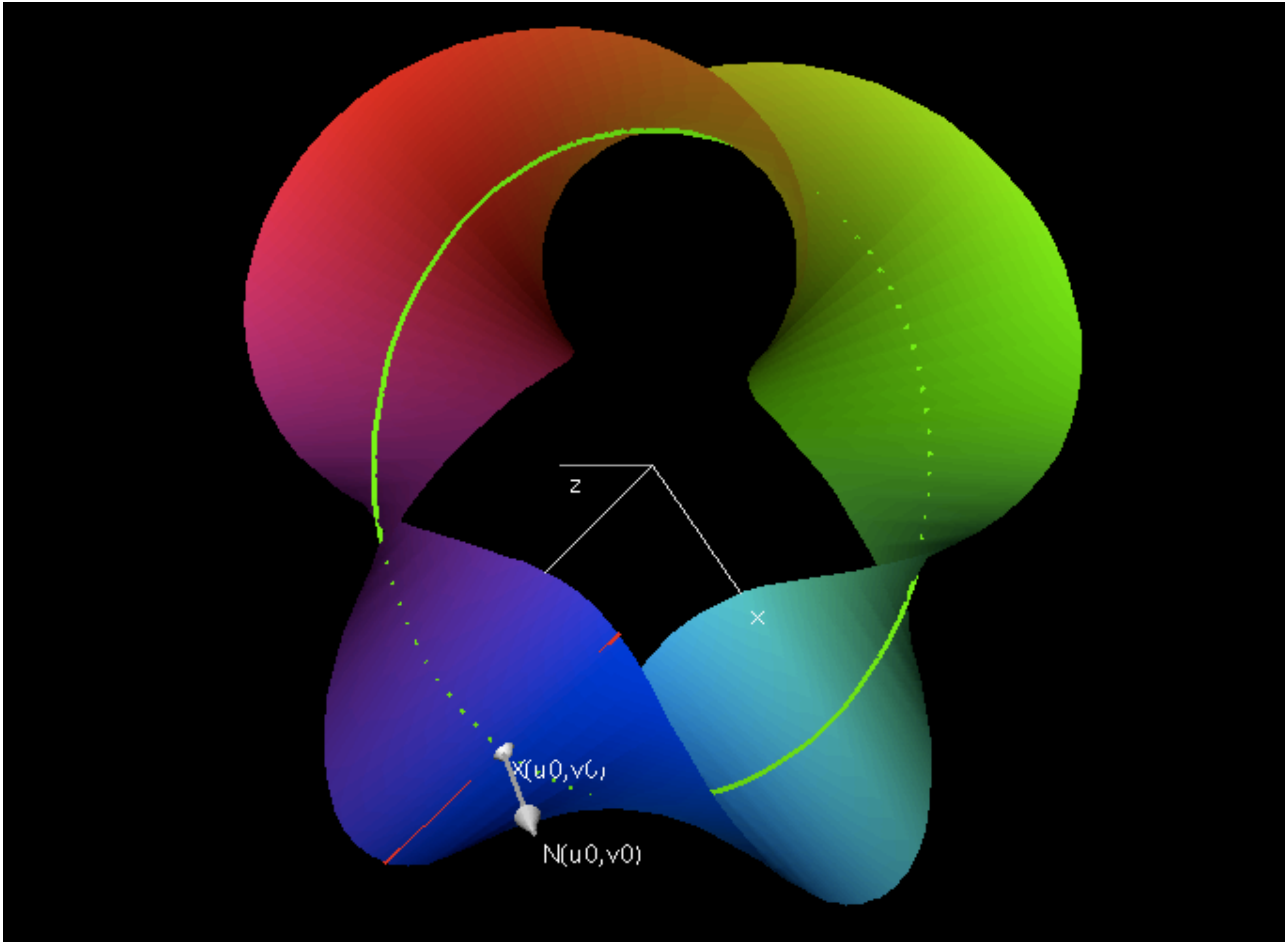
Differential Geometry in the Large

Seminar Lectures New York University 1946 and Stanford University 1956

Second Edition


z
X(u0,v0)
N(u0,v0)


z
X(u0,v0)
N(u0,v0)


z
N(u0,v0)

CHASE

Characteristics:	Torus	Möbius
• #(boundary curves)	2	1

Characteristics:	Torus	Möbius
• #(boundary curves)	2	1
• #(pieces of complement of X)	2	1

Folds

Intersections

Inflections

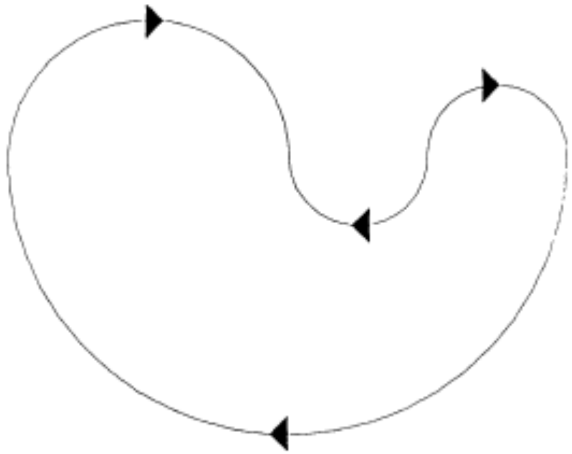


FIGURE 1a

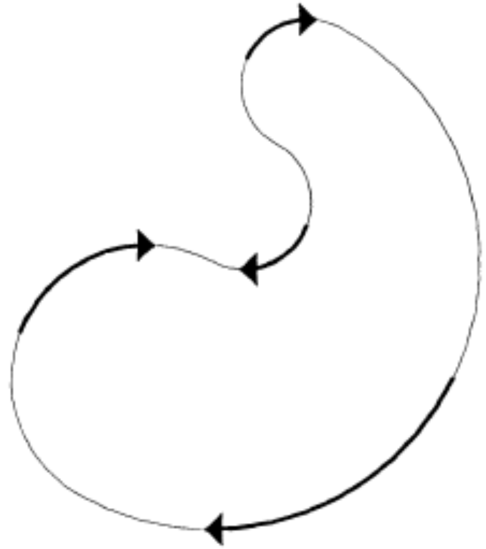


FIGURE 1b

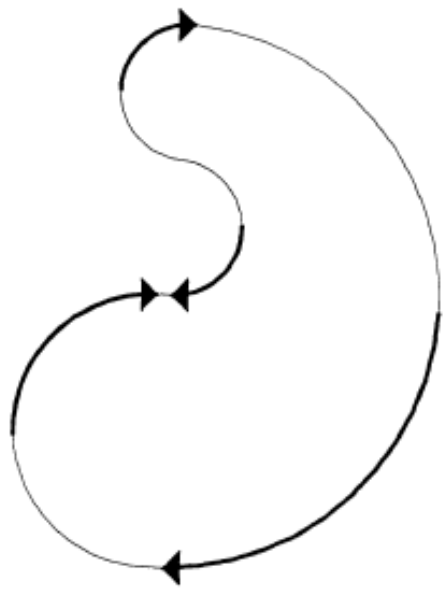


FIGURE 1c

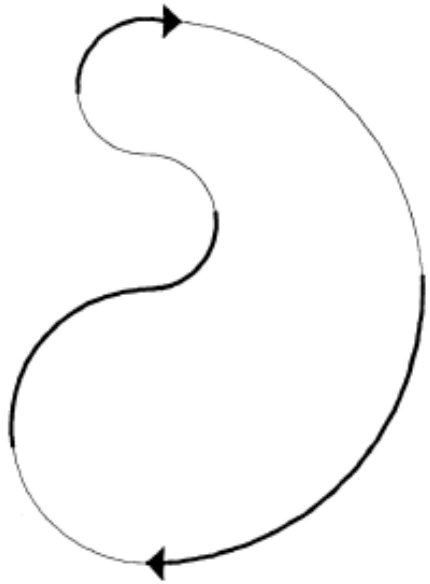
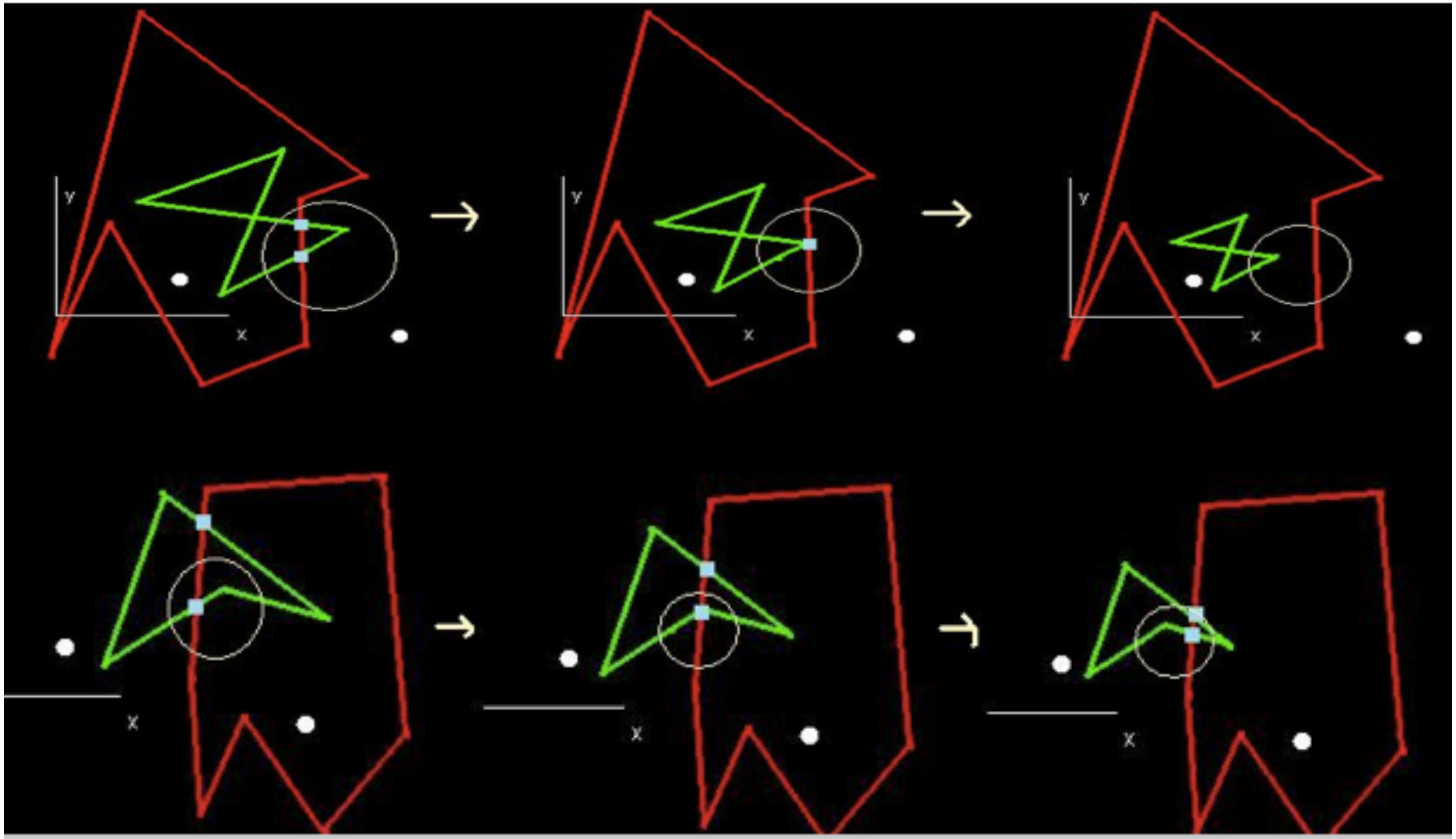


FIGURE 1d

PARITY LEMMA: $\#(X \cap Y) \equiv 0 \pmod{2}$

Deformation Argument: Changes are even.



CYLINDER

- (Drawing on Blackboard)

MöBIUS BAND

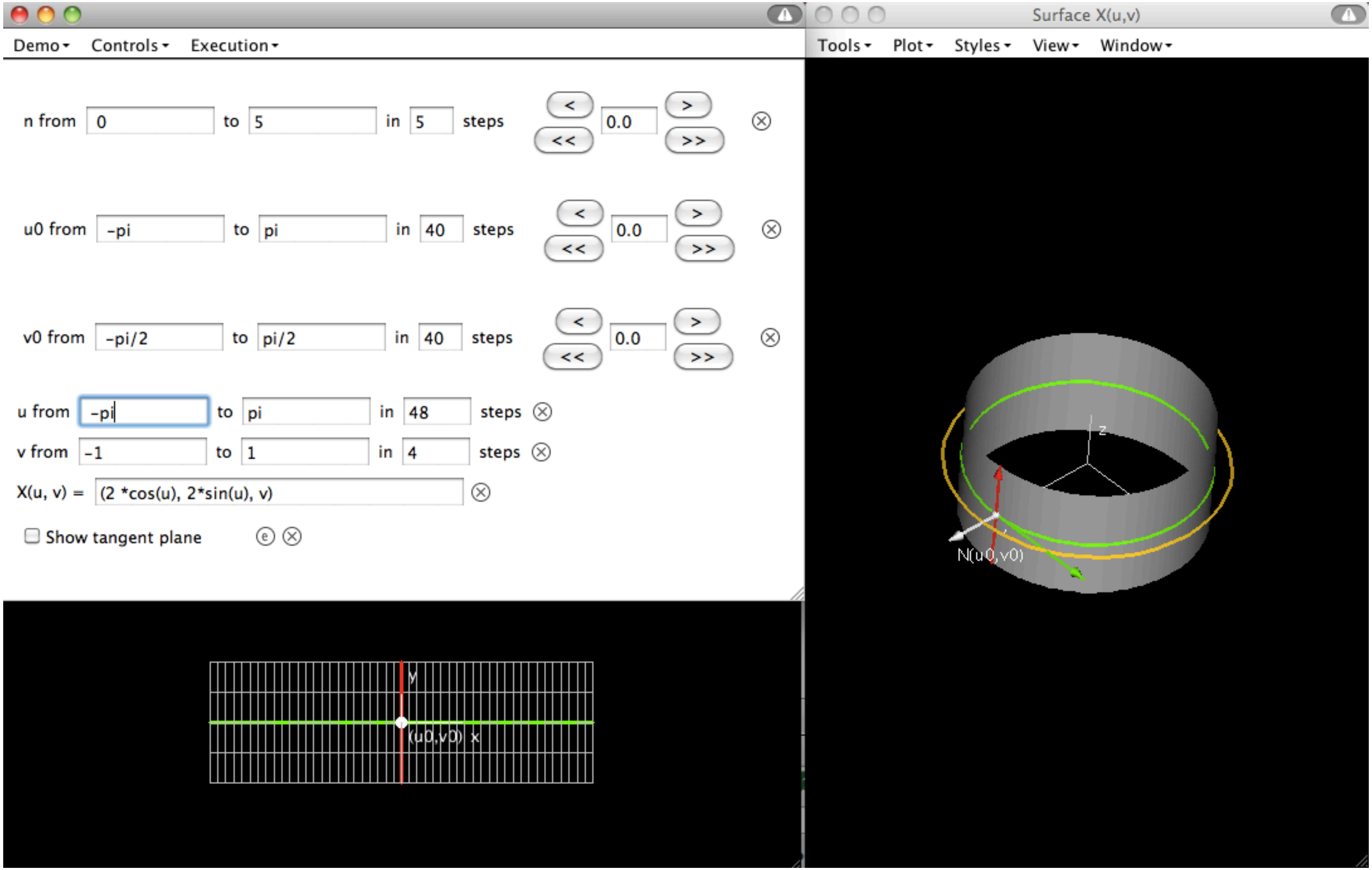
- (Drawing on Blackboard)

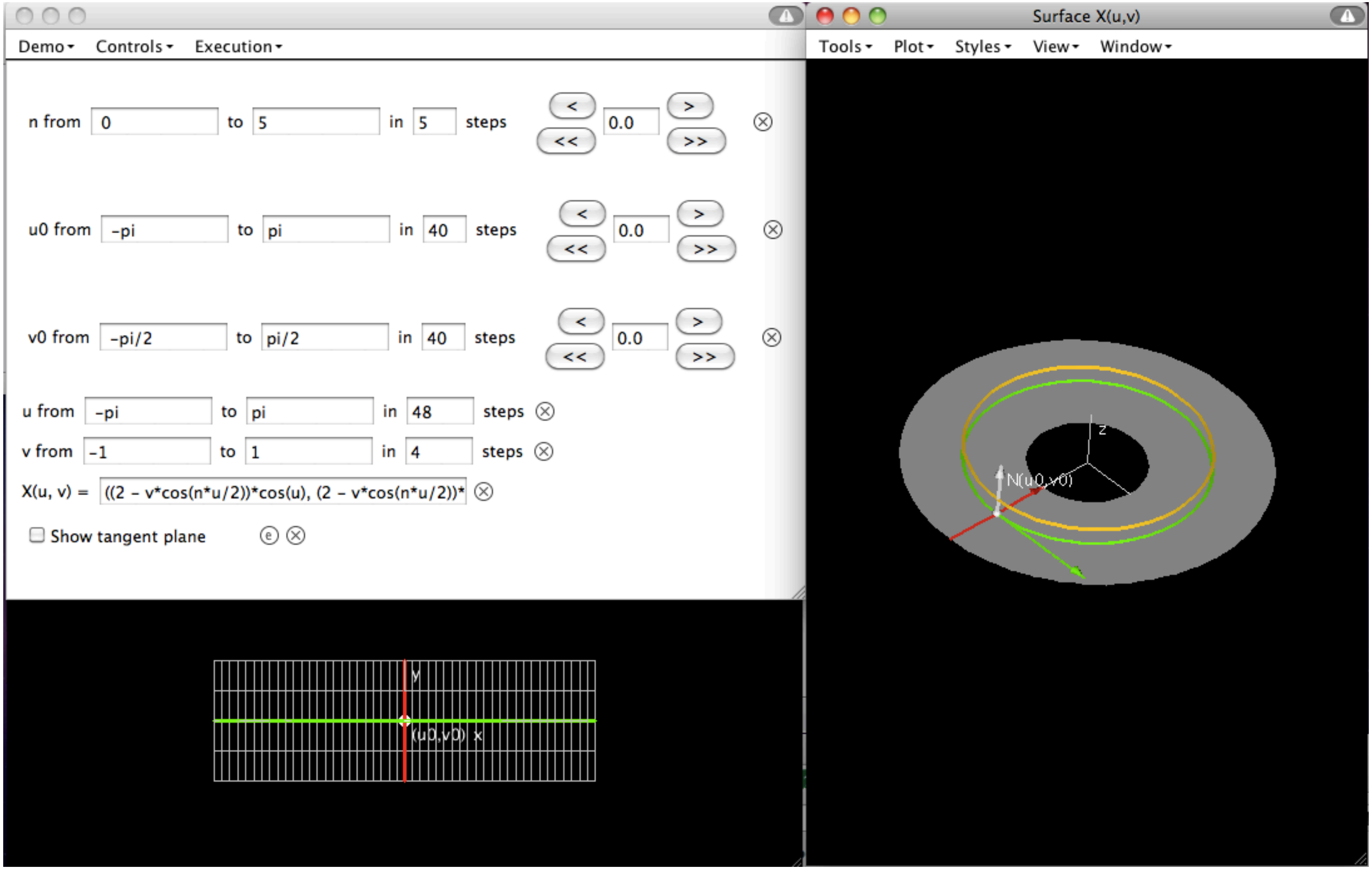
Characteristics:	Torus	Möbius
• #(boundary curves)	2	1
• #(pieces of complement of X)	2	1
• #(X ∩ tangential variation)	Even	Odd

Characteristics:	Torus	Möbius
• #(boundary curves)	2	1
• #(pieces of complement of X)	2	1
• #(X ∩ tangential variation)	Even	Odd
• #(X ∩ normal variation)	Even	Odd

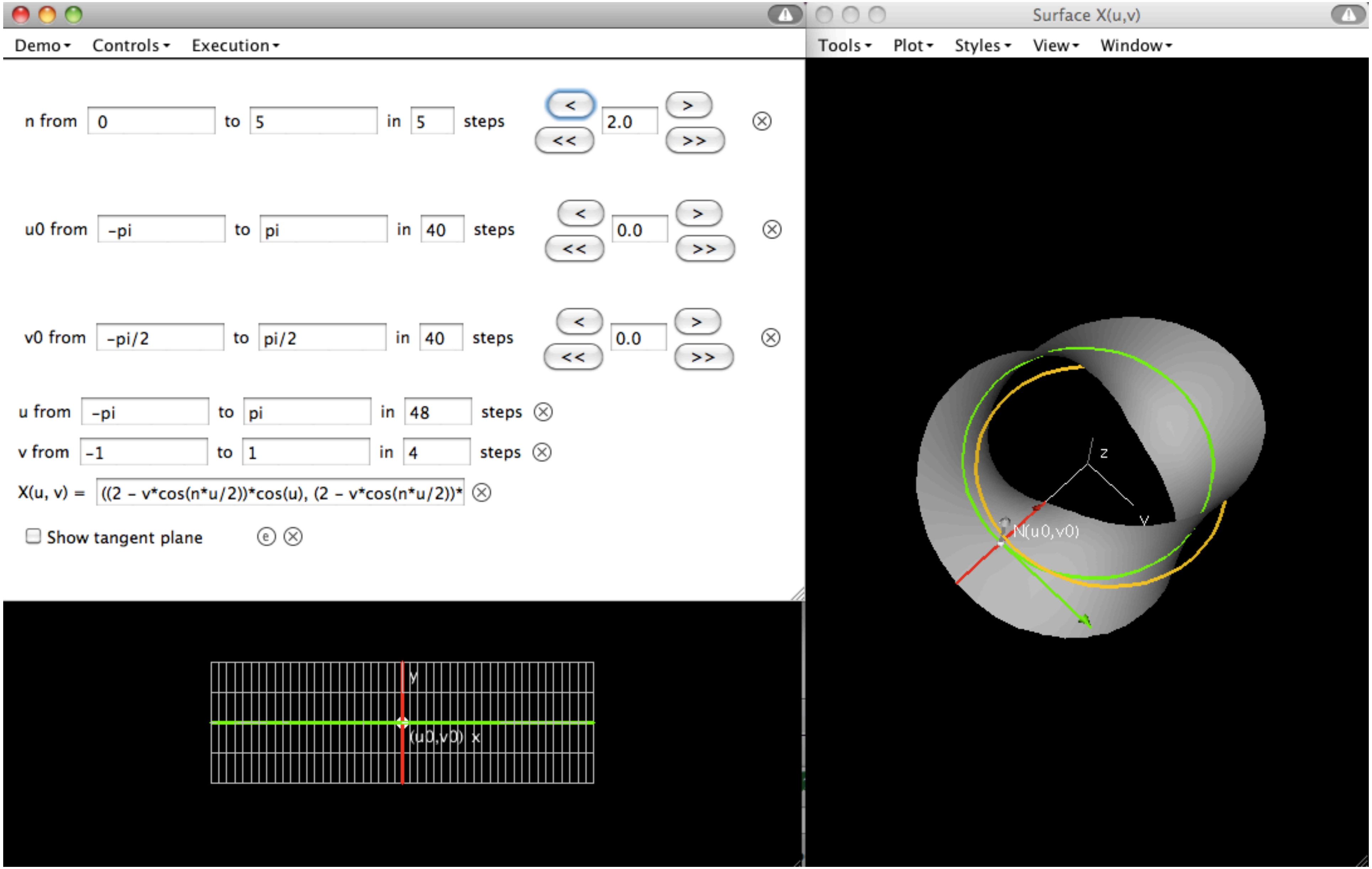
Whitney Duality Level 1

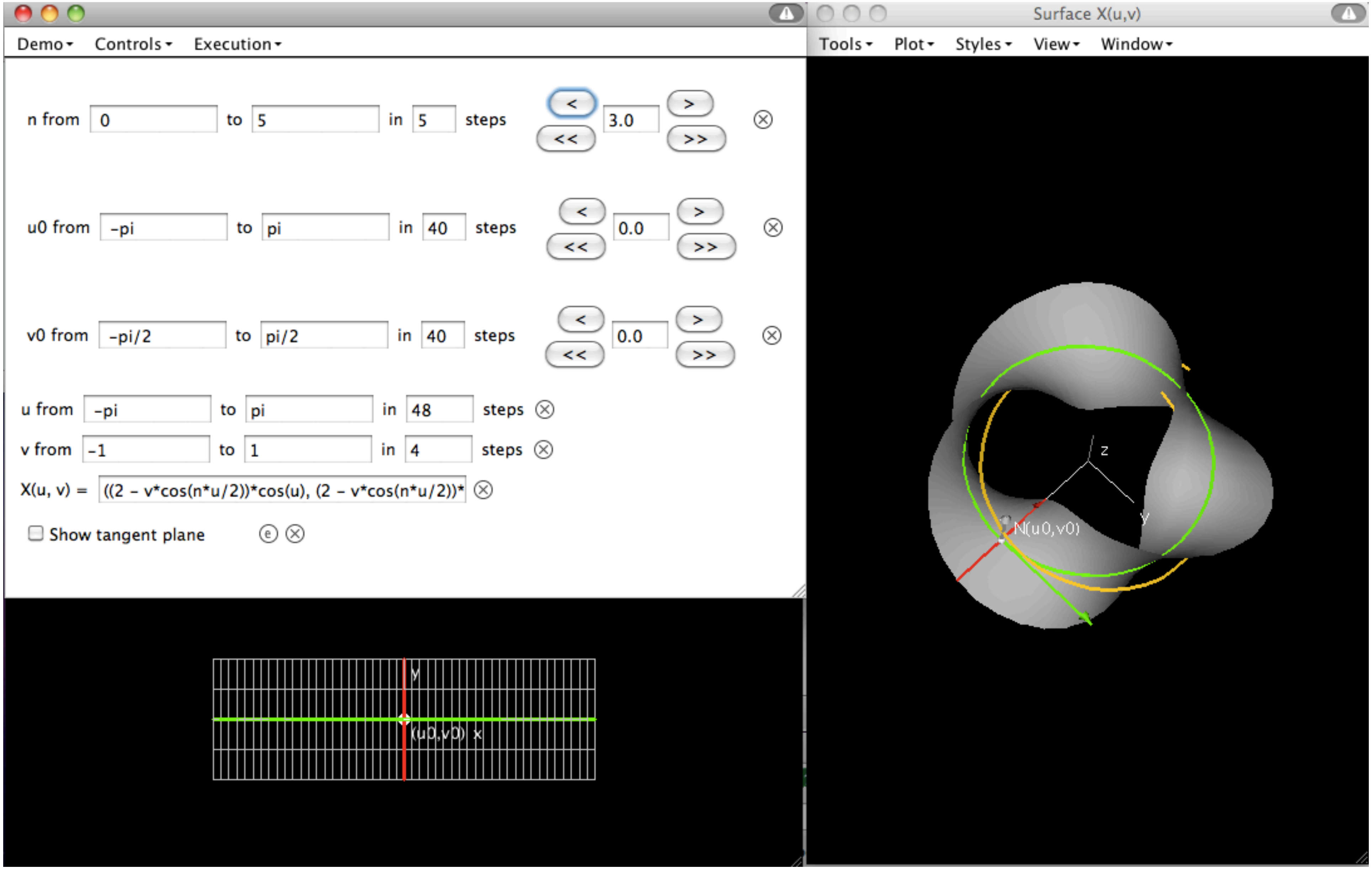
- X: [a,b] → M , a curve on a surface
- T(t) = unit tangent vector = X'(t)/|X'(t)|
- N(t) = unit normal vector
- U(t) = N(t) x T(t)
- Cylinder strip: N(b) = N(a), U(b) = U(a)
- Möbius strip: N(b) = -N(a), U(b) = -U(a)
- #(X ∩ tangential var.) ≡ #(X ∩ normal var.)
- $DW_1 = D\underline{W}_1$


Demo
Controls
Execution
n from 0 to 5 in 5 steps
0.0
u0 from -pi to pi in 40 steps
0.0
v0 from -pi/2 to pi/2 in 40 steps
0.0
u from -pi to pi in 48 steps
v from -1 to 1 in 4 steps
X(u, v) = (2 *cos(u), 2*sin(u), v)
Show tangent plane
(u0,v0)
x
Surface X(u,v)
Tools
Plot
Styles
View
Window
z
N(u0,v0)

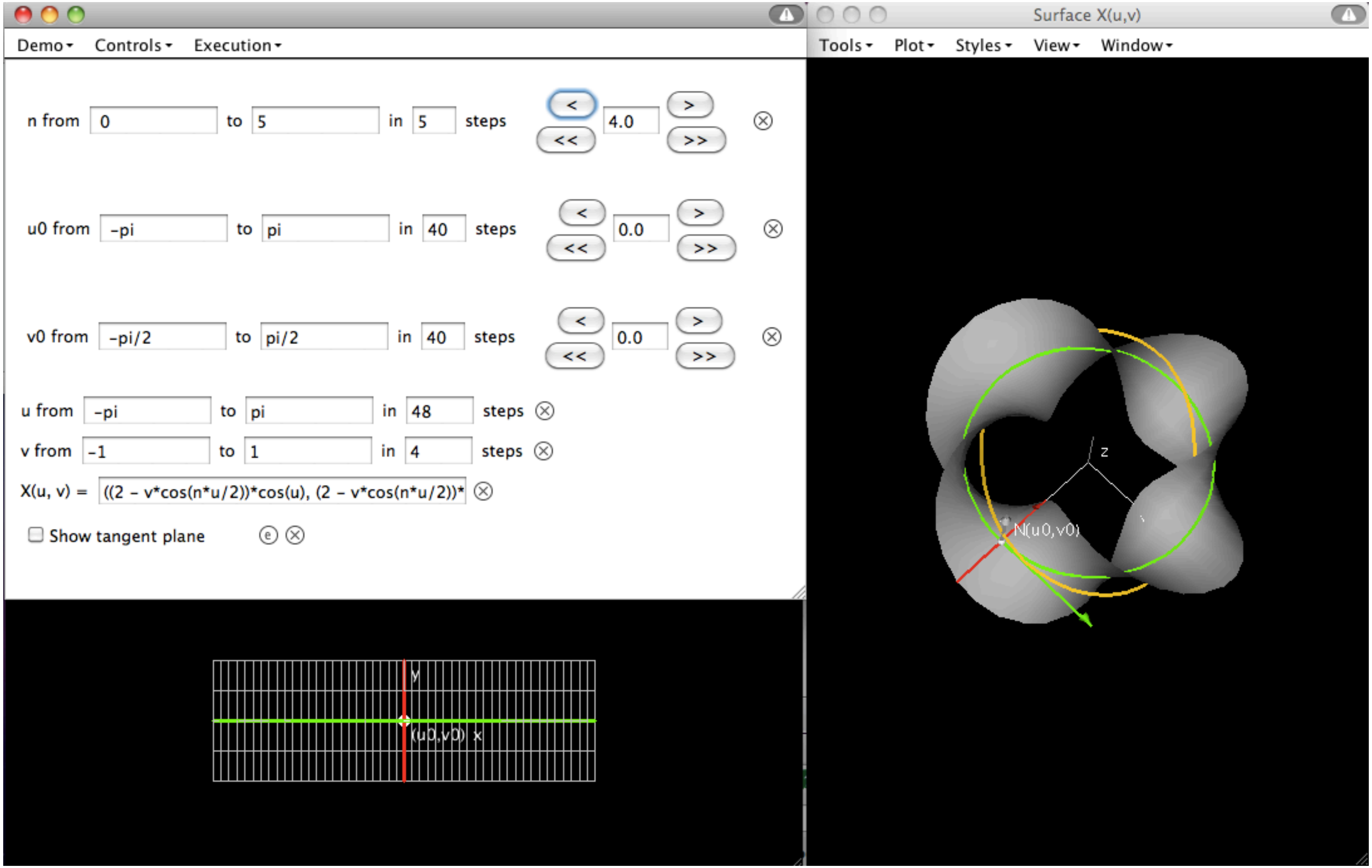


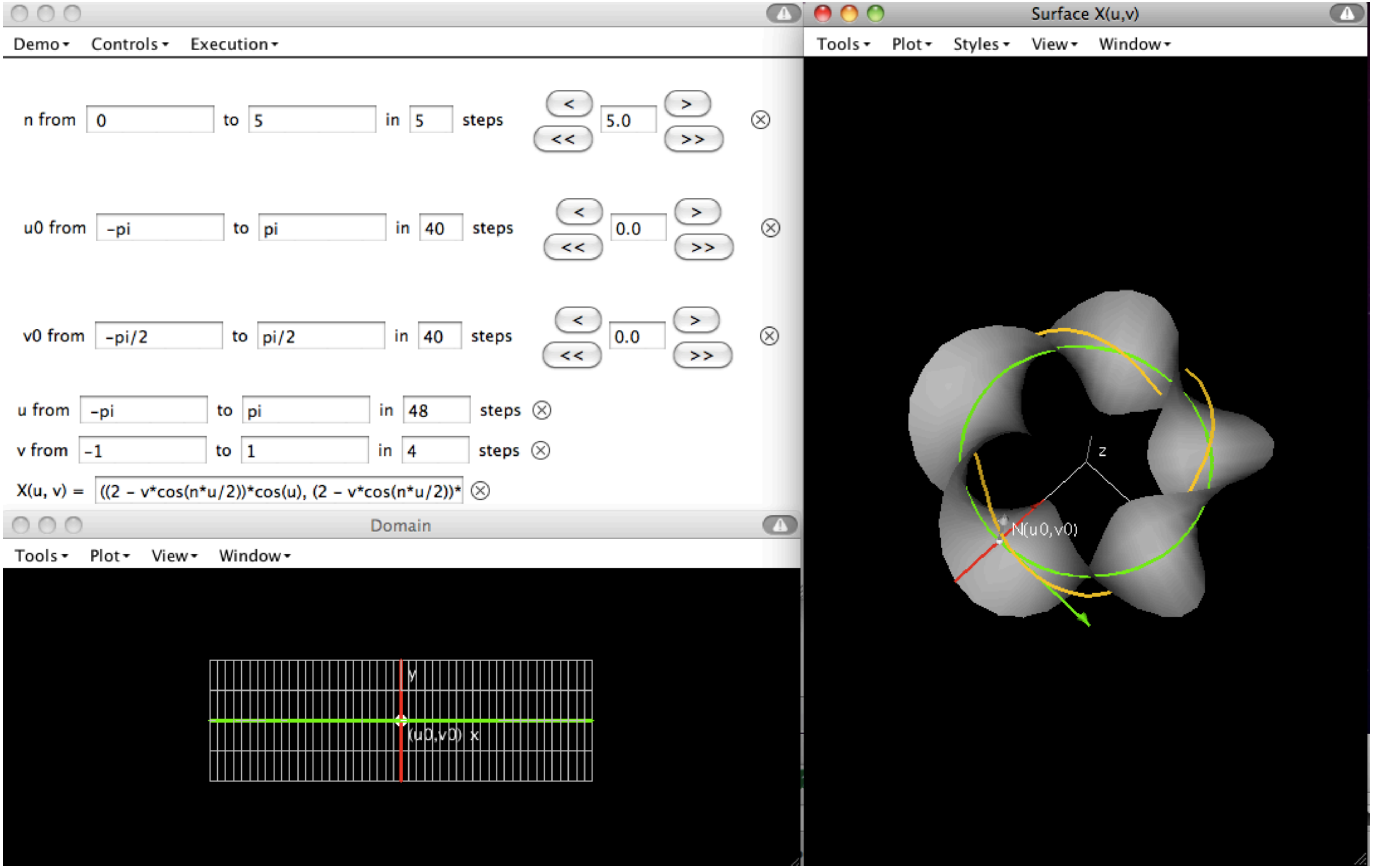
Demo
Controls
Execution
n from
0
to
5
in
5
steps
0.0
u0 from
-pi
to
pi
in
40
steps
0.0
v0 from
-pi/2
to
pi/2
in
40
steps
0.0
u from
-pi
to
pi
in
48
steps
v from
-1
to
1
in
4
steps
X(u, v) =
((2 - v*cos(n*u/2))*cos(u), (2 - v*cos(n*u/2))*
Show tangent plane
y
(u0,v0)
x
Surface X(u,v)
Tools
Plot
Styles
View
Window
z
N(u0,v0)


Demo Controls Execution
n from 0 to 5 in 5 steps 2.0
u0 from -pi to pi in 40 steps 0.0
v0 from -pi/2 to pi/2 in 40 steps 0.0
u from -pi to pi in 48 steps
v from -1 to 1 in 4 steps
X(u, v) = ((2 - v*cos(n*u/2))*cos(u), (2 - v*cos(n*u/2))*
Show tangent plane
(u0,v0) x
Surface X(u,v)
Tools Plot Styles View Window
z
y
N(u0,v0)



Demo Controls Execution
n from 0 to 5 in 5 steps
< 3.0 >
<< >>
u0 from -pi to pi in 40 steps
< 0.0 >
<< >>
v0 from -pi/2 to pi/2 in 40 steps
< 0.0 >
<< >>
u from -pi to pi in 48 steps
v from -1 to 1 in 4 steps
X(u, v) = ((2 - v*cos(n*u/2))*cos(u), (2 - v*cos(n*u/2))*
Show tangent plane
y
(u0,v0) x
Surface X(u,v)
Tools Plot Styles View Window
z
y
N(u0,v0)

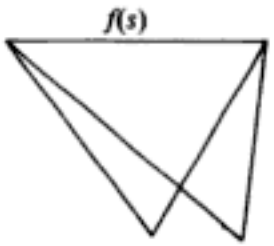

Demo Controls Execution
n from 0 to 5 in 5 steps
4.0
u0 from -pi to pi in 40 steps
0.0
v0 from -pi/2 to pi/2 in 40 steps
0.0
u from -pi to pi in 48 steps
v from -1 to 1 in 4 steps
X(u, v) = ((2 - v*cos(n*u/2))*cos(u), (2 - v*cos(n*u/2))*
Show tangent plane
y
(u0,v0) x
Surface X(u,v)
Tools Plot Styles View Window
z
N(u0,v0)


Demo Controls Execution
n from 0 to 5 in 5 steps
5.0
u0 from -pi to pi in 40 steps
0.0
v0 from -pi/2 to pi/2 in 40 steps
0.0
u from -pi to pi in 48 steps
v from -1 to 1 in 4 steps
X(u, v) = ((2 - v*cos(n*u/2))*cos(u), (2 - v*cos(n*u/2))*
Domain
Tools Plot View Window
y
(u0,v0) x
Surface X(u,v)
Tools Plot Styles View Window
z
N(u0,v0)

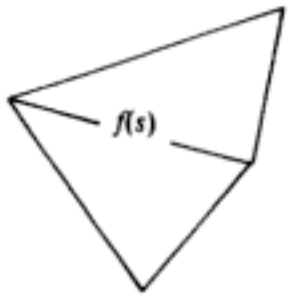
Characteristics:	Torus	Möbius
• #(boundary curves)	2	1
• #(pieces of complement of X)	2	1
• #(X ∩ tangential variation)	Even	Odd
• #(X ∩ normal variation)	Even	Odd
• #(X ∩ fold cycle of plane map)	Even	Odd

Properties of Fold Cycles

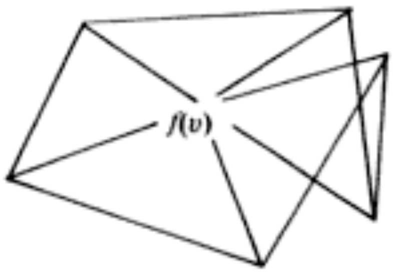
- For an immersed surface $F: M^2 \rightarrow R^3$, any two singularity sets of projections to planes together bound a region.
- For an abstract simplicial surface, any two mappings to the plane with vertices in general position are homologous by a sequence of elementary moves.



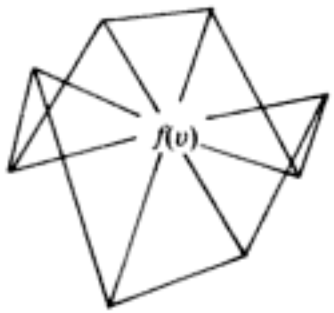
s IS A FOLD FOR *f*



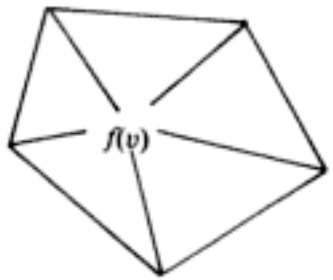
s IS ORDINARY FOR *f*



2 FOLD EDGES



4 FOLD EDGES



0 FOLD EDGES

Whitney Duality Theorem

Cycle Level Duality for SW Classes

- $F: M^2 \rightarrow R^3$, immersion
- $P_i: R^3 \rightarrow R^1{}_j = [E_j]$,
- $P_3F: M^2 \rightarrow R^1{}_3$ χ
- $S(P_3F) = \{x \text{ in } M \mid T_X F \perp R_3\}$
- $W_2(M^2) \equiv \chi(M^2)$ (mod 2)

Cycle Level Duality for SW Classes

- F: $M^2 \rightarrow R^3$, immersion

χ

Cycle Level Duality for SW Classes

- $F: M^2 \rightarrow R^3$, immersion
- $P_i: R^3 \rightarrow R^1{}_j = [E_j]$,

χ

Cycle Level Duality for SW Classes

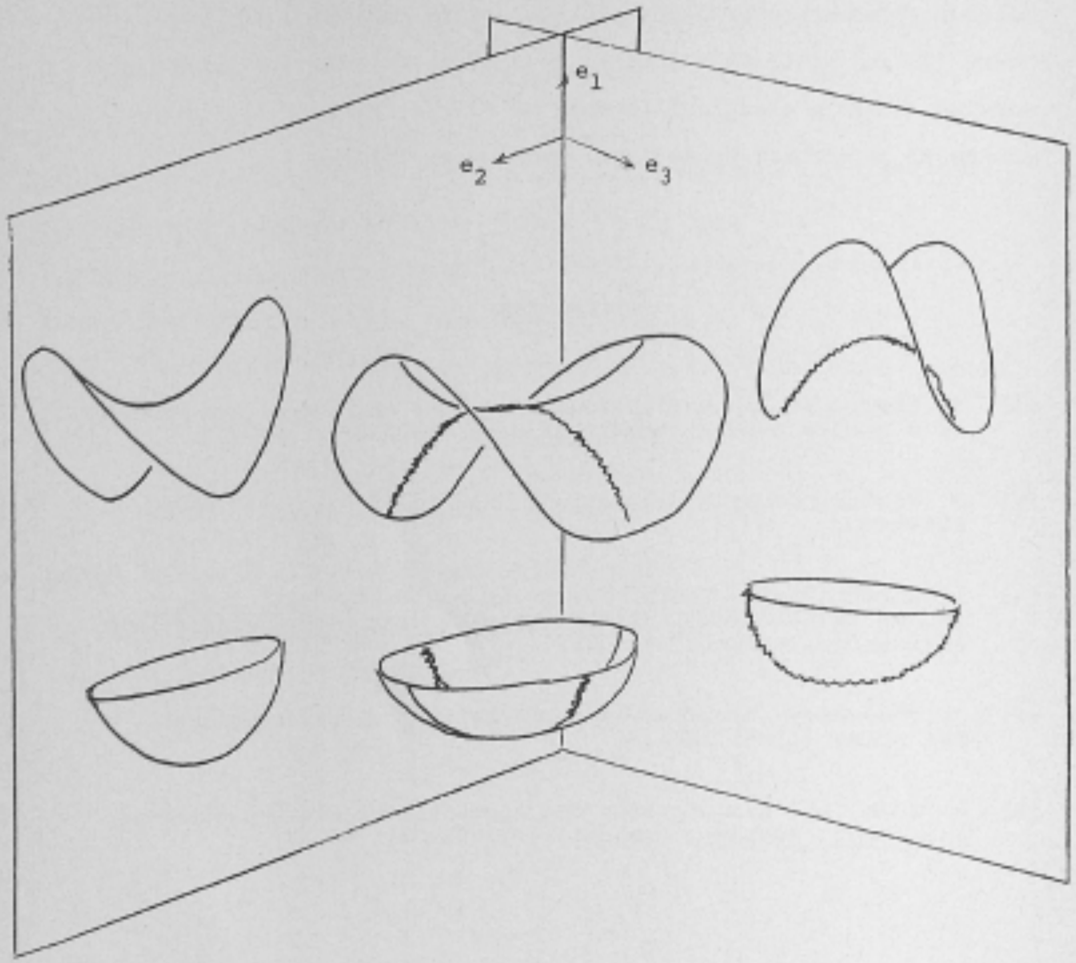
- $F: M^2 \rightarrow R^3$, immersion
- $P_i: R^3 \rightarrow R^1{}_j = [E_j]$,
- $P_3F: M^2 \rightarrow R^1{}_3$ χ

Cycle Level Duality for SW Classes

- F: $M^2 \rightarrow R^3$, immersion
- P_i: $R^3 \rightarrow R^1{}_j = [E_j]$,
- P_3F: $M^2 \rightarrow R^1{}_3$ x
- $S(P_3F) = \{x$ in $M \mid T_XF \perp R_3\}$

Cycle Level Duality for SW Classes

- $F: M^2 \rightarrow R^3$, immersion
- $P_i: R^3 \rightarrow R^1{}_j = [E_j]$,
- $P_3F: M^2 \rightarrow R^1{}_3$ $\quad\chi$
- $S(P_3F) = \{x \text{ in } M \mid T_xF \perp R_3\}$
- $W_2(M^2) \equiv \chi(M^2) \pmod 2$


e1
e2
e3

Cycle Level Duality for SW Classes

- F: $M^2 \rightarrow R^3$, immersion

Cycle Level Duality for SW Classes

- $F: M^2 \rightarrow R^3$, immersion
- $P_{ij}: R^3 \rightarrow R^2_{ij} = [E_i] \oplus [E_j]$

Cycle Level Duality for SW Classes

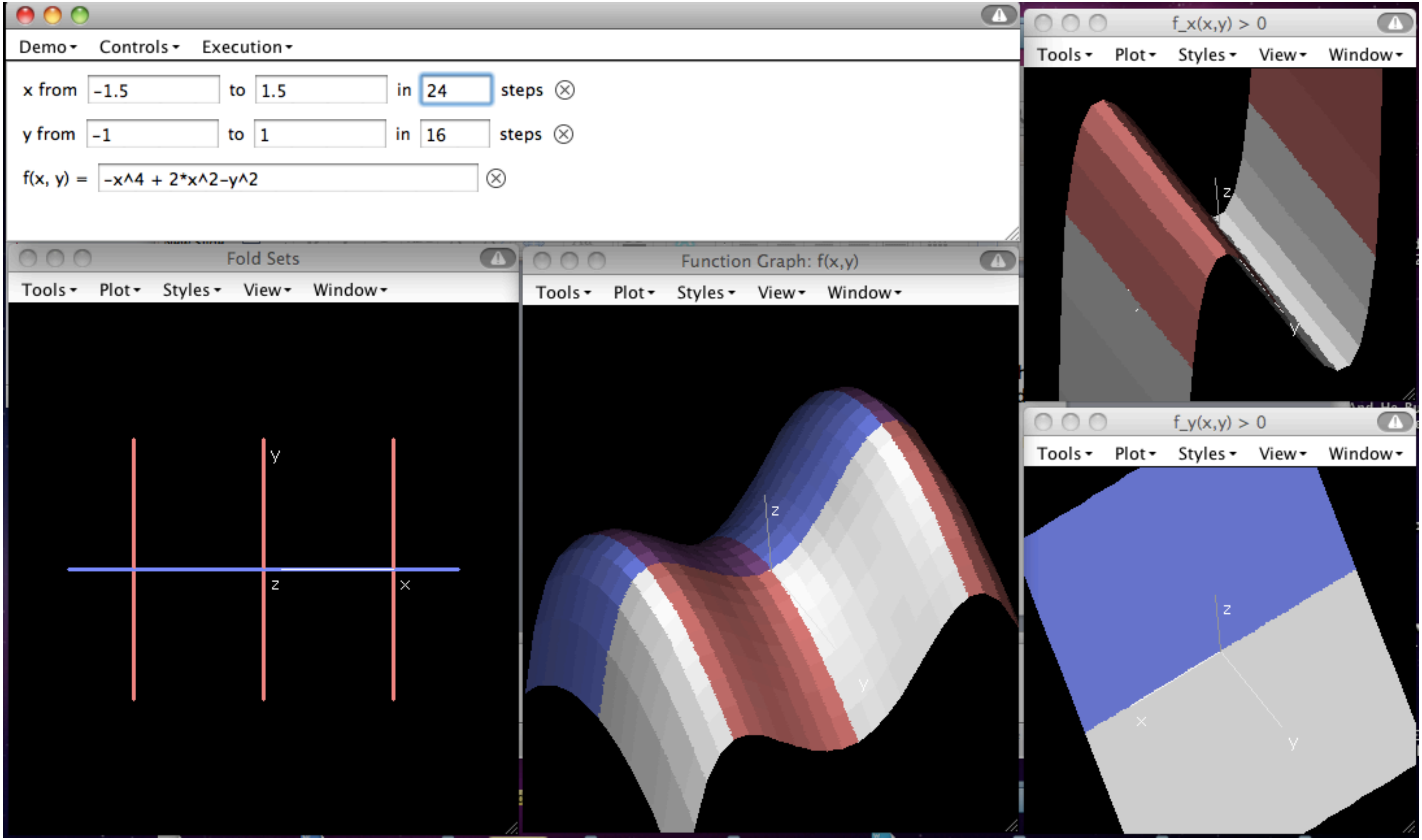
- $F: M^2 \rightarrow R^3$, immersion
- $P_{ij}:R^3 \rightarrow R^2_{ij} = [E_i] \oplus [E_j]$
- $S(P_{ij}F) = \{ x \text{ in } M \mid T_xF \perp R^2_{ij} \}$

Cycle Level Duality for SW Classes

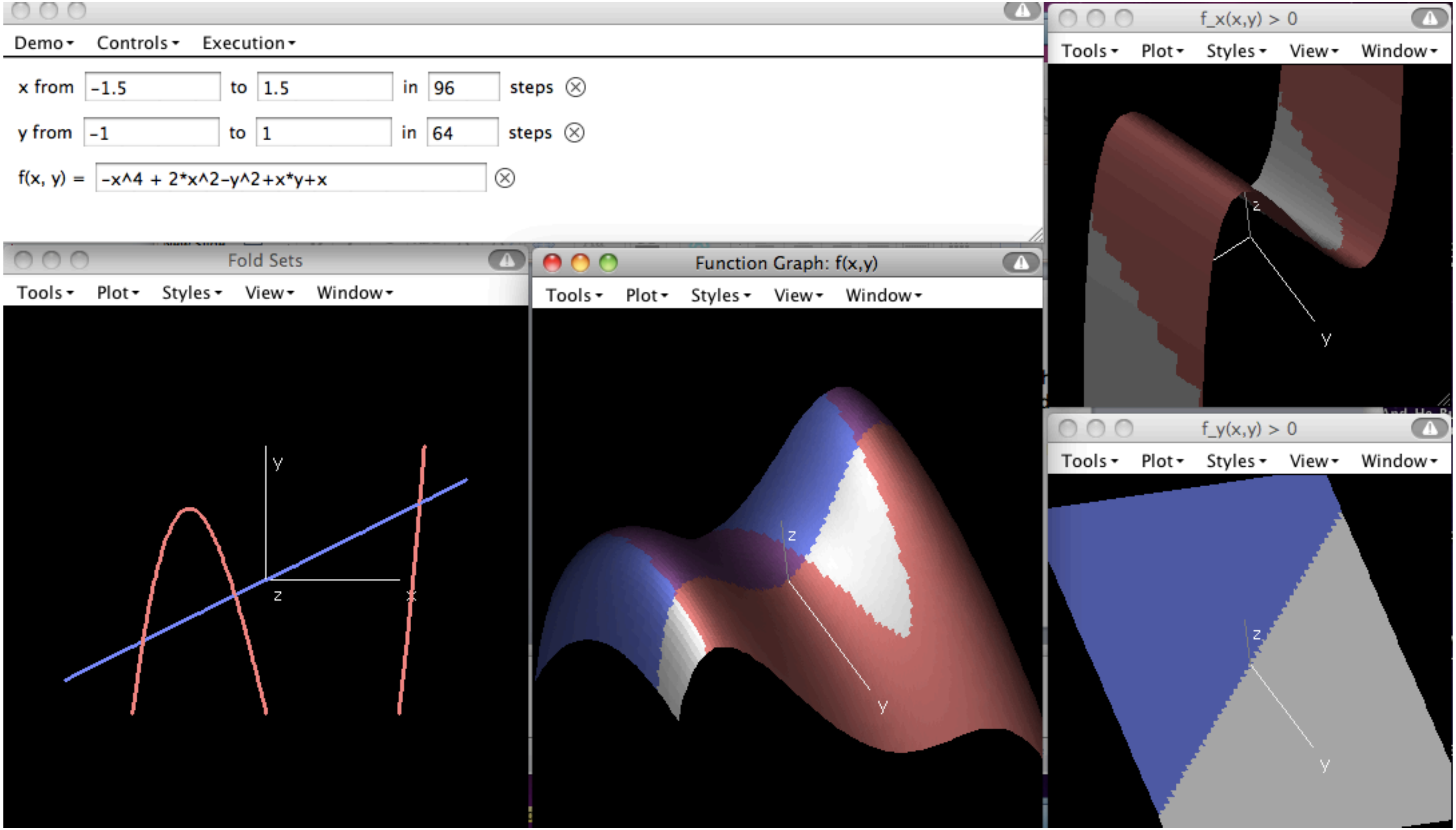
- $F: M^2 \rightarrow R^3$, immersion
- $P_{ij}:R^3 \rightarrow R^2_{ij} = [E_i] \oplus [E_j]$
- $S(P_{ij}F) = \{ x \text{ in } M \mid T_xF \perp R^2_{ij} \}$
- $S(P_{13}F) \cap S(P_{23}F) = S(P_3F)$

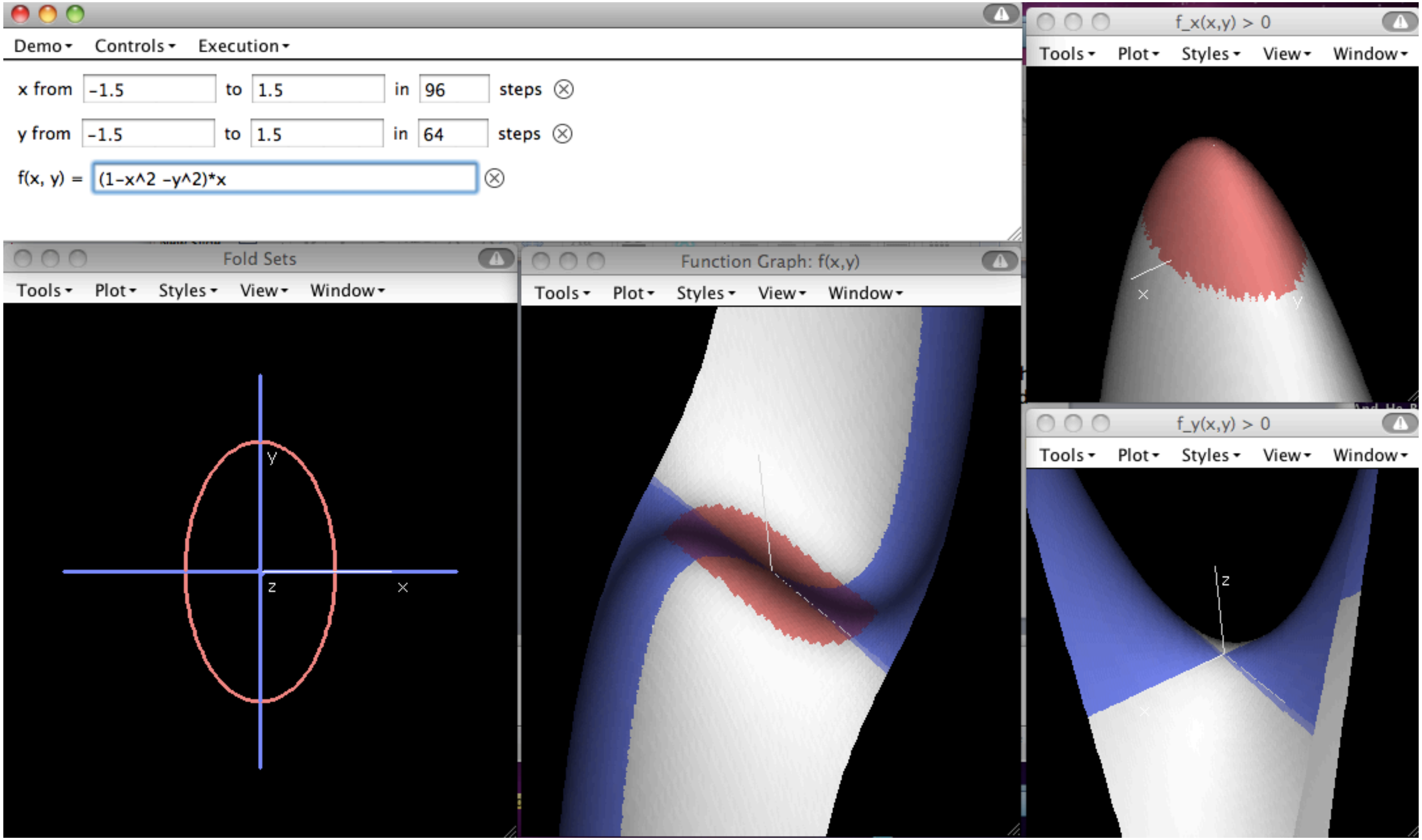
Cycle Level Duality for SW Classes

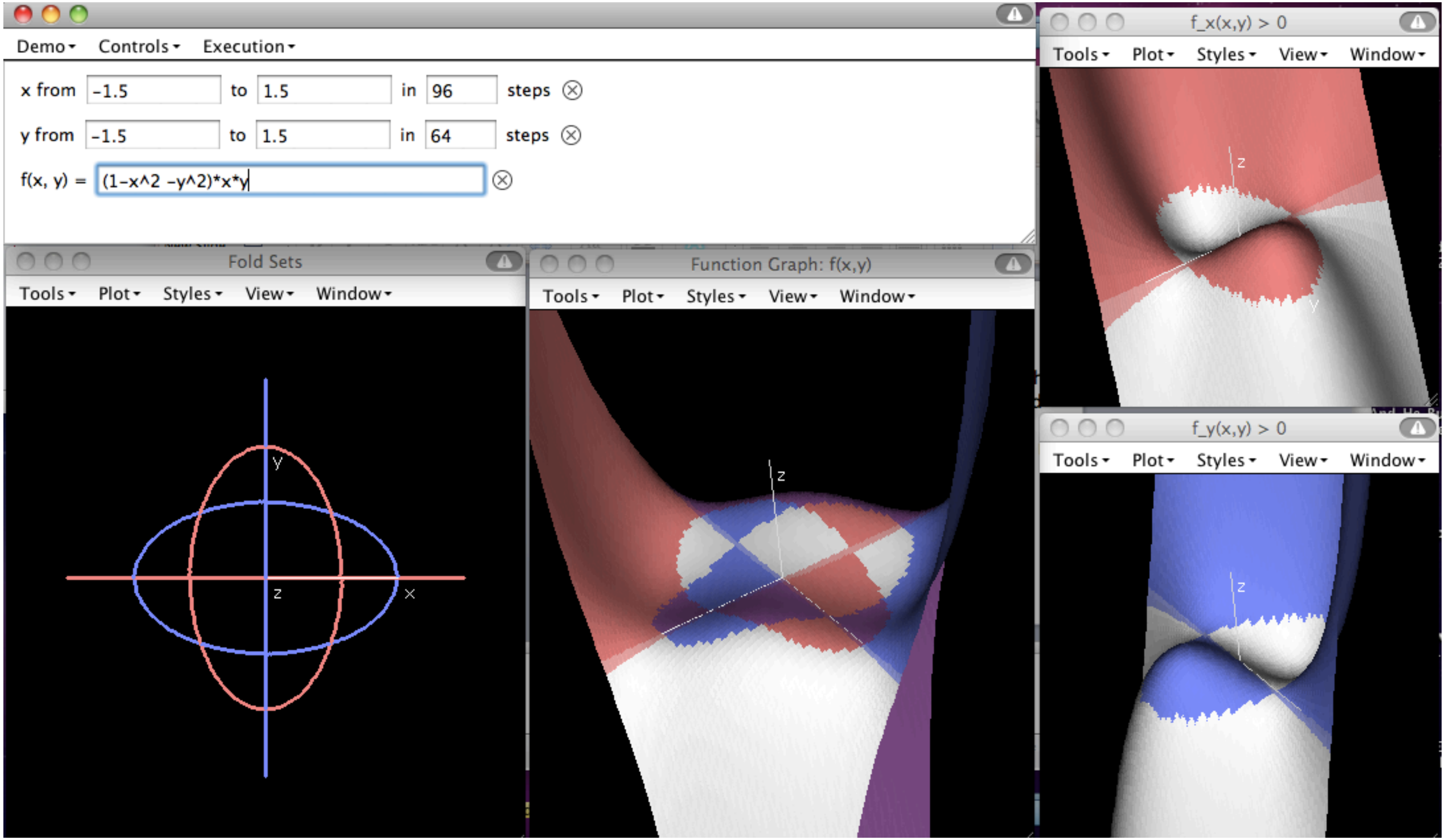
- $F: M^2 \rightarrow R^3$, immersion
- $P_{ij}: R^3 \rightarrow R^2_{ij} = [E_i] \oplus [E_j]$
- $S(P_{ij}F) = \{ x \text{ in } M \mid T_xF \perp R^2_{ij} \}$
- $S(P_{13}F) \cap S(P_{23}F) = S(P_3F)$
- $W_1 \cup \underline{W_1} = W_2$



Demo Controls Execution
x from -1.5 to 1.5 in 24 steps
y from -1 to 1 in 16 steps
f(x, y) = -x^4 + 2*x^2-y^2
Fold Sets
Tools Plot Styles View Window
y
z
x
Function Graph: f(x,y)
Tools Plot Styles View Window
z
y
f_x(x,y) > 0
Tools Plot Styles View Window
z
y
f_y(x,y) > 0
Tools Plot Styles View Window
z
x
y

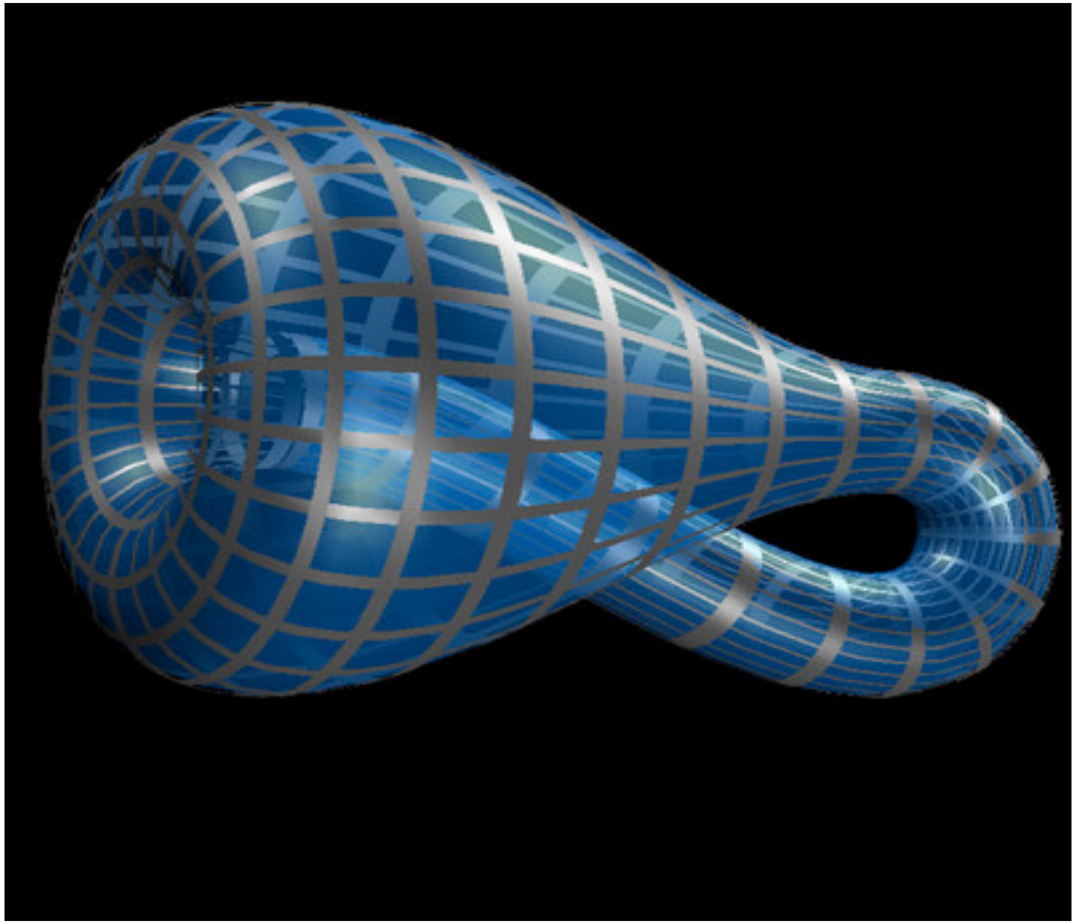

Demo
Controls
Execution
x from -1.5 to 1.5 in 96 steps
y from -1 to 1 in 64 steps
f(x, y) = -x^4 + 2*x^2-y^2+x*y+x
f_x(x,y) > 0
Tools
Plot
Styles
View
Window
Fold Sets
Function Graph: f(x,y)
f_y(x,y) > 0
y
z
x


Demo Controls Execution
x from -1.5 to 1.5 in 96 steps
y from -1.5 to 1.5 in 64 steps
f(x, y) = (1-x^2 -y^2)*x
Fold Sets
Function Graph: f(x,y)
f_x(x,y) > 0
f_y(x,y) > 0
Tools Plot Styles View Window
y
z
x

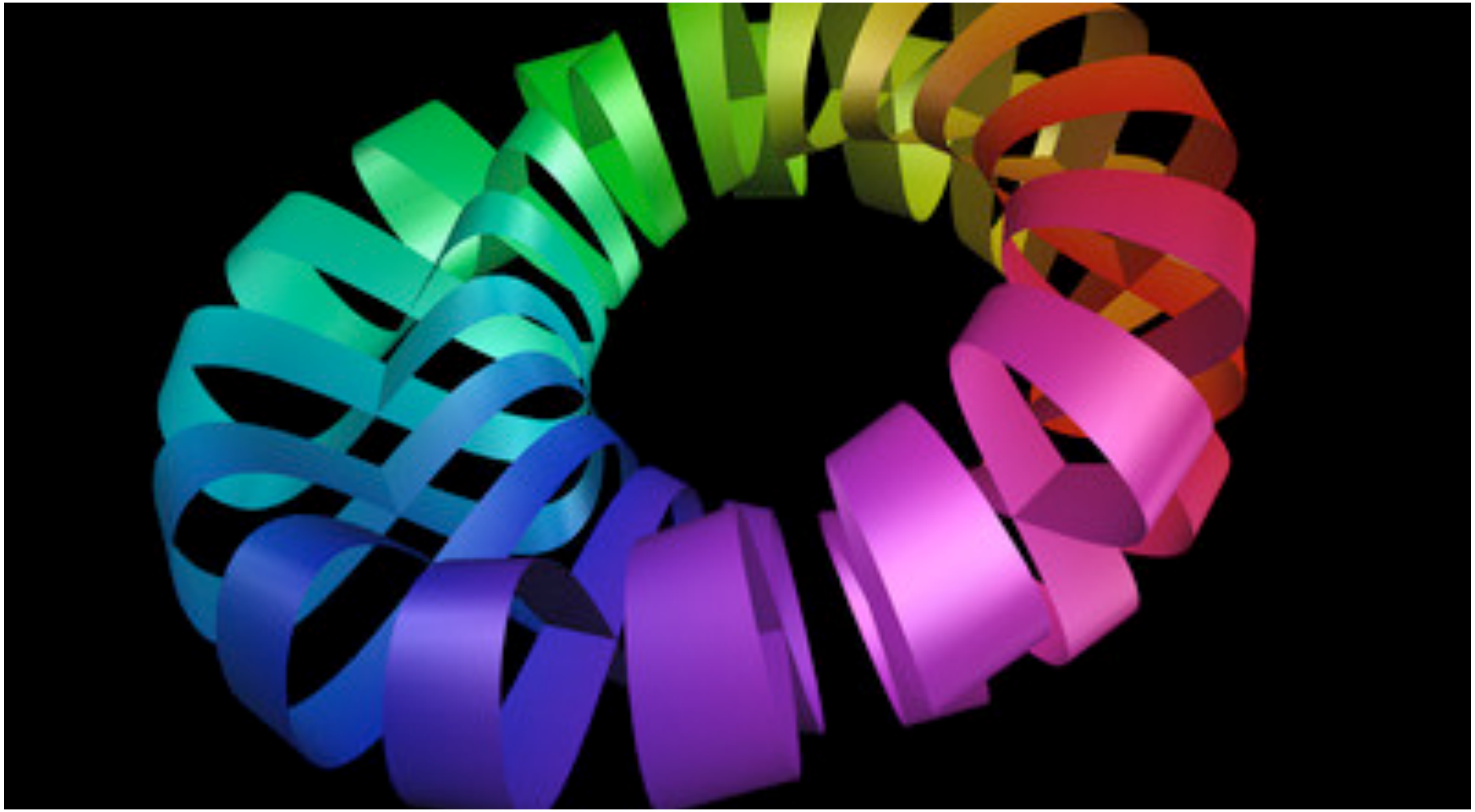


Demo
Controls
Execution
x from -1.5 to 1.5 in 96 steps
y from -1.5 to 1.5 in 64 steps
f(x, y) = (1-x^2 -y^2)*x*y
Fold Sets
Tools
Plot
Styles
View
Window
y
z
x
Function Graph: f(x,y)
Tools
Plot
Styles
View
Window
z
f_x(x,y) > 0
Tools
Plot
Styles
View
Window
z
y
f_y(x,y) > 0
Tools
Plot
Styles
View
Window
z

GLASSBLOWERS KLEIN BOTTLE



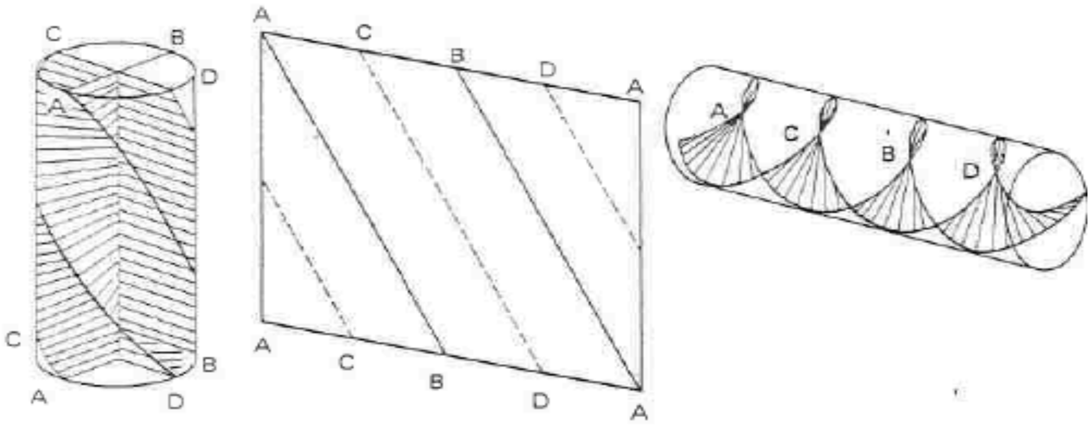
CANDY STRIPE KLEIN BOTTLE



Minimal Surfaces

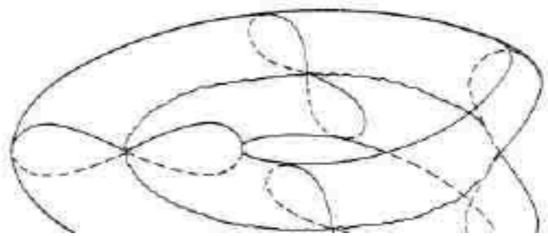
In the

Bicylinder Boundary

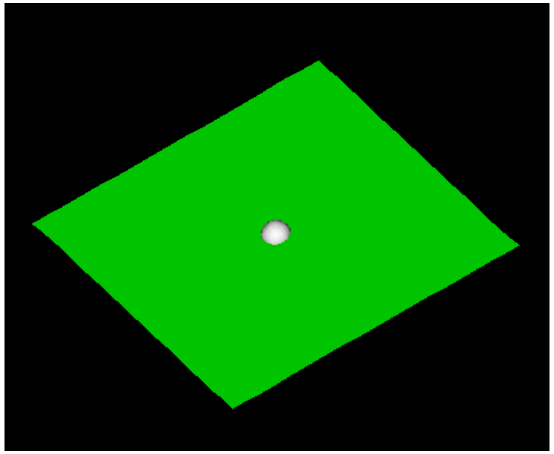


Remark. The minimal immersions obtained in this way for the tor and the Klein bottle are not flat — the helicoid pieces are of negative curv ture, and there is a positive curvature contribution at the edges where t surface meets the torus "rim."

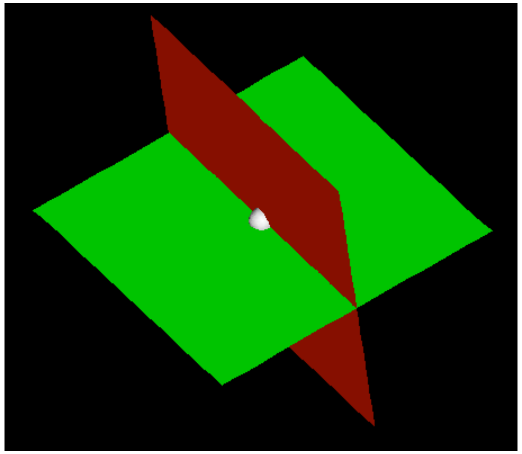
Remark. This immerson of the Klein bottle into C^3 leads to an imm sion into $\mathbb{R}^3$ which is qualitatively different from the "usual" immersic To describe this immersion, we take an immersed circle in the $x - z$ pla in the form of symmetric "figure eight" with its intersection point on t circle $x^2 + y^2 = 1$, $z = 0$. We then construct a "twisted surface of revol tion" by letting the intersection with the half-space bounded by the z-a: and making an angle of $\theta(0 \leq \theta < 2)$ with the half-space $y = 0$, $x \geq 0$ given by rotating the original curve about its center through an angle of θ When we return to the original position, the two halves of the figure eig have been interchanged so we obtain a Klein bottle. The top and botto points of the eight together describe a single closed curve and the complime of this curve is the union of two Mobius bands intersecting along the circ $x^2 + y^2 = 1$, $z = 0$.



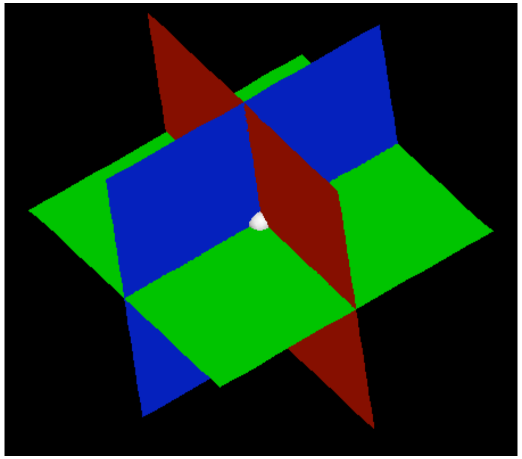
Multiple Points



Single point



Double point



Triple point

Über die Curvatura integra und die Topologie geschlossener Flächen.

Von

Werner Boy in Leipzig.

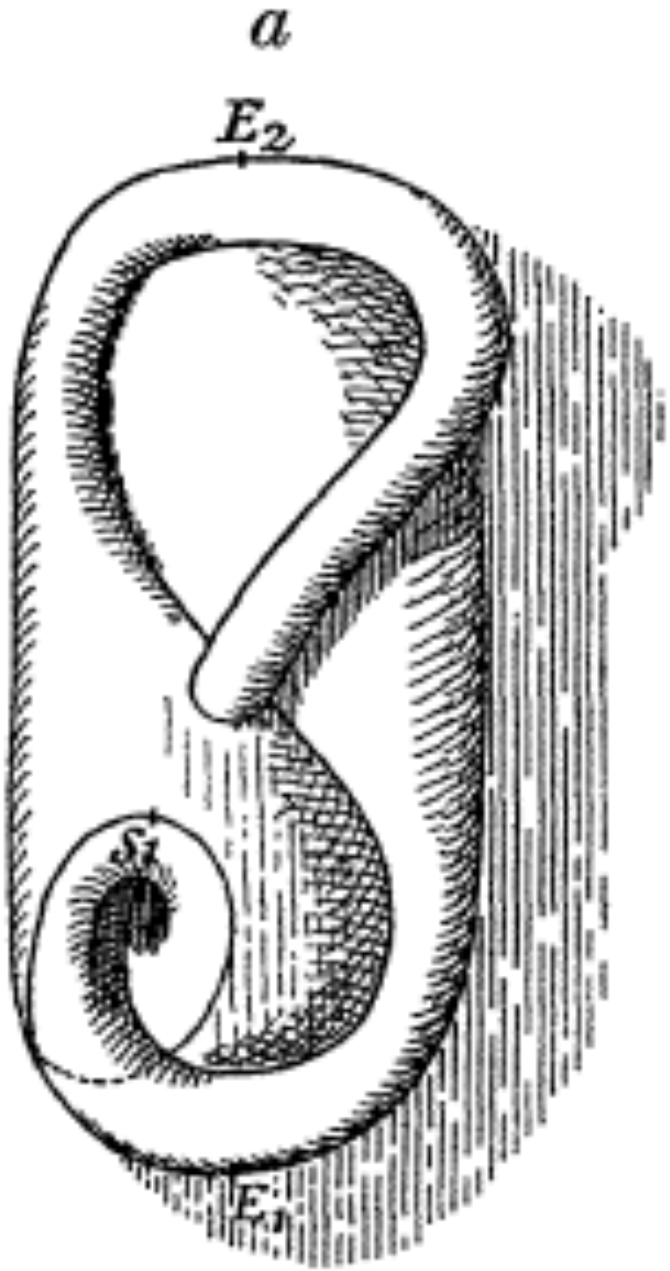


Fig. 17 a.

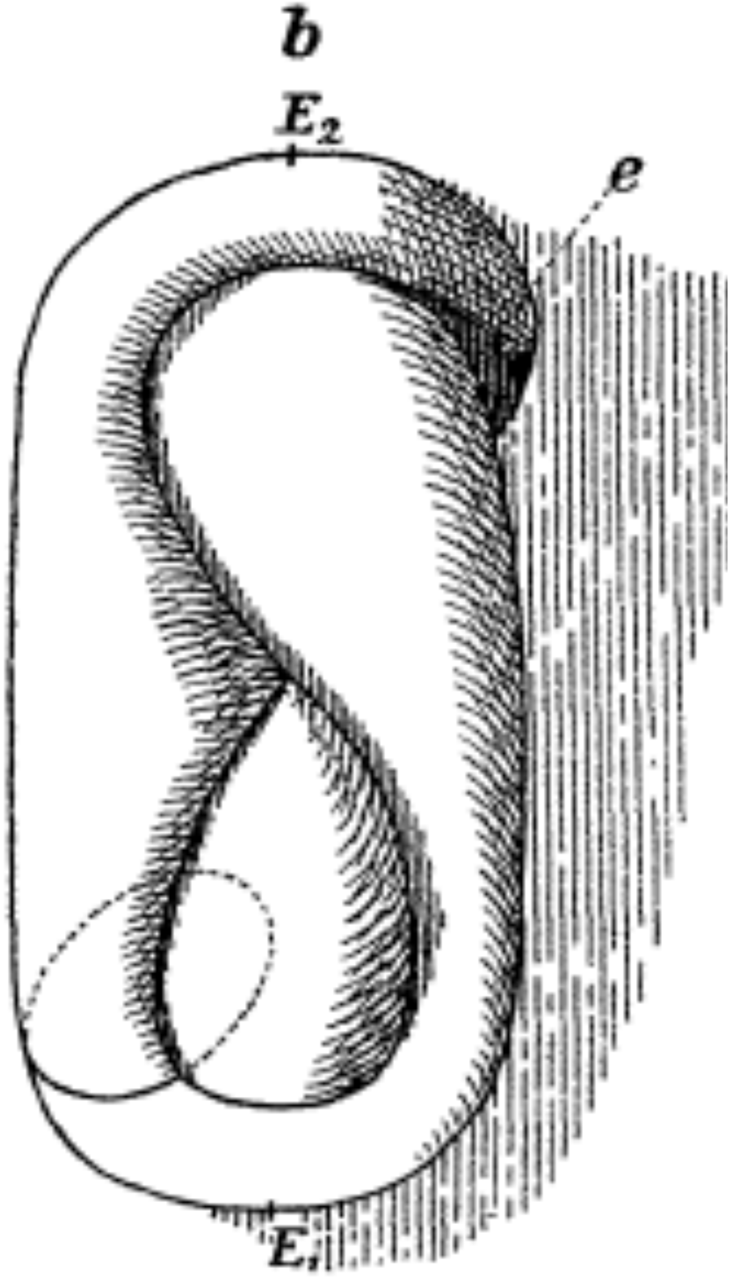
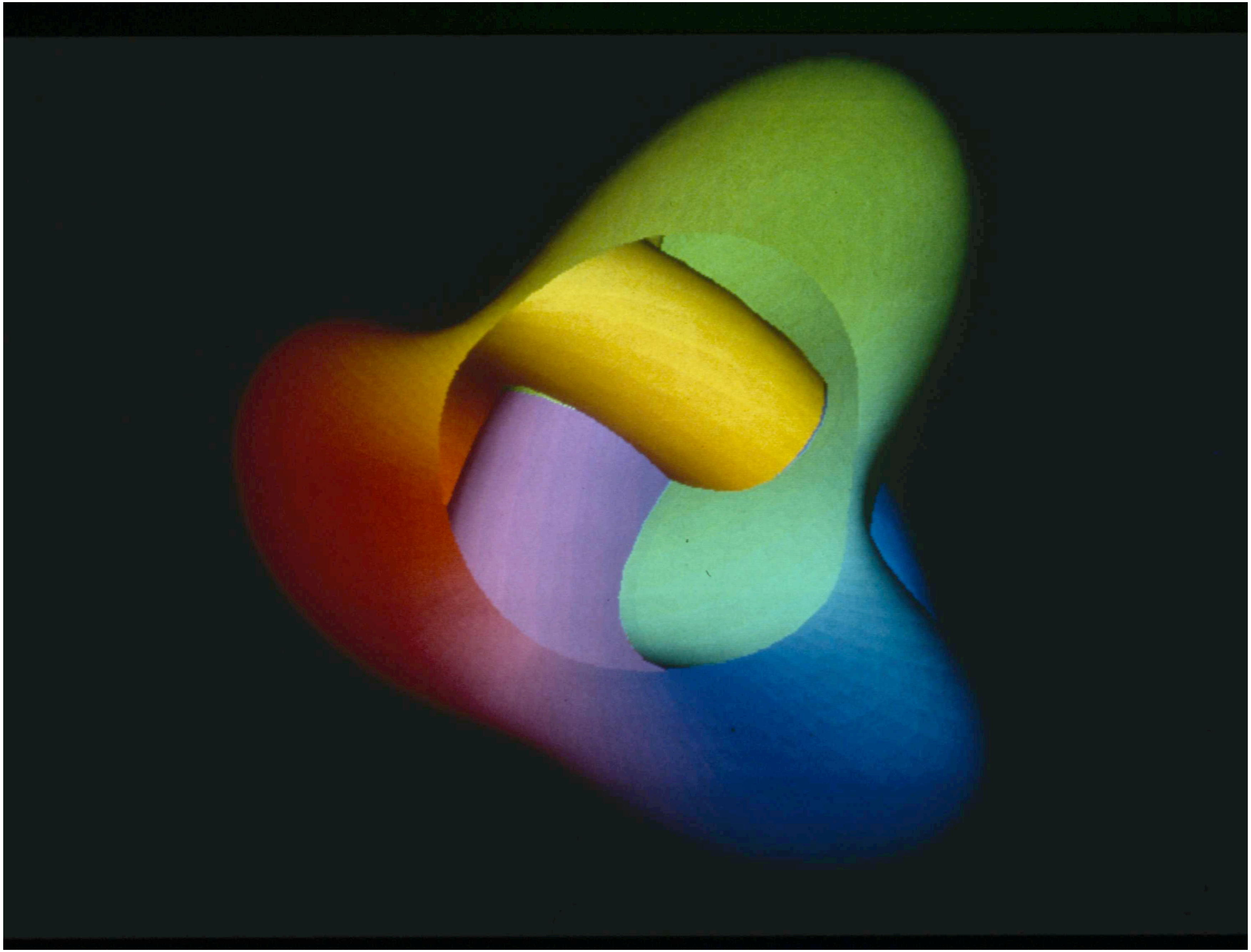
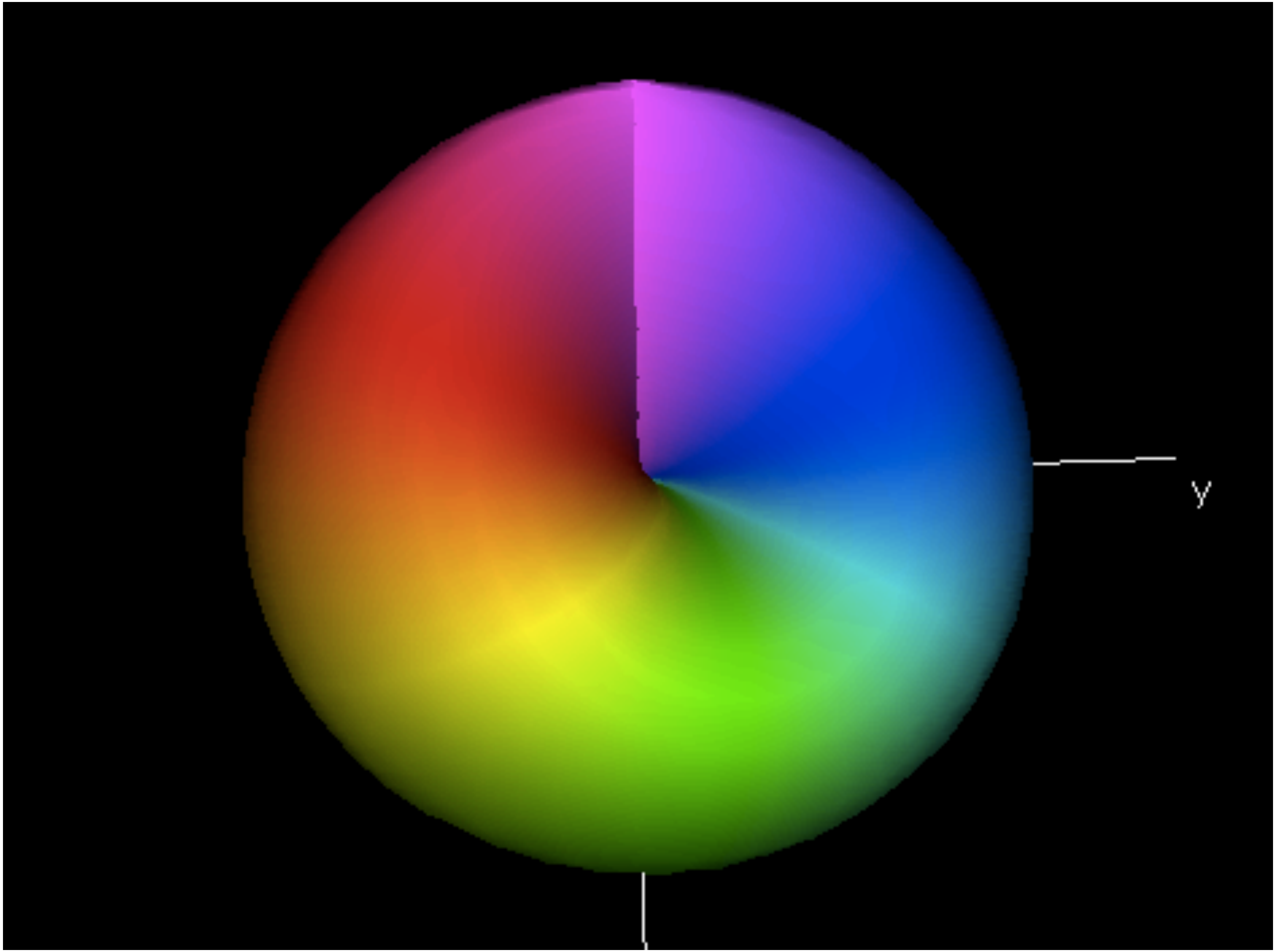
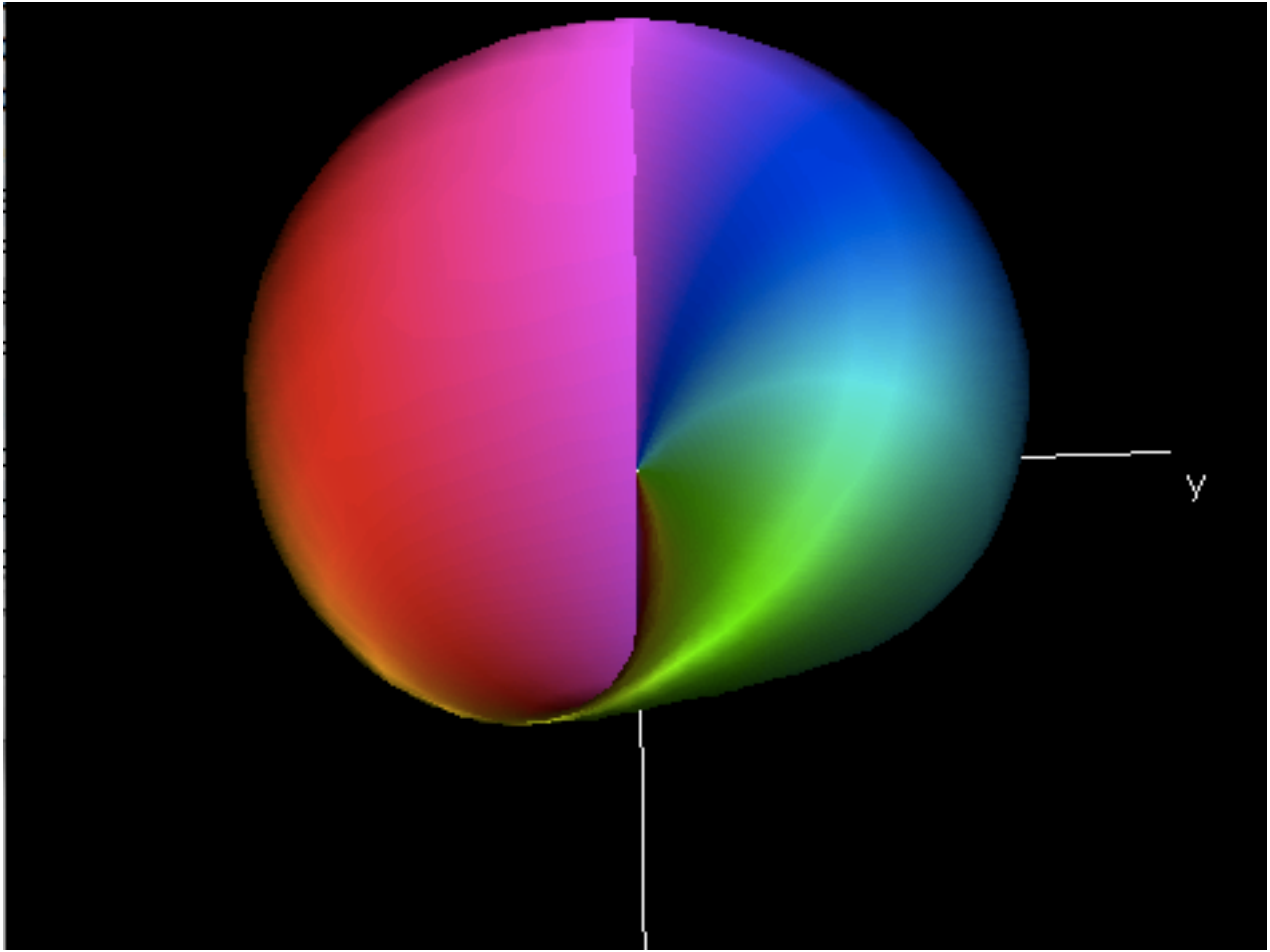


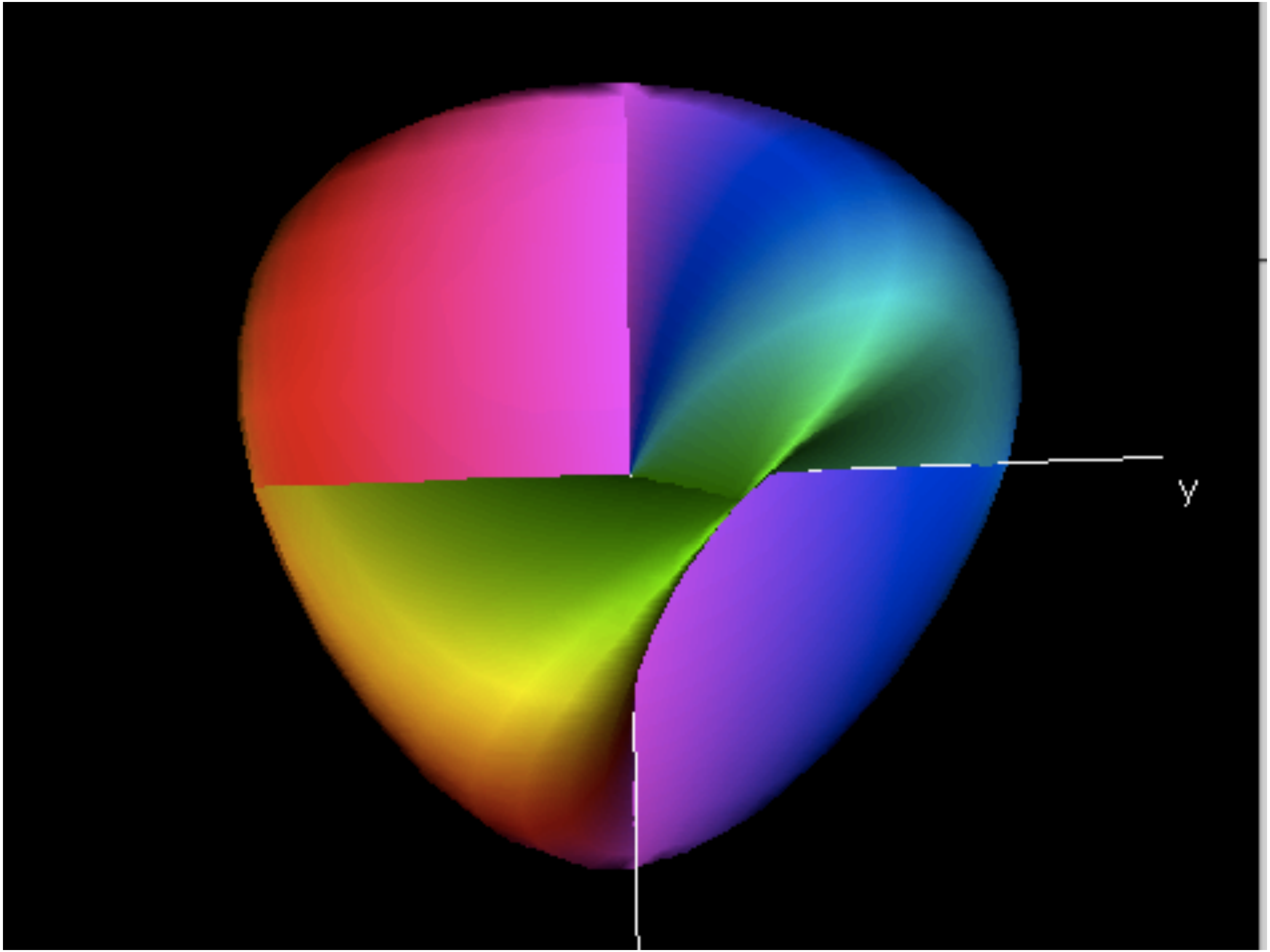
Fig. 17 b.



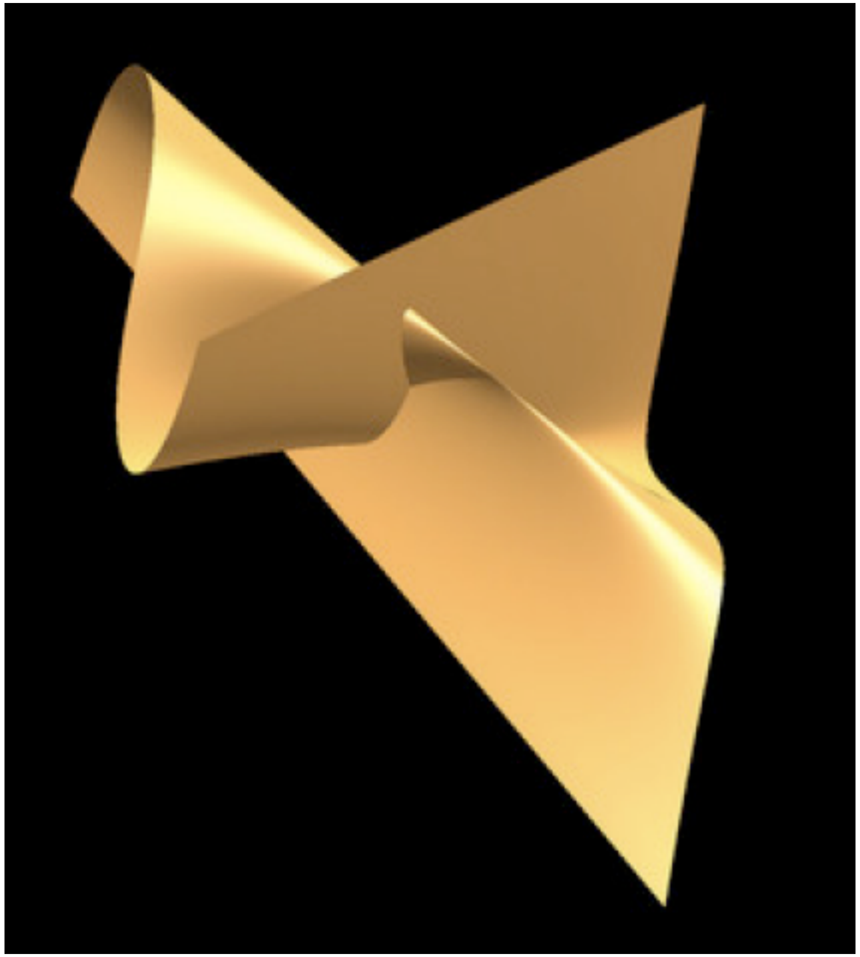


y


y


y

Triple Point Twist



Triple Point Theorem

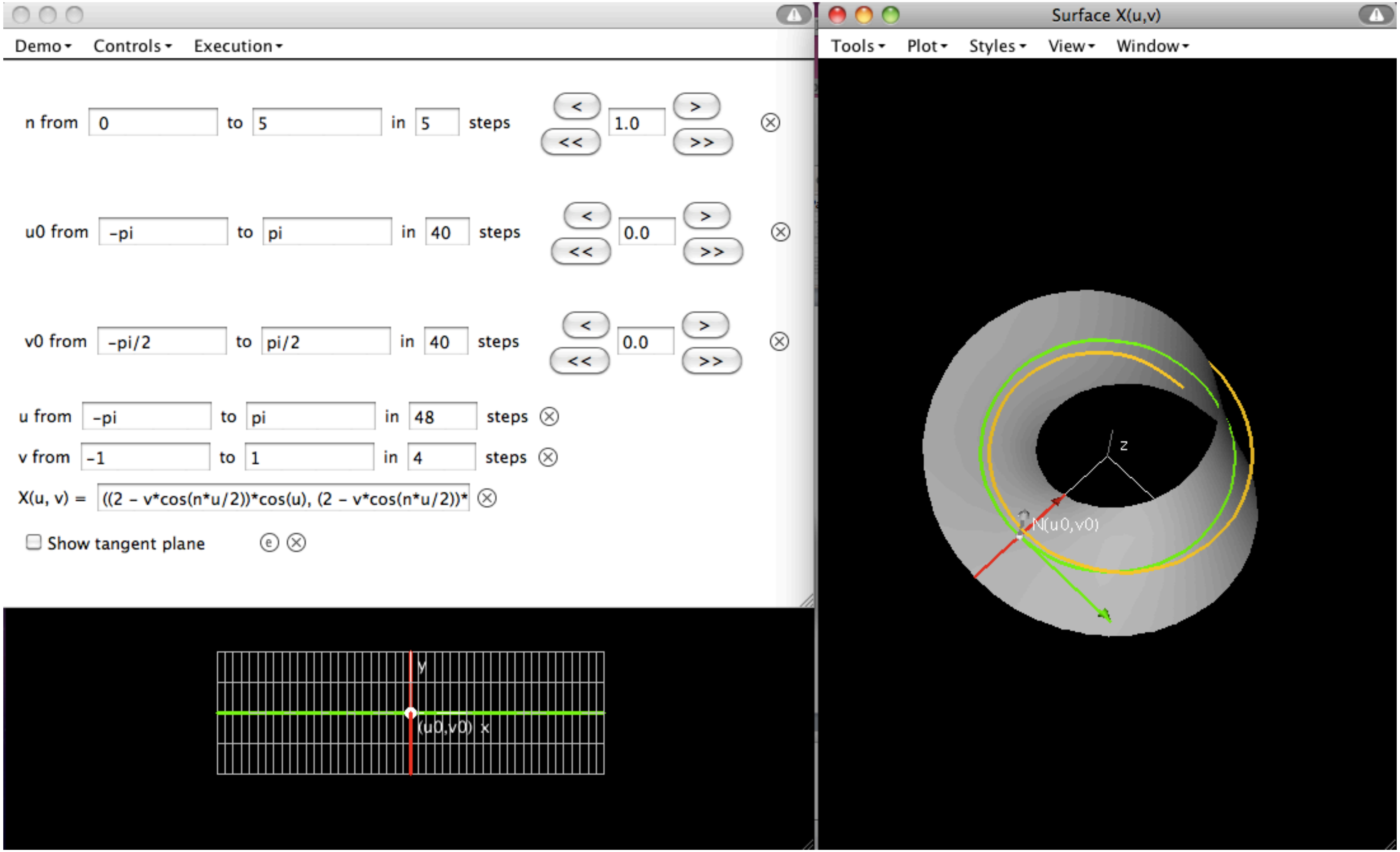
- Theorem: For an immersion, $f{:}M^2 \rightarrow R^3$, with $T(f)$ triple points,

  the Euler characteristic $\chi(M)$

  is congruent modulo 2

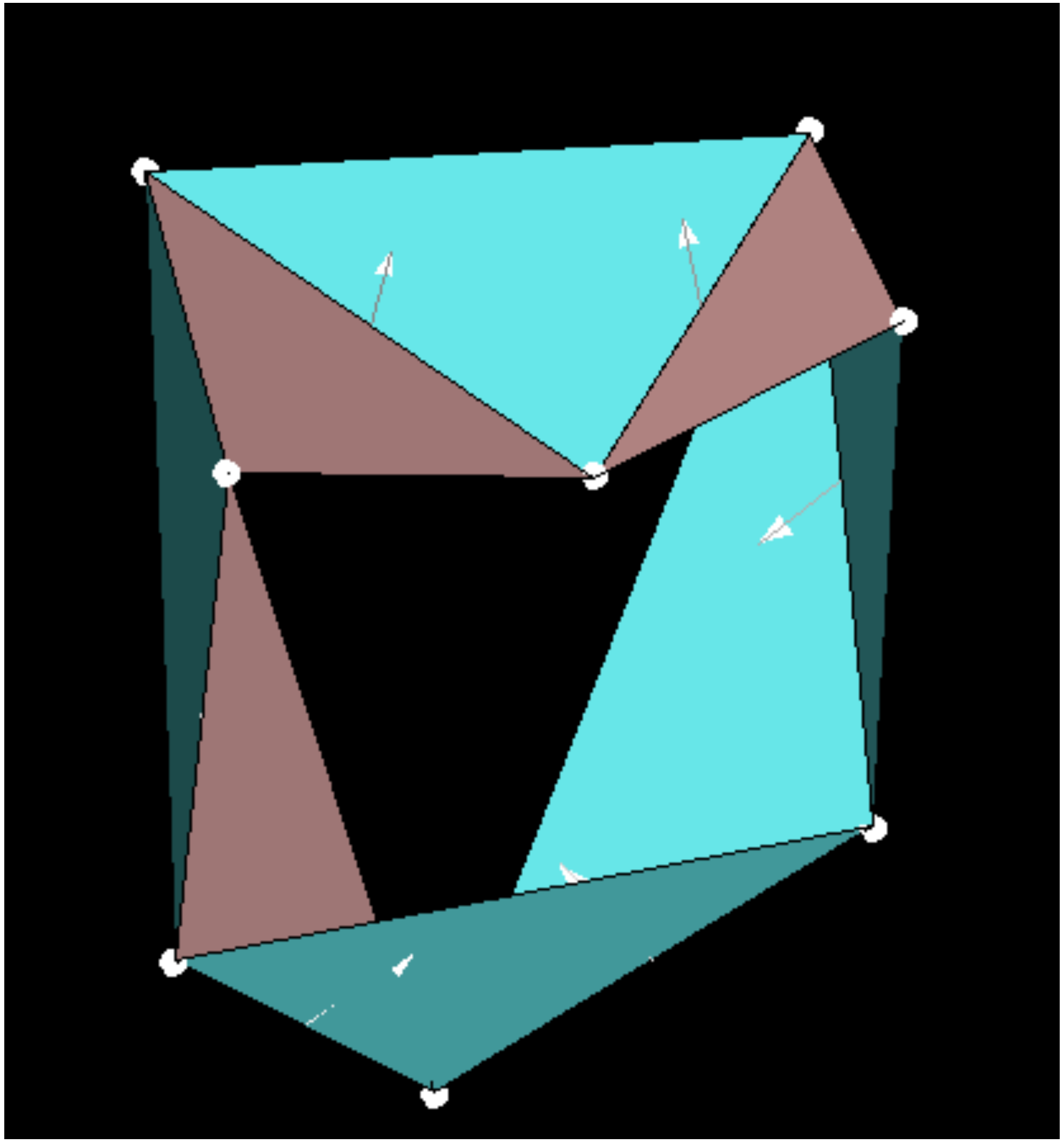
  to the number of triple points:

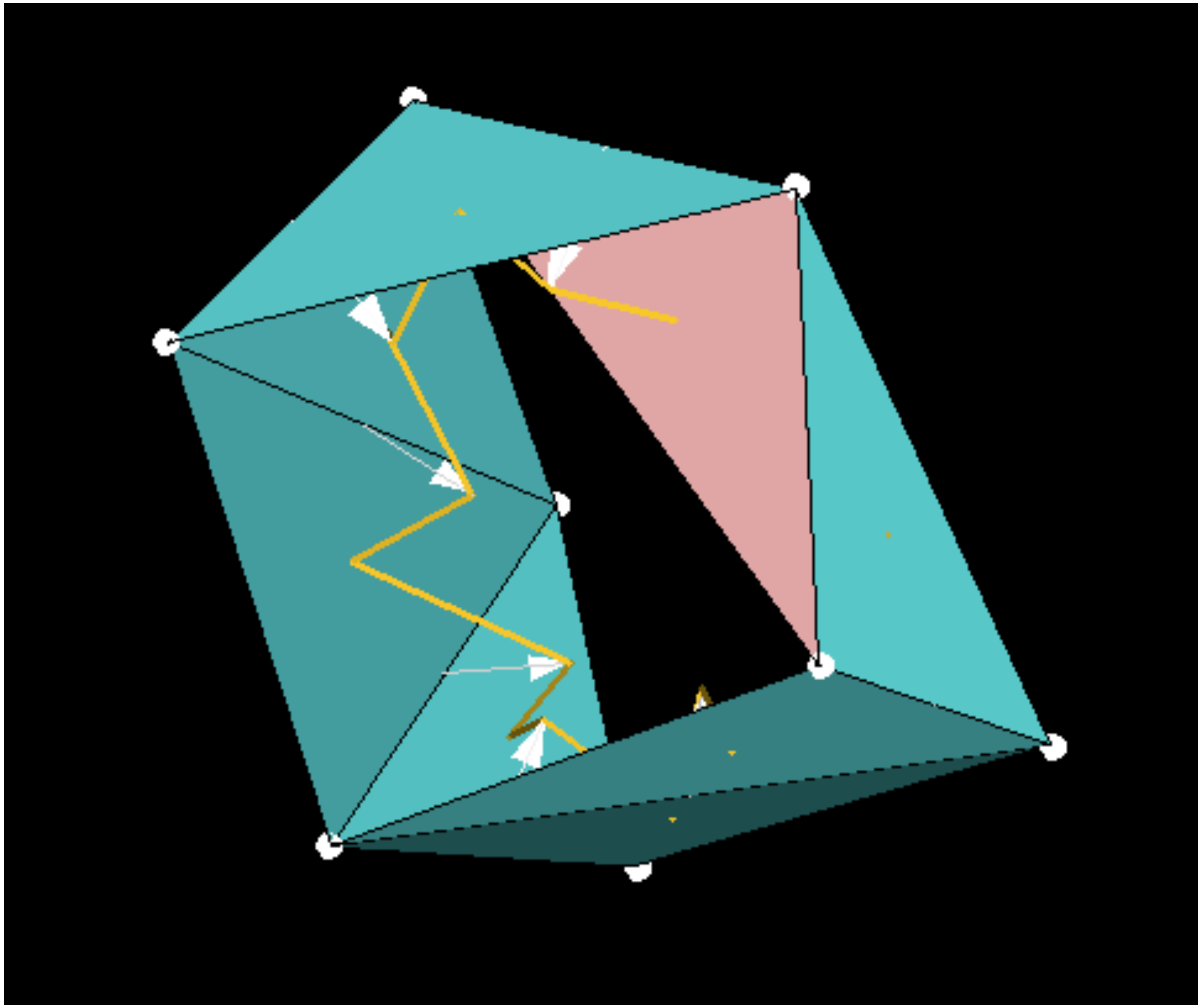
  $\chi(M) \equiv T(f) \bmod(2)$.

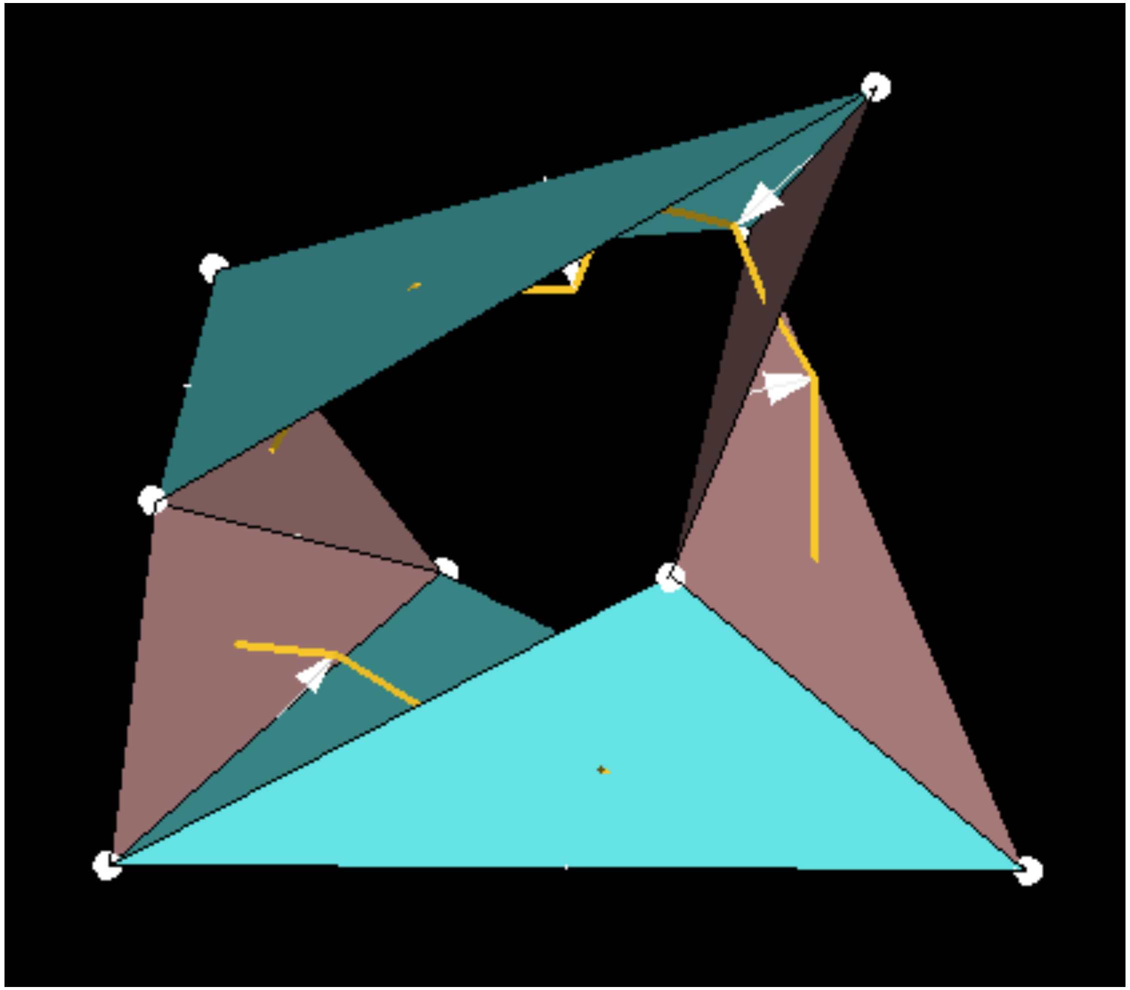


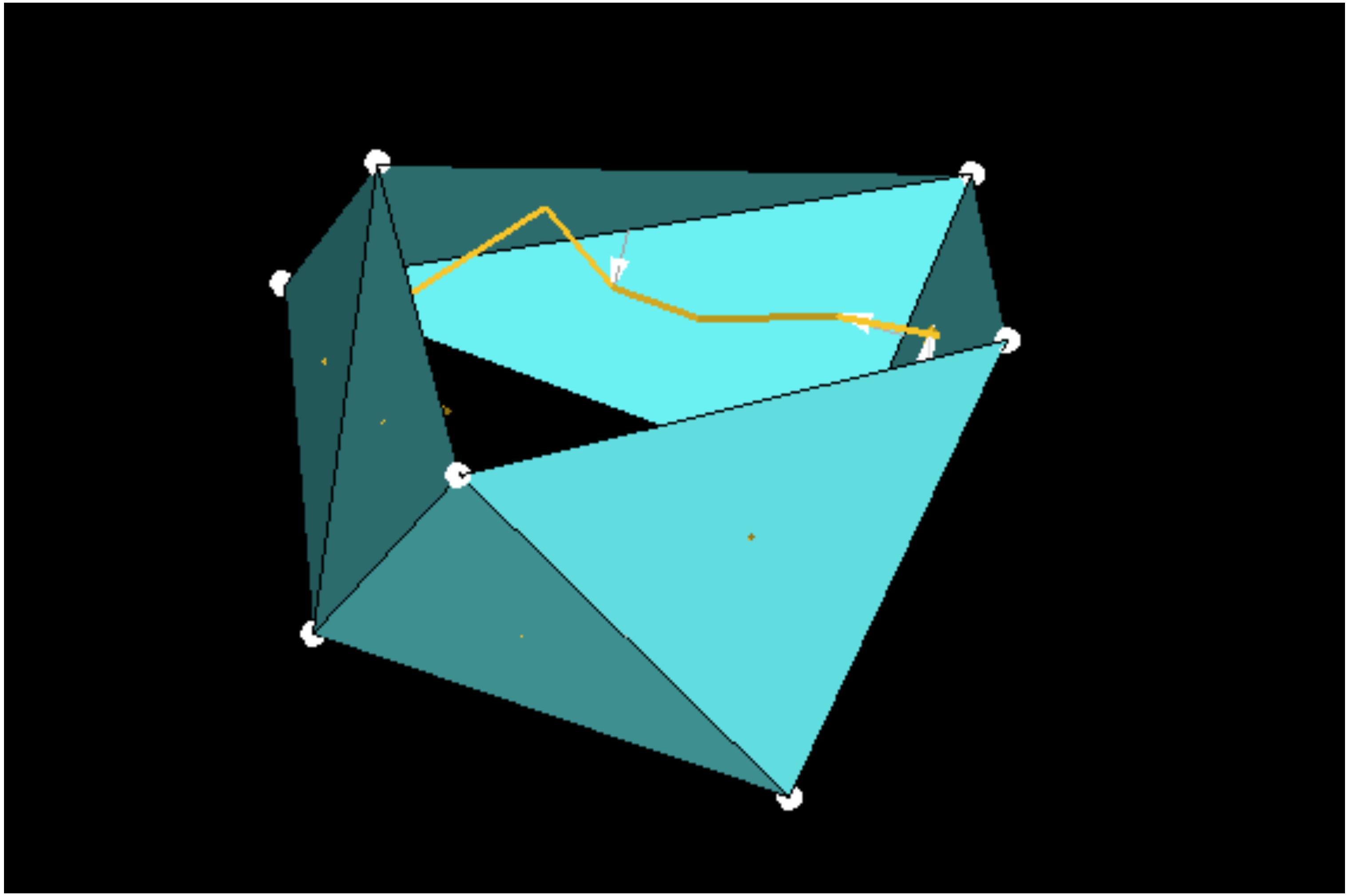
Demo
Controls
Execution
n from 0 to 5 in 5 steps
1.0
u0 from -pi to pi in 40 steps
0.0
v0 from -pi/2 to pi/2 in 40 steps
0.0
u from -pi to pi in 48 steps
v from -1 to 1 in 4 steps
X(u, v) = ((2 - v*cos(n*u/2))*cos(u), (2 - v*cos(n*u/2))*
Show tangent plane
y
(u0,v0)
x
Surface X(u,v)
Tools
Plot
Styles
View
Window
z
N(u0,v0)

Characteristics:	Torus	Möbius
• #(boundary curves)	2	1
• #(pieces of complement of X)	2	1
• #(X ∩ tangential variation)	Even	Odd
• #(X ∩ normal variation)	Even	Odd
• #(X ∩ fold cycle of plane map)	Even	Odd
• #(X ∩ double point preimage)	Even	Odd









Characteristics:	Torus	Möbius
• #(boundary curves)	2	1
• #(pieces of complement of X)	2	1
• #(X ∩ tangential variation)	Even	Odd
• #(X ∩ normal variation)	Even	Odd
• #(X ∩ fold cycle of plane map)	Even	Odd
• #(X ∩ double point preimage)	Even	Odd
• #(X ∩ inflection chain or H = 0)	Even	Odd

Characteristics:	Torus	Möbius
• #(boundary curves)	2	1
• #(pieces of complement of X)	2	1
• #(X ∩ tangential variation)	Even	Odd
• #(X ∩ normal variation)	Even	Odd
• #(X ∩ fold cycle of plane map)	Even	Odd
• #(X ∩ double point preimage)	Even	Odd
• #(X ∩ inflection chain or H = 0)	Even	Odd

BMS Student Collaborators

- Barbara Jablonska (John Sullivan)
- Felix Günther (Alexander Bobenko)
- Charles Gunn (Ulrich Pinkall)

Barbara Jablonska

Einladung

zu der mündlichen Prüfung von:

Frau Master of Science Barbara Jablonska
am Mittwoch, dem 23. Mai 2012, um 14 Uhr, s. t.
im Raum MA 415 des Mathematikgebäudes,
10623 Berlin, Straße des 17. Juni 136

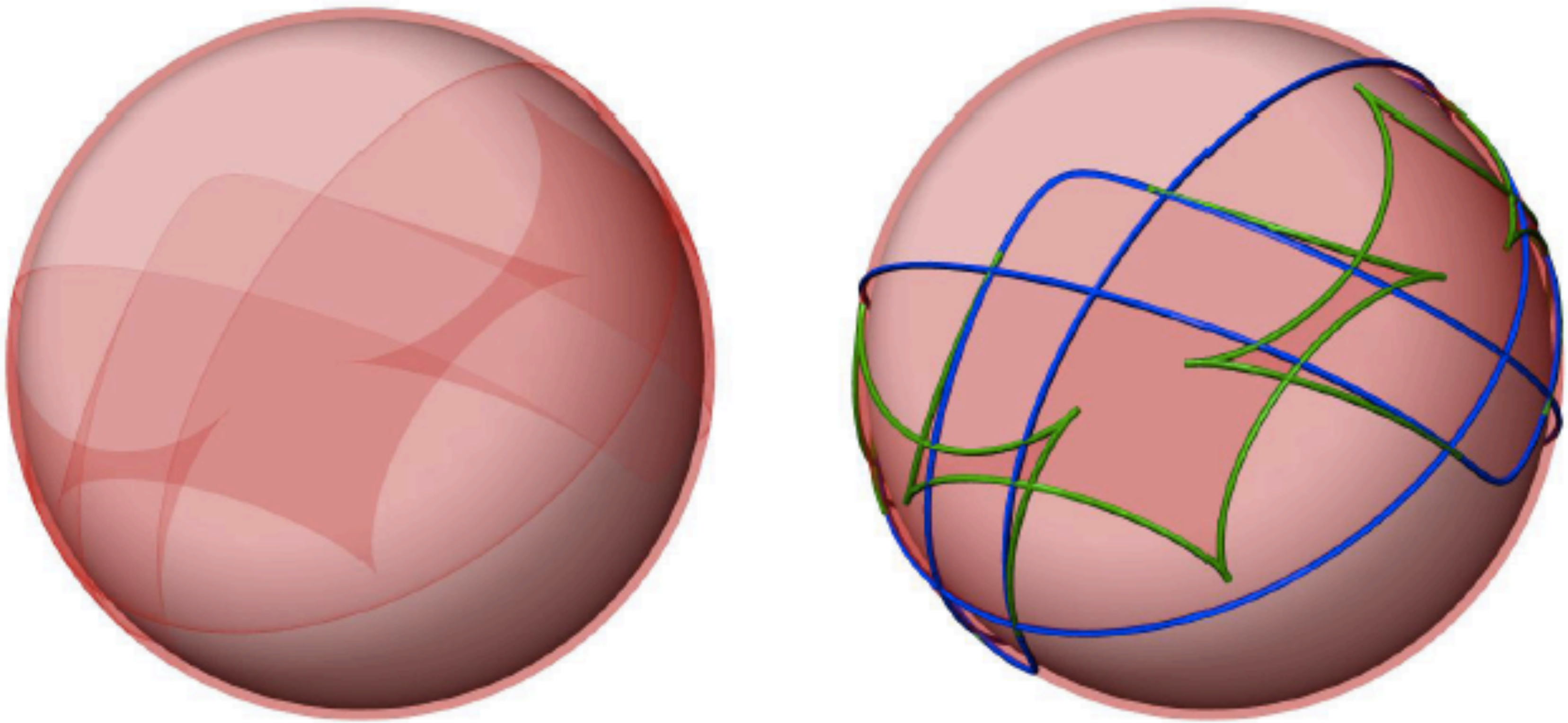
zur Erlangung des akademischen Grades "Doktorin der Naturwissenschaften"

Thema der Dissertation:

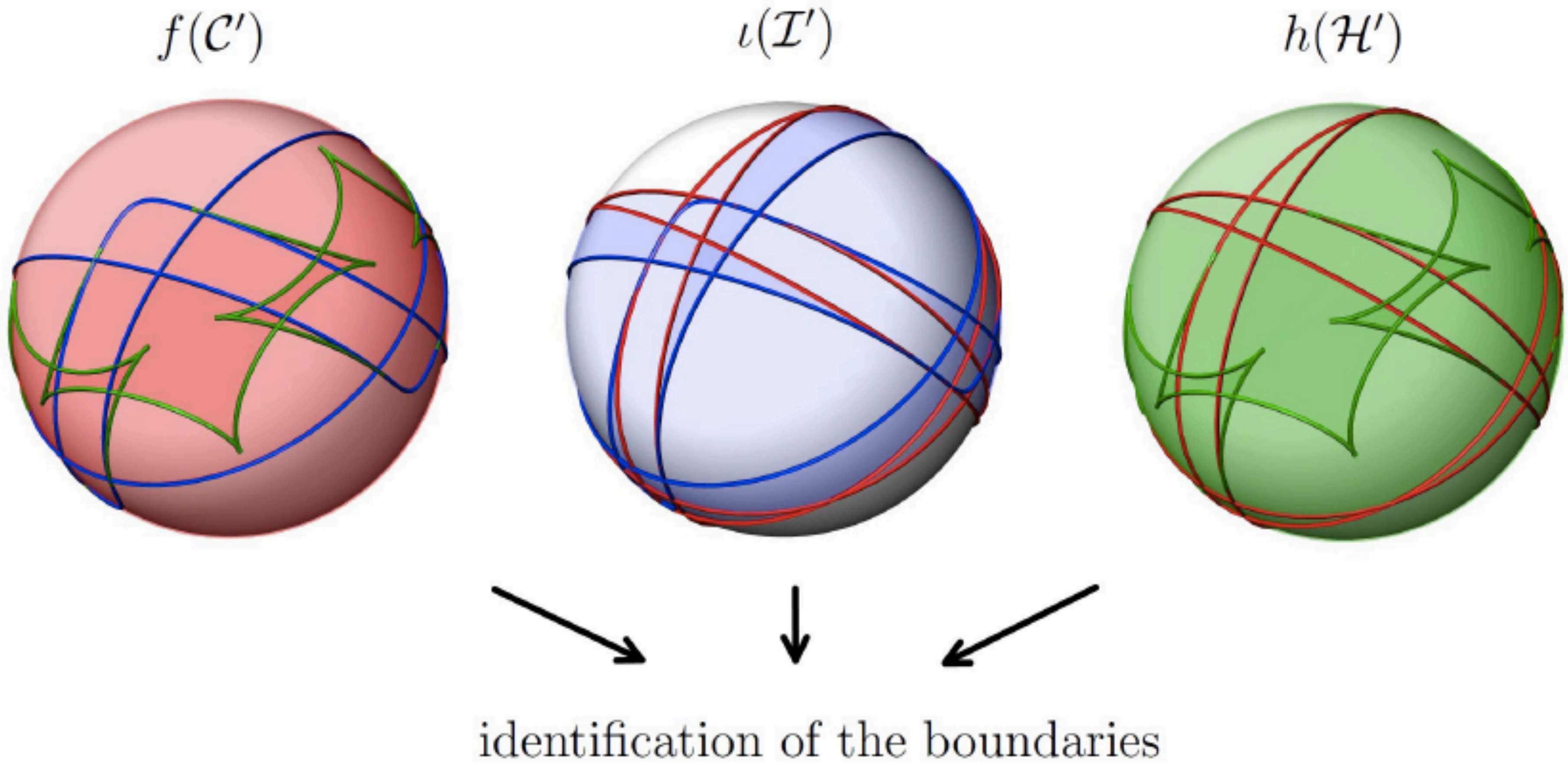
"Surfaces associated to a space curve:
A new proof of Fabricius-Bjerre's Formula"

Promotionsausschuss:

Vorsitzender:	**Prof. Dr. F. Tröltzsch**
Gutachter:	**Prof. Dr. J. M. Sullivan**
	Prof. Dr. T. Banchoff (Brown University USA)



Barbara Jablonska Ph. D. Thesis


$f(\mathcal{C}')$
$\iota(\mathcal{I}')$
$h(\mathcal{H}')$
identification of the boundaries

Felix Guenther

The maximum number of intersections of two simple polygons

Felix Günther*

Abstract

We determine the maximum number of intersections between two simple polygons with p and q vertices, respectively, in the plane. The case where p or q is even is quite easy and already known, but when p and q are both odd, the problem is more difficult. We prove that the conjectured maximum $(p-1)(q-1)+2$ is correct for all odd p and q.

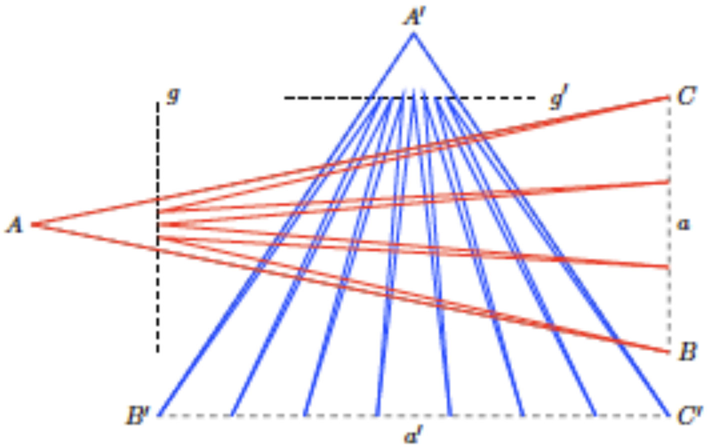


Figure 1: Two simple polygons with 8 and 16 vertices intersecting 128 times

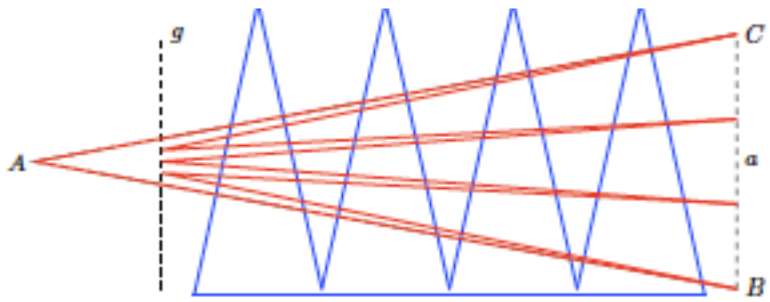


Figure 2: Two simple polygons with 8 and 9 vertices intersecting 64 times

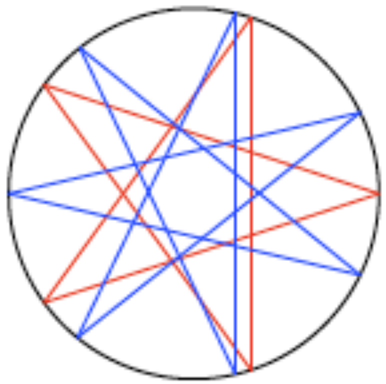


Figure 3: Two non-simple polygons with 5 and 7 vertices intersecting 28 times

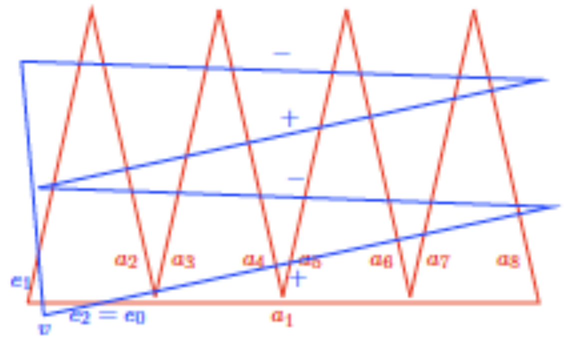
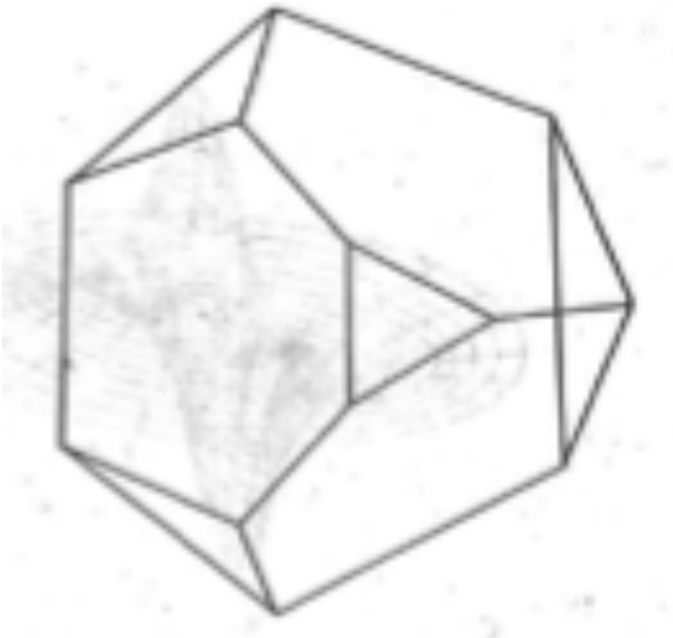


Figure 5: Two simple polygons with 5 and 9 vertices intersecting 34 times

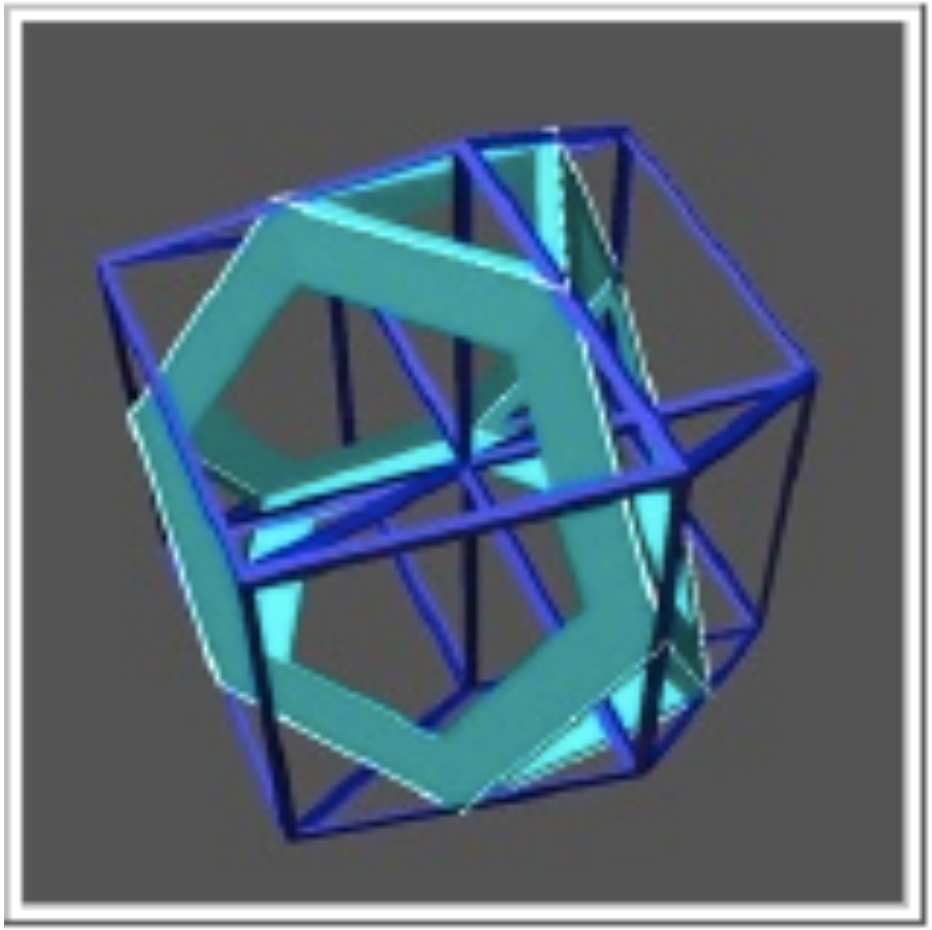
Charles Gunn

Hopf Circles on the Flat Torus in S^3

Truncated Tetrahedron in R^4



Reinhardtsbrunn
1982



TFB Conference 2003

Truncated Tetrahedron

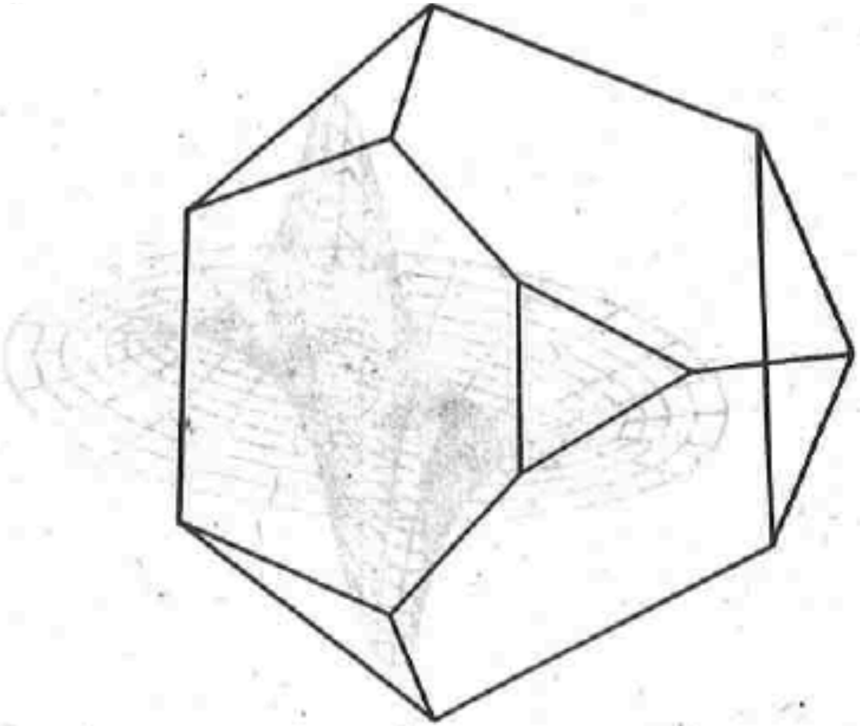
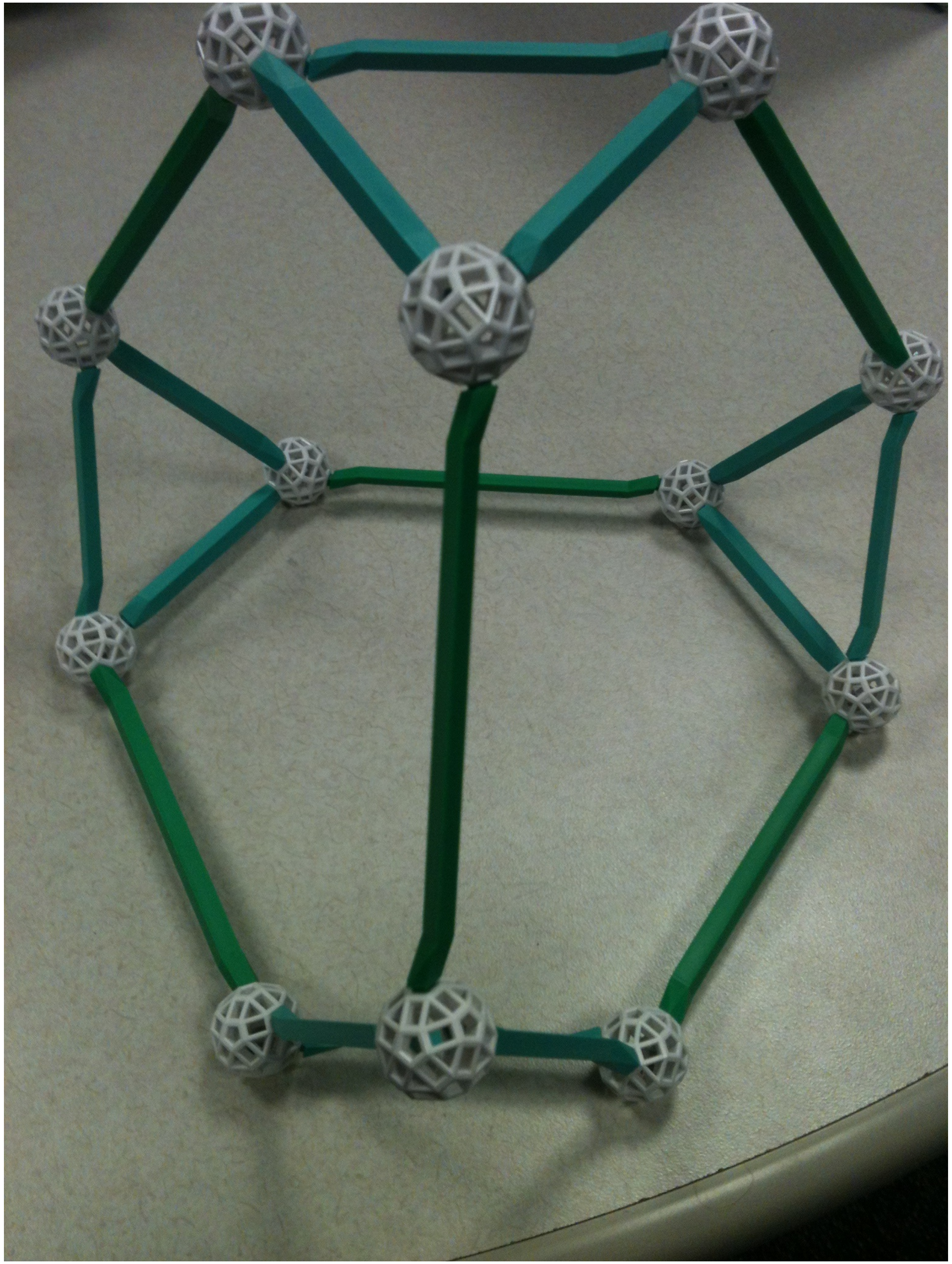
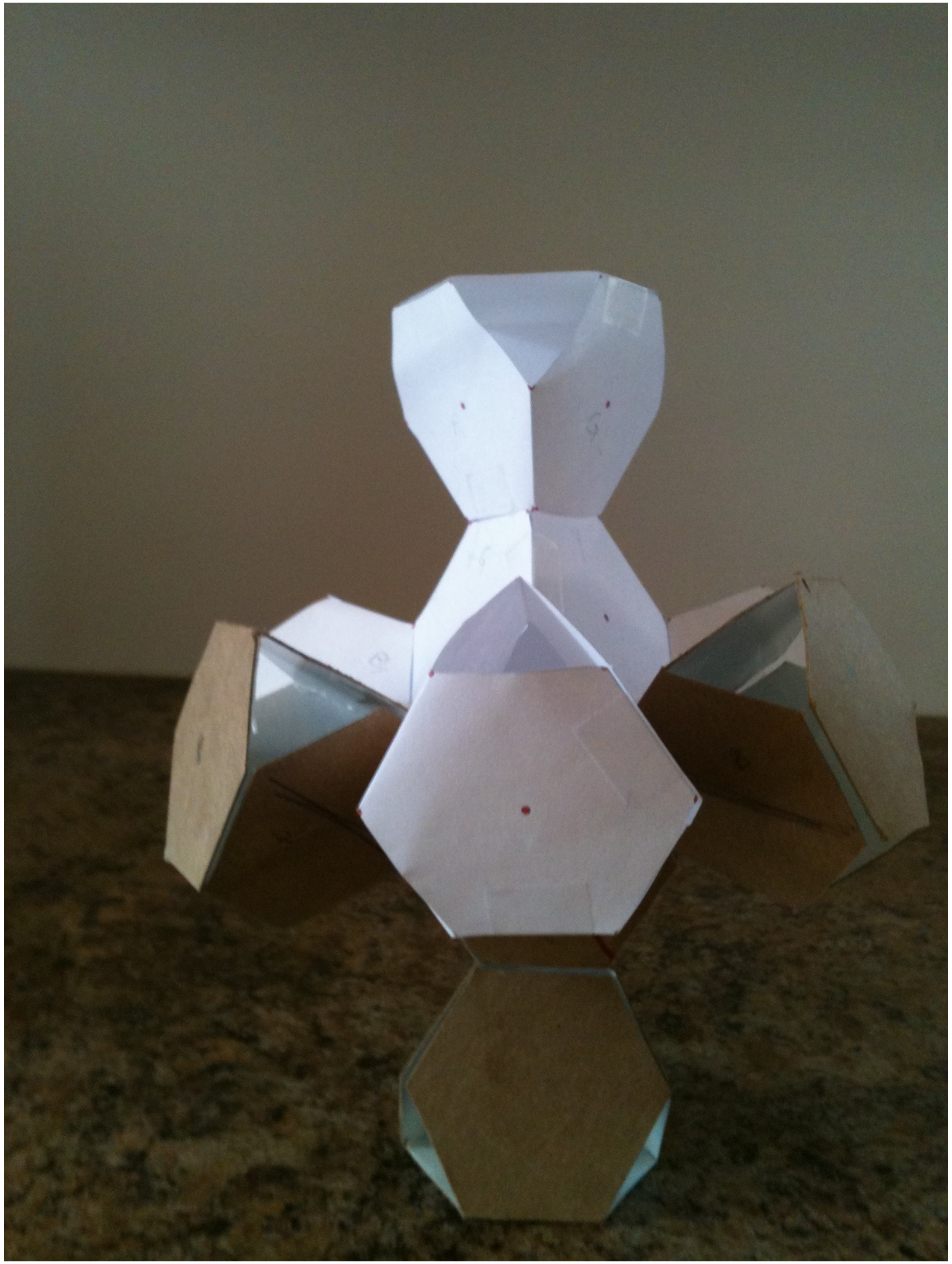
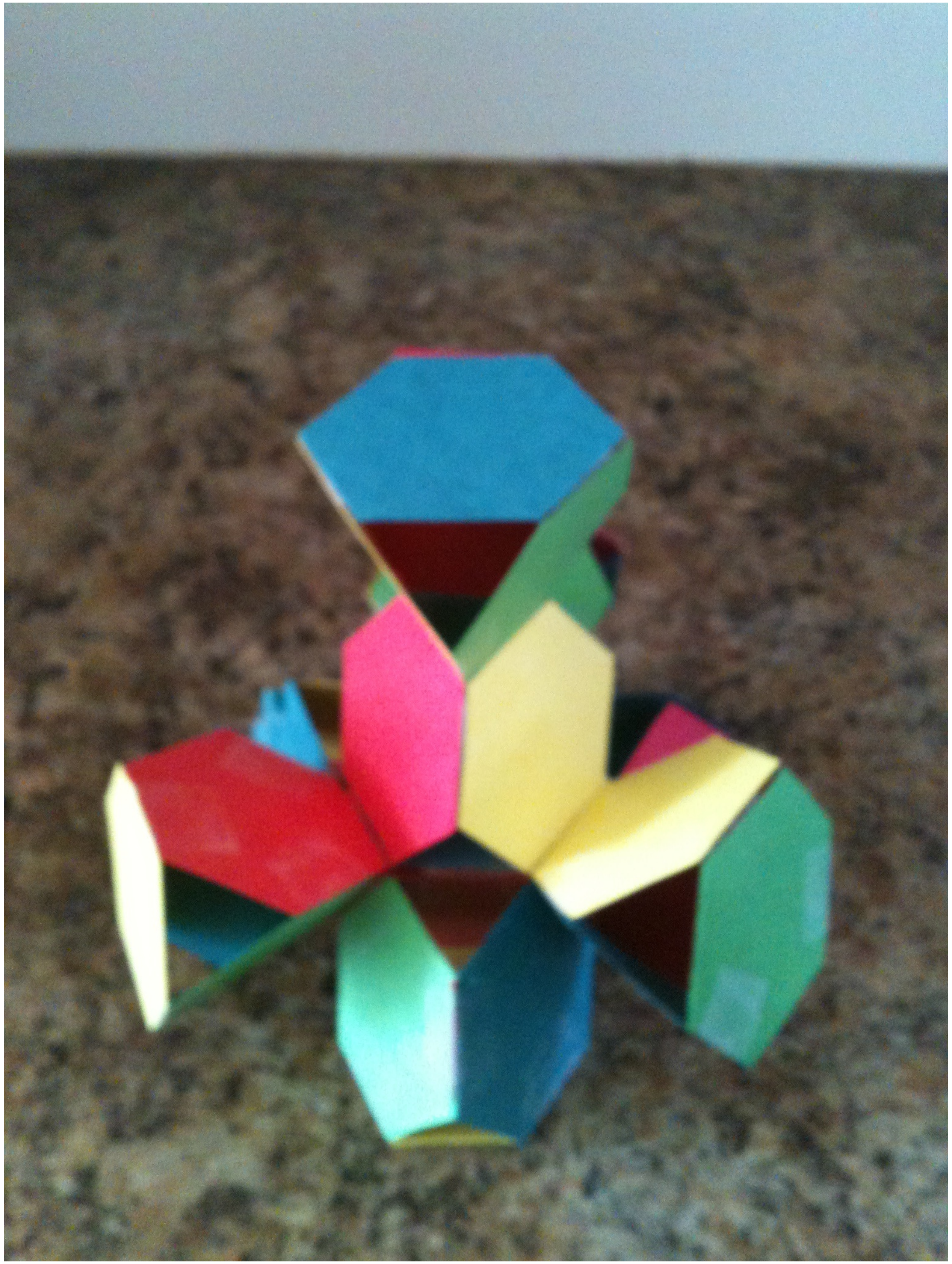


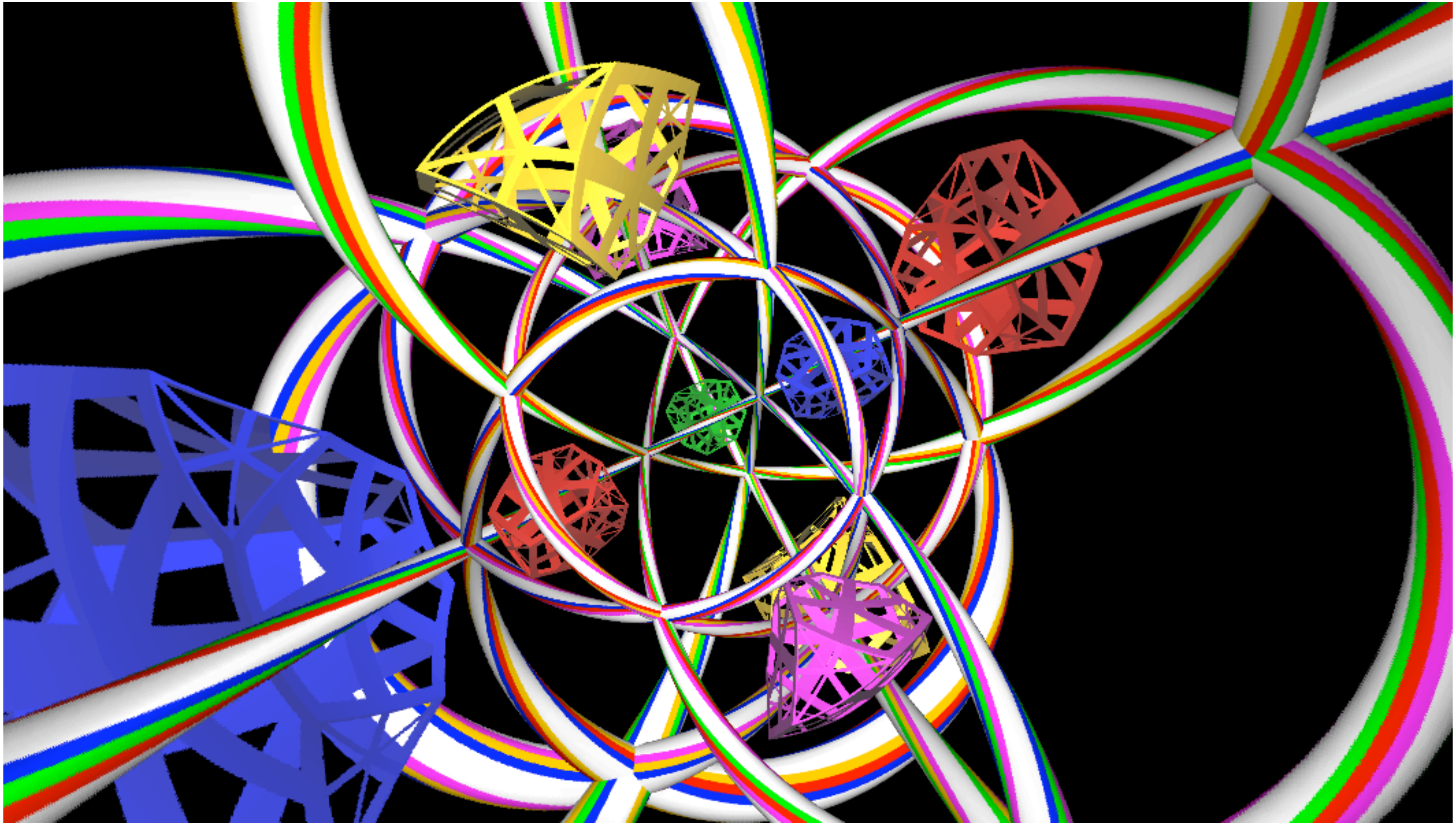
Figure 4: Hyperplane slice of 4-cube

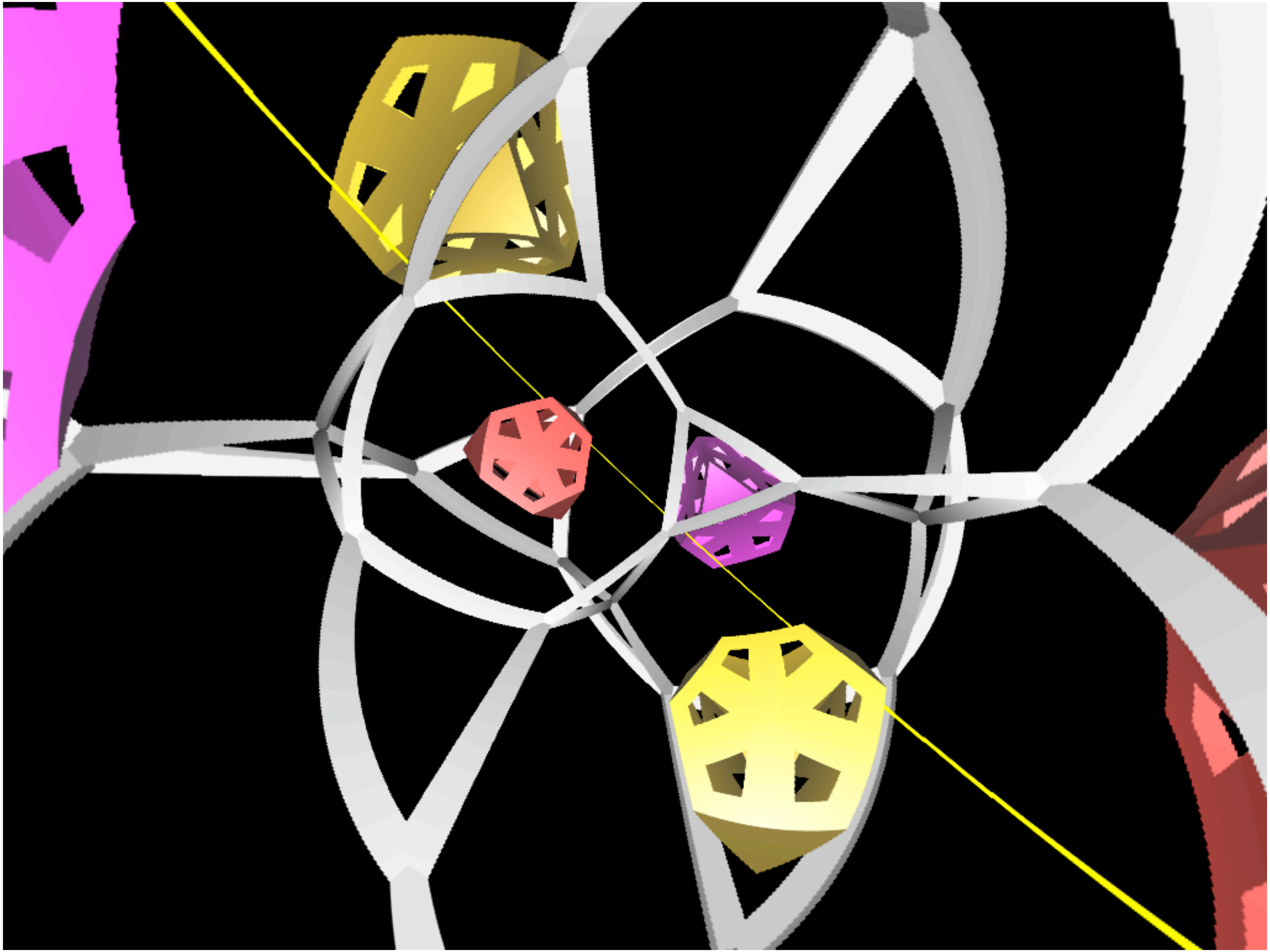
324

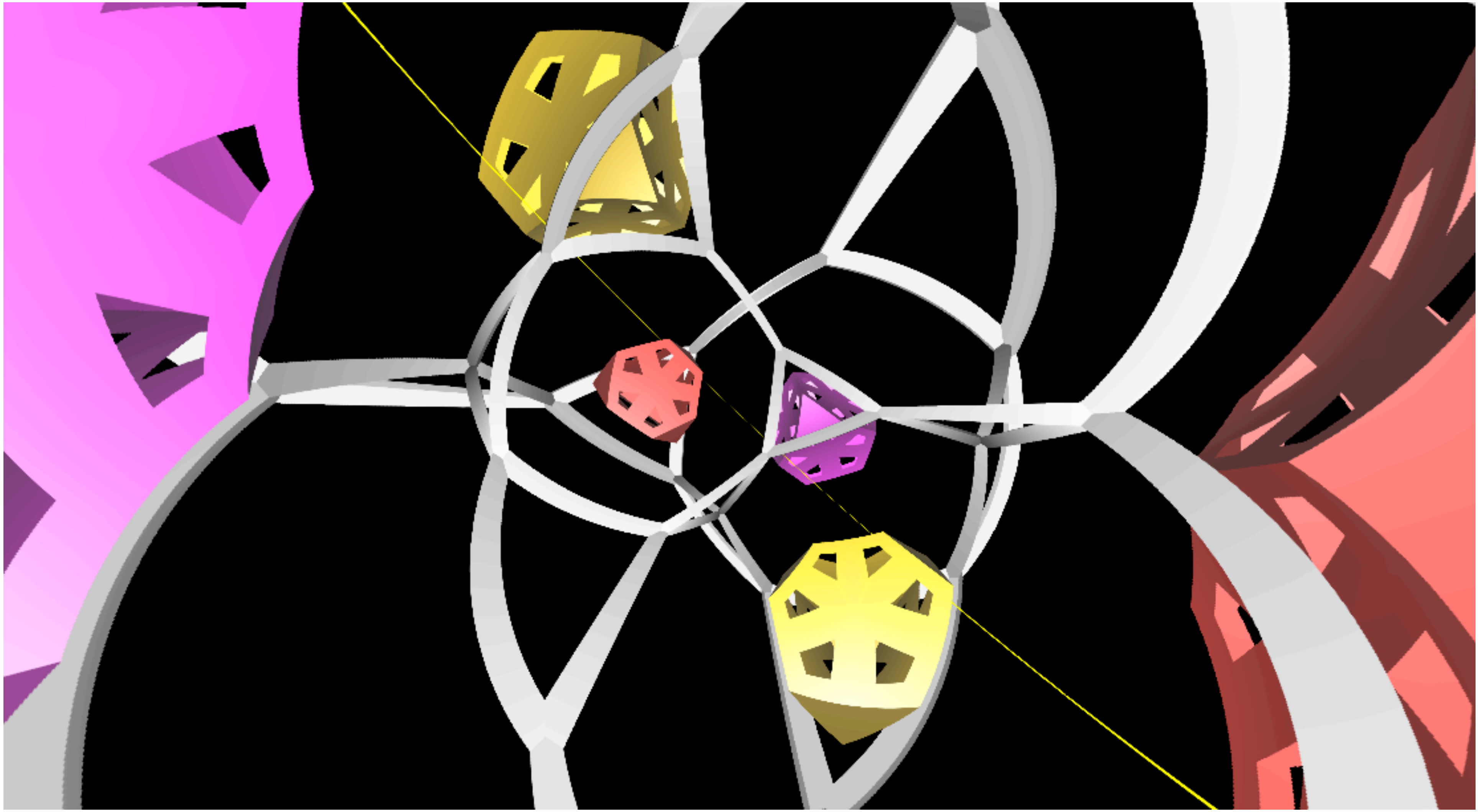


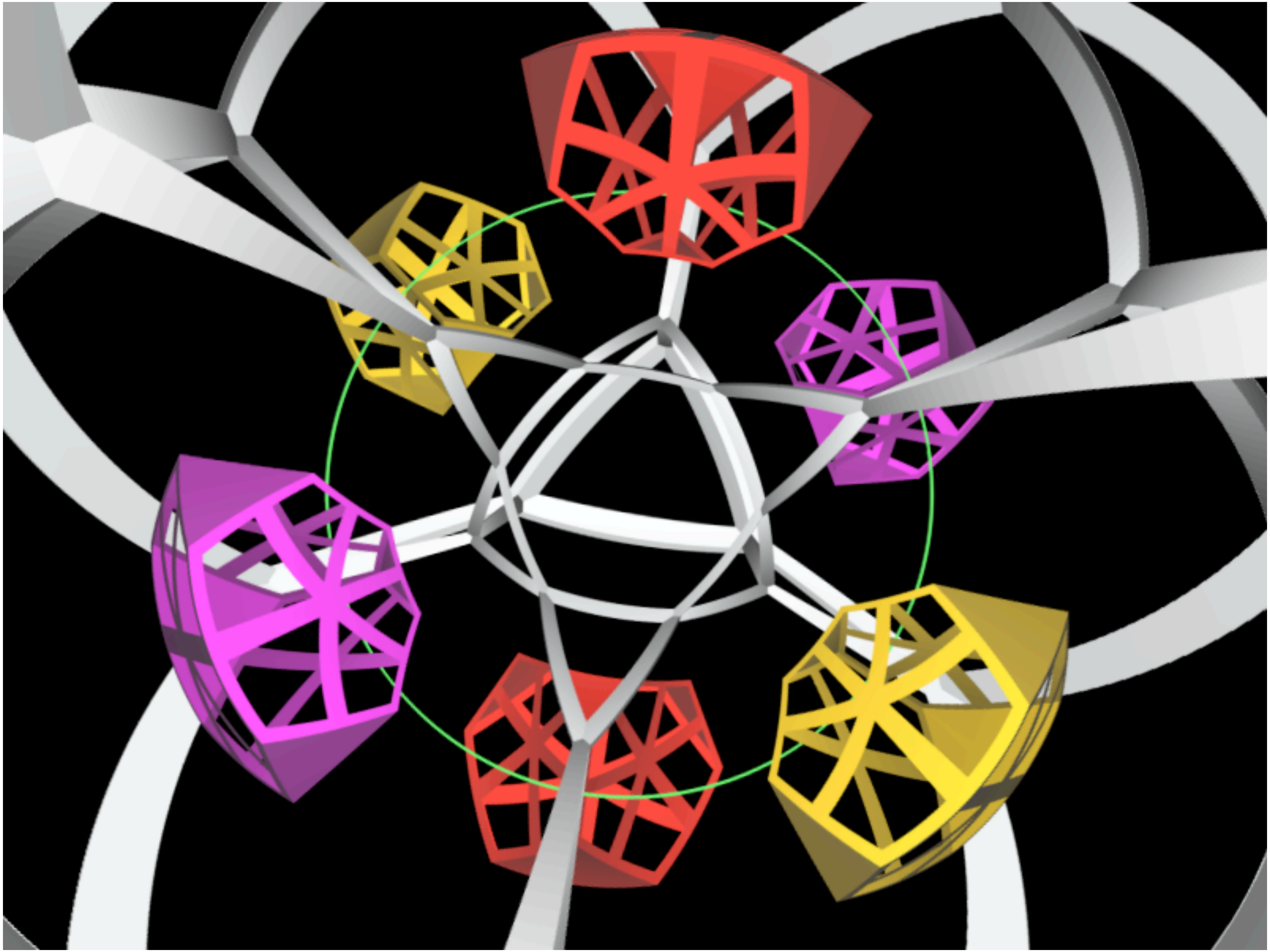


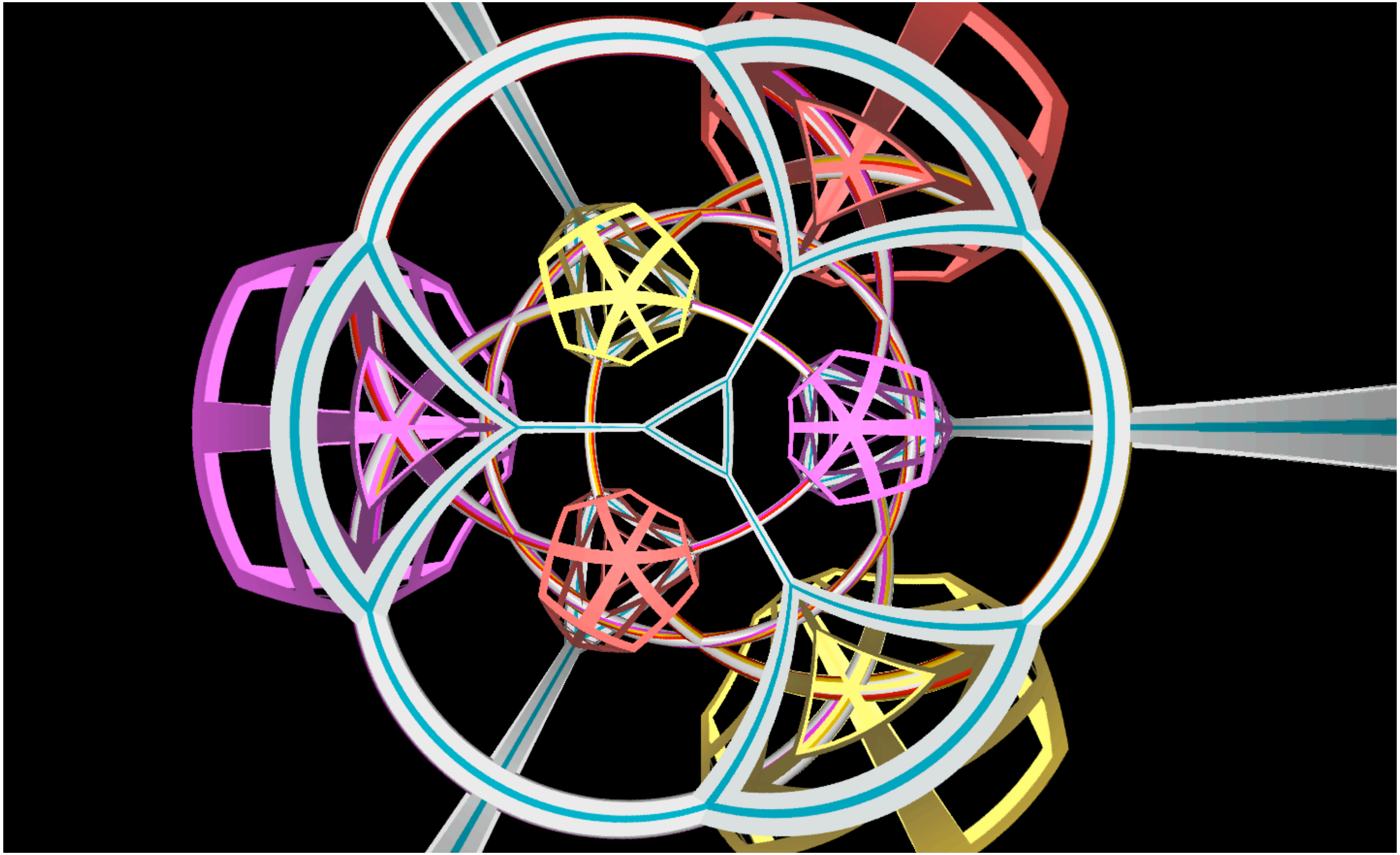












- Additional slides for questions after the talk:

Cycle Level Duality for SW Classes

- $F: M^2 \rightarrow R^3$, immersion
- $P_{ij}:R^3 \rightarrow R^2_{ij} = [E_i] \oplus [E_j]$
- $S(P_{ij}F) = \{ x \text{ in } M \mid T_xF \perp R^2_{ij} \}$
- $S(P_{13}F) \cap S(P_{23}F) = S(P_3F)$
- $W_1 \cup \underline{W_1} = W_2$

Cycle Level Duality for SW Classes

- $F: M^2 \rightarrow R^3$
- $P_i: R^3 \rightarrow R^1{}_j = [E_j]$, $P_3F: M^2 \rightarrow R^1{}_3$
- $S(P_3F) = \{x \text{ in } M \mid T_X F \perp R_3\}$
- $P_{ij}: R^3 \rightarrow R^2{}_{ij} = [E_i] \oplus [E_j]$
- $S(P_{ij}F) = \{ x \text{ in } M \mid T_x F \perp R^2{}_{ij} \}$

Cycle Level Duality for SW Classes

- $F: M^2 \rightarrow R^3$
- $P_i: R^3 \rightarrow R^1{}_j = [E_j]$, $P_3F: M^2 \rightarrow R^1{}_3$
- $S(P_3F) = \{x \text{ in } M \mid T_XF \perp R_3\}$
- $P_{ij}: R^3 \rightarrow R^2{}_{ij} = [E_i] \oplus [E_j]$
- $S(P_{ij}F) = \{ x \text{ in } M \mid T_xF \perp R^2{}_{ij} \}$
- $S(P_{13}F) \cap S(P_{23}F) = S(P_3F)$

Cycle Level Duality for SW Classes

- $F: M^2 \rightarrow R^4$, stable map from R^4
- $P_{ijk}: R^4 \rightarrow R^3_{ijk} = [E_i] \oplus [E_j] \oplus [E_k]$
- $S(P_{ijk}F) = \{ x \text{ in } M \mid T_xF \perp R^2_{ijk} \}$
- $S(P_{13}F) \cap S(P_{23}F) = S(P_3F) \cup S(P_{123}F)$
- $W_1 \cup \underline{W_1} = W_2 + \underline{W}_2$